The GIT Boundary of Quintic Threefolds (Announcement of Results)

Yasutaka Shibata

Abstract

We give an explicit description of the strictly semistable boundary of the GIT moduli space of quintic threefolds. More precisely, for the natural action of SL(5) on $\mathbb{P}(\mathrm{Sym}^5\mathbb{C}^5)$, we classify the 38 boundary components arising from the maximal strictly semistable supports and construct closed-orbit normal forms for the general polystable member in each component. The closed-orbit verification is carried out uniformly by Luna's centralizer reduction: toric centralizers are treated by the convex-hull criterion, while non-toric centralizers are handled using the Casimiro–Florentino criterion.

We also announce the pairwise non-inclusion of the 38 boundary families and record the structure of its computational verification. The argument uses a successive system of semicontinuity and stabilizer obstructions, including component dimensions, apolar Hilbert functions and graded Betti tables, Hilbert functions of singular schemes, singular one-cycle data together with Hessian ranks, and generic stabilizer tori.

For each boundary component, we determine the singular locus of a general closed-orbit representative and compute the corresponding local minimal exponents. The positive-dimensional singular loci include lines and line arrangements, conics, cuspidal plane curves, planes, and smooth quartic surfaces. The isolated singularities that occur on the boundary are quasi-homogeneous and fall into eleven analytic types, with local minimal exponent equal to 1. As a consequence, the global minimal exponent of a general strictly semistable closed-orbit quintic threefold in every boundary component is

$$\widetilde{\alpha}(X) = 1 = \frac{4+1}{5},$$

the critical value appearing in the stability criterion for quintic hypersurfaces in $\mathbb{P}^4$.

Finally, we compute the codimension-one wall-adjacency relation among the boundary components by an explicit slice-matching calculation. The resulting adjacency graph has 38 vertices and 184 edges; it is connected and has no isolated vertices. In particular, any two boundary components can be connected by a chain of at most four codimension-one adjacencies. This paper is an announcement of our results. The detailed proofs and the complete case-by-case computations will appear in a forthcoming full-length paper.

1 Introduction

This paper studies the Geometric Invariant Theory compactification of the moduli space of quintic threefolds. We consider the natural action of

$$G = \mathrm{SL}(5)$$

on

$$\mathbb{P}(W), \qquad W = \mathrm{Sym}^5\,\mathbb{C}^5,$$

and the corresponding GIT quotient

$$\mathcal{M}^{\mathrm{GIT}} := \mathbb{P}(W)^{ss}/\!/G.$$

The open locus of smooth quintic hypersurfaces in $\mathbb{P}^4$ is one of the basic moduli spaces in Calabi–Yau geometry. Its dimension is

$$\dim \mathbb{P}(\mathrm{Sym}^5 \mathbb{C}^5) - \dim \mathrm{PGL}(5) = 125 - 24 = 101.$$

The GIT compactification adds boundary points represented by strictly semistable quintics. The purpose of this paper is to give an explicit description of this boundary.

This paper is an announcement of results. We record the final classification of the boundary components, the closed-orbit normal forms for their general polystable representatives, the singular-locus and minimal-exponent tables, the pairwise non-inclusion statement, and the codimension-one wall-adjacency graph. The full certified proofs, including the complete case-by-case computations, exact specializations, and reproducibility details, will appear in a forthcoming full-length paper.

An important antecedent to the present work is Lakhani's study of the GIT compactification of quintic threefolds [Lak10]. We regard that work as a foundational reference for the GIT compactification problem in this setting. The present announcement develops this direction further by giving a concrete atlas of the strictly semistable boundary: we enumerate the maximal strictly semistable supports, pass from the torus data to the full SL(5)-quotient, construct closed-orbit normal forms, analyze the resulting singular loci and minimal exponents, prove pairwise non-inclusion of the quotient-side boundary families, and compute the codimension-one wall-adjacency relation.

We first fix the notation for the boundary. Let

$$\pi : \mathbb{P}(W)^{ss} \longrightarrow \mathcal{M}^{\mathrm{GIT}}$$

be the good quotient map, and put

$$\Sigma^{ss} := \mathbb{P}(W)^{ss} \setminus \mathbb{P}(W)^{s}.$$

Thus Σ^{ss} is the strictly semistable locus before taking the quotient. The GIT boundary in the quotient is

$$\partial\mathcal{M}^{\mathrm{GIT}} := \pi(\Sigma^{ss}) \subset \mathcal{M}^{\mathrm{GIT}}.$$

Throughout this paper we distinguish carefully between objects before and after taking the quotient. Tilded symbols will denote prequotient loci inside $\mathbb{P}(W)^{ss}$, while untilded symbols will denote their images in the GIT quotient.

Let $T \subset G$ be the diagonal maximal torus. We write

$$I := \{u = (u_0, \dots, u_4) \in \mathbb{Z}^5_{\geq 0} \mid u_0 + \cdots + u_4 = 5\}$$

for the set of exponent vectors of degree-five monomials, and

$$x^u := x_0^{u_0} \cdots x_4^{u_4}.$$

For $r \in \mathbb{Z}^5$ with $r_0 + \cdots + r_4 = 0$, set

$$I(r)_{\geq 0} := \{u \in I \mid r \cdot u \geq 0\}, \qquad I(r)_{=0} := \{u \in I \mid r \cdot u = 0\}.$$

The Hilbert–Mumford numerical criterion, in its convex-geometric form for the fixed torus T, reduces the enumeration of maximal strictly semistable supports to a finite search. The result is a list of thirty-eight maximal T-strictly semistable supports

$$S_k = I(r_k)_{\geq 0}, \qquad k = 1, \dots, 38,$$

where the vectors r_k are listed in Section 2.

For each support S_k, define

$$V_k := \mathrm{Span}_{\mathbb{C}}\{x^u \mid u \in S_k\} \subset W.$$

The corresponding prequotient family is

$$\widetilde{\Phi}_k := \overline{G \cdot \mathbb{P}(V_k)} \cap \Sigma^{ss} \subset \mathbb{P}(W)^{ss},$$

where the closure is taken in $\mathbb{P}(W)$. The quotient-side boundary family is then defined by

$$\Phi_k := \pi(\widetilde{\Phi}_k) \subset \partial\mathcal{M}^{\mathrm{GIT}}.$$

Thus $\widetilde{\Phi}_k$ is the G-saturated family before taking the quotient, whereas Φ_k is the corresponding subvariety of the GIT boundary. In particular, whenever we speak of a boundary component Φ_k, or of an inclusion

$$\Phi_k \subset \Phi_\ell,$$

the statement is made inside the quotient $\mathcal{M}^{\mathrm{GIT}}$. Similarly, all dimensions denoted $\dim \Phi_k$ are quotient dimensions, not dimensions of $\widetilde{\Phi}_k$ inside $\mathbb{P}(W)^{ss}$.

The distinction between $\widetilde{\Phi}_k$ and Φ_k is important. The locus Σ^{ss} remembers all strictly semistable forms before identifying S-equivalent points, while $\partial\mathcal{M}^{\mathrm{GIT}}$ records only the corresponding closed polystable orbits. A general member of Φ_k is represented by a closed orbit lying in the closure of the support family $\mathbb{P}(V_k)$. The construction of this closed orbit is one of the main computations of the paper.

The case of quintic threefolds is especially natural from the viewpoint of minimal exponents. For hypersurfaces of degree d in $\mathbb{P}^n$, the threshold appearing in Park's stability criterion is

$$\frac{n+1}{d}.$$

For quintic threefolds, $(n, d) = (4, 5)$, so the threshold is

$$\frac{4+1}{5} = 1.$$

One of the main features of the classification announced below is that the general closed-orbit representative in every strictly semistable boundary component realizes this equality:

$$\widetilde{\alpha}(X) = 1.$$

Thus the boundary components described here are not merely a list of GIT degenerations; they also give a concrete collection of quintic threefolds lying exactly at the critical minimal-exponent threshold.

We now summarize the main results.

Theorem A (Boundary families and irreducible components). For the natural action of $\mathrm{SL}(5)$ on $\mathbb{P}(\mathrm{Sym}^5 \mathbb{C}^5)$, the strictly semistable locus is covered by the thirty-eight prequotient families:

$$\Sigma^{ss} = \widetilde{\Phi}_1 \cup \cdots \cup \widetilde{\Phi}_{38}.$$

Consequently, the GIT boundary is

$$\partial\mathcal{M}^{\mathrm{GIT}} = \Phi_1 \cup \cdots \cup \Phi_{38}.$$

The families Φ_k arise from the thirty-eight maximal strictly semistable supports

$$S_k = I(r_k)_{\geq 0}, \qquad k = 1, \ldots, 38,$$

listed in Section 2. Moreover, no two distinct quotient-side families contain one another:

$$\Phi_k \not\subset \Phi_\ell \qquad (k \neq \ell).$$

Hence the thirty-eight subvarieties

$$\Phi_1, \dots, \Phi_{38} \subset \partial\mathcal{M}^{\mathrm{GIT}}$$

are precisely the irreducible components of the GIT boundary.

Theorem B (Closed-orbit normal forms). For each boundary component Φ_k, a general point is represented by an explicit closed-orbit normal form

$$\phi_k^{\mathrm{nf}} \in \mathrm{Sym}^5 \mathbb{C}^5.$$

These normal forms are recorded in Section 4. They are obtained as follows. Starting from a general quintic f_k supported on $S_k = I(r_k)_{\geq 0}$, we take the one-parameter-subgroup limit

$$\varphi_k := \lim_{t \to 0} \lambda_{r_k}(t) \cdot f_k.$$

This limit lies in the fixed subspace for

$$H_k := \lambda_{r_k}(\mathbb{G}_m).$$

By Luna's centralizer reduction, the closedness of the SL(5)-orbit is reduced to a closed-orbit problem for the centralizer $C_G(H_k)$ acting on the H_k-fixed subspace. If $C_G(H_k)$ is toric, closedness is checked by the convex-hull criterion. If $C_G(H_k)$ is non-toric, closedness is checked by the Casimiro–Florentino criterion. This gives a uniform method for constructing the general polystable representative in each of the thirty-eight components.

Theorem C (Singular loci on the boundary). Let

$$X_k := V(\phi_k^{\mathrm{nf}}) \subset \mathbb{P}^4$$

be a general closed-orbit representative of the k-th boundary component. Then the singular locus

$$\mathrm{Sing}(X_k)$$

is explicitly determined for every $k = 1, \dots, 38$. The positive-dimensional parts that occur are low-degree configurations: lines and line arrangements, smooth conics, cuspidal plane curves, planes, and smooth quartic surfaces. In several cases these positive-dimensional components are accompanied by isolated singular points. All isolated singularities appearing for general boundary representatives are quasi-homogeneous and fall into exactly eleven analytic types. These eleven types are listed in Section 5 and are referred to as the extremal quintic threefold singularities.

Theorem D (Minimal exponents on the boundary). For every $k = 1, \dots, 38$, the global minimal exponent of a general closed-orbit quintic threefold

$$X_k = V(\phi_k^{\mathrm{nf}})$$

is equal to the critical value

$$\widetilde{\alpha}(X_k) = 1 = \frac{4+1}{5}.$$

More precisely, every isolated boundary singularity listed in Section 5 is weighted homogeneous with local minimal exponent equal to 1. For the positive-dimensional singular loci, the corresponding local computations along the general and special points again give global minimum 1.

Thus every boundary component contains a general polystable representative whose singularities realize exactly the equality case of the stability threshold.

Theorem E (Codimension-one wall-adjacency). The codimension-one wall-adjacency relation among the thirty-eight boundary components is determined by an explicit mutual slice-matching computation. The resulting wall-adjacency graph has 38 vertices and 184 edges. It is connected, has no isolated vertices, and has diameter 4. In particular, any two boundary components can be connected by a chain of at most four codimension-one wall adjacencies. The full neighbor sets are recorded in Section 7.

We briefly explain the structure of the verification. The starting point is the Hilbert–Mumford numerical criterion. After fixing the diagonal maximal torus $T \subset \mathrm{SL}(5)$, the criterion becomes a convex-geometric condition on monomial supports. If

$$\eta = (1, 1, 1, 1, 1)$$

is the barycenter of the degree-five simplex, then T-semistability is equivalent to the condition

$$\eta \in \mathrm{Conv}(\mathrm{Supp}(f)).$$

Strict T-semistability means that η lies on the boundary of this convex hull. This reduces the enumeration of maximal strictly semistable supports to a finite convex-geometric search. The outcome is the list

$$S_k = I(r_k)_{\geq 0}, \qquad k = 1, \ldots, 38.$$

The next step is to pass from the torus classification to the full SL(5)-quotient. For each support S_k, the prequotient family

$$\widetilde{\Phi}_k = \overline{G \cdot \mathbb{P}(V_k)} \cap \Sigma^{ss}$$

records all strictly semistable quintics obtained from the support family by SL(5)-saturation. Its quotient image

$$\Phi_k = \pi(\widetilde{\Phi}_k)$$

is the actual boundary family in $\partial \mathcal{M}^{\mathrm{GIT}}$. The main point is that these thirty-eight quotient-side families do not collapse into one another. This is the pairwise non-inclusion theorem.

The non-inclusion verification is organized as a sequence of monotone obstructions to specialization. Starting from the

$$38 \cdot 37 = 1406$$

ordered candidate inclusions $\Phi_k \subset \Phi_\ell$, $k \neq \ell$, we apply successive filters using quotient dimensions, apolar Hilbert functions and graded Betti tables, Hilbert functions of singular schemes, singular one-cycle data together with Hessian ranks, and finally generic connected stabilizer tori. Each filter rules out a subset of the remaining ordered pairs, and together the filters eliminate all 1406 candidates. In this announcement we record the structure and the complete numerical output of this verification; the forthcoming full-length paper will give the certified computational details.

The closed-orbit computation is also uniform. For each k, the one-parameter-subgroup limit

$$\varphi_k = \lim_{t \to 0} \lambda_{r_k}(t) \cdot f_k$$

lies in the H_k-fixed subspace of W. Luna's centralizer reduction replaces the closed-orbit question for $G = \mathrm{SL}(5)$ by a residual closed-orbit question for $C_G(H_k)$. This yields the normal forms

$$\phi_k^{\mathrm{nf}}$$

listed in Section 4. These normal forms are the reference representatives for all subsequent singularity and minimal-exponent computations.

The singularity analysis is then performed directly from the normal forms. For

$$X_k = V(\phi_k^{\mathrm{nf}}) \subset \mathbb{P}^4,$$

we compute the saturated Jacobian scheme

$$\mathrm{Sing}(X_k) = V\left(\left(\frac{\partial \phi_k^{\mathrm{nf}}}{\partial x_0}, \dots, \frac{\partial \phi_k^{\mathrm{nf}}}{\partial x_4}\right) : (x_0, \dots, x_4)^\infty\right).$$

This determines the positive-dimensional singular components and the isolated singular points. At each isolated point, local coordinates put the germ into a weighted-homogeneous normal form. These local normal forms give the eleven extremal singularity types and show that each has local minimal exponent equal to 1. The positive-dimensional cases are handled by explicit local analysis along their general and special points, giving the same global minimum

$$\widetilde{\alpha}(X_k) = 1.$$

Finally, the adjacency computation records a combinatorial wall-adjacency skeleton of the boundary. For a wall $I(r_i)_{=0}$, we compare codimension-one slices and test whether such a slice is simultaneously realized from another maximal state r_j. A mutual slice match gives an undirected wall-adjacency edge. One-sided slice matches are also useful as auxiliary incidence data, but they are not counted as codimension-one wall-adjacencies. The resulting graph has 38 vertices, 184 edges, and diameter 4.

We end by describing the organization of the paper. Section 2 recalls the Hilbert–Mumford numerical criterion and enumerates the thirty-eight maximal strictly semistable supports for the fixed maximal torus. Section 3 defines the prequotient families $\widetilde{\Phi}_k$, the quotient-side boundary families Φ_k, and states the boundary-component and non-inclusion theorems. Section 4 gives the closed-orbit normal forms and the closedness tests via Luna's centralizer reduction. Section 5 determines the singular loci of the general closed-orbit representatives and lists the extremal quintic threefold singularities. Section 6 summarizes the pairwise non-inclusion verification by the successive-filter argument. Section 7 computes the codimension-one wall-adjacency graph. Section 8 explains the computational framework underlying the announced results and indicates how the full case-by-case verification is organized in the forthcoming full-length paper.

2 Numerical criterion and maximal strictly semistable supports

2.1 Numerical criterion

We work throughout with

$$W = \mathrm{Sym}^5 \mathbb{C}^5, \qquad G = \mathrm{SL}(5), \qquad \mathbb{P}(W) = \mathbb{P}(\mathrm{Sym}^5 \mathbb{C}^5),$$

and with homogeneous coordinates

$$[x_0 : x_1 : x_2 : x_3 : x_4]$$

on $\mathbb{P}^4$. Let $T \subset G$ be the diagonal maximal torus.

For a vector $u = (u_0, \dots, u_4) \in \mathbb{Q}^5$, set

$$\mathrm{wt}(u) := u_0 + \cdots + u_4.$$

We use the notation

$$\mathbb{Z}^5(d) := \{u \in \mathbb{Z}^5 \mid \mathrm{wt}(u) = d\}, \qquad \mathbb{Z}^5_{\geq 0} := \{u \in \mathbb{Z}^5 \mid u_i \geq 0 \text{ for all } i\},$$

and

$$I := \mathbb{Z}^5(5) \cap \mathbb{Z}^5_{\geq 0}.$$

Thus I is the set of exponent vectors of degree-five monomials in $x_0, \ldots, x_4$. We write

$$x^u := x_0^{u_0} x_1^{u_1} x_2^{u_2} x_3^{u_3} x_4^{u_4} \qquad (u \in I).$$

The barycenter of the degree-five simplex is

$$\eta = (1, 1, 1, 1, 1).$$

For a vector $r \in \mathbb{Q}^5$, define

$$I(r)_{\geq 0} := \{u \in I \mid r \cdot u \geq 0\}, \quad I(r)_{>0} := \{u \in I \mid r \cdot u > 0\}, \quad I(r)_{=0} := \{u \in I \mid r \cdot u = 0\}.$$

A nonzero vector $r \in \mathbb{Z}^5$ is called *reduced* if

$$\gcd(r_0, r_1, r_2, r_3, r_4) = 1.$$

Let

$$f = \sum_{u \in I} a_u x^u \in W.$$

The support of f is

$$\operatorname{Supp}(f) := \{u \in I \mid a_u \neq 0\}.$$

A one-parameter subgroup of T is written, after choosing the above basis, as

$$\lambda_r(t) = \operatorname{diag}(t^{r_0}, t^{r_1}, t^{r_2}, t^{r_3}, t^{r_4}), \qquad r = (r_0, \ldots, r_4) \in \mathbb{Z}^5(0).$$

Definition 2.1. Let $s \subset I$. We say that s is *not stable with respect to* T if

$$s \subset I(r)_{\geq 0}$$

for some nonzero $r \in \mathbb{Z}^5(0)$. We say that s is *unstable with respect to* T if

$$s \subset I(r)_{>0}$$

for some nonzero $r \in \mathbb{Z}^5(0)$.

For $0 \neq f \in W$, we say that f is not stable, respectively unstable, with respect to T if $\operatorname{Supp}(f)$ has the corresponding property.

Equivalently, for the fixed torus T, the convex-geometric form of the criterion is

$$f \text{ is } T\text{-semistable} \iff \eta \in \operatorname{Conv}(\operatorname{Supp}(f)),$$

and

$$f \text{ is } T\text{-stable} \iff \eta \in \operatorname{relint} \operatorname{Conv}(\operatorname{Supp}(f)),$$

where the relative interior is taken inside the affine hyperplane $\mathrm{wt} = 5$. Thus f is strictly T-semistable precisely when

$$\eta \in \operatorname{Conv}(\operatorname{Supp}(f)) \quad \text{but} \quad \eta \notin \operatorname{relint} \operatorname{Conv}(\operatorname{Supp}(f)).$$

Theorem 2.2 (Hilbert–Mumford numerical criterion)**.** *Let $0 \neq f \in W = \operatorname{Sym}^5 \mathbb{C}^5$. The point $[f] \in \mathbb{P}(W)$, equivalently the quintic threefold $X_f = V(f) \subset \mathbb{P}^4$, is not stable, respectively unstable, for the $G = \mathrm{SL}(5)$-action if and only if there exists $\sigma \in \mathrm{SL}(5)$ such that $\sigma \cdot f$ is not stable, respectively unstable, with respect to the fixed torus T.*

In particular, $[f]$ is strictly semistable for the $\mathrm{SL}(5)$-action if and only if

1. *there exists $\sigma \in \mathrm{SL}(5)$ such that $\sigma \cdot f$ is not stable with respect to T, and*
2. *for every $\sigma \in \mathrm{SL}(5)$, the transform $\sigma \cdot f$ is semistable with respect to T.*

Proof. This is the Hilbert–Mumford numerical criterion for the action of $\mathrm{SL}(5)$ on $\mathbb{P}(\operatorname{Sym}^5 \mathbb{C}^5)$, written in the language of supports. Since every one-parameter subgroup of $\mathrm{SL}(5)$ is conjugate to one contained in the fixed maximal torus T, it is enough to test diagonal one-parameter subgroups after an $\mathrm{SL}(5)$-change of coordinates. The support conditions above are exactly the inequalities imposed by the Hilbert–Mumford weights. $\square$

2.2 Maximal strictly semistable supports for the fixed torus

We now enumerate the maximal supports that can occur for strictly semistable quintic threefolds with respect to the fixed maximal torus T. Put

$$\mathcal{S} := \{I(r)_{\geq 0} \mid r \in \mathbb{Z}^5(0),\ r \neq 0\},$$

ordered by inclusion. Since different vectors r may define the same subset $I(r)_{\geq 0}$, it is useful to enumerate supporting hyperplanes rather than all one-parameter subgroups directly.

Lemma 2.3. *Let $I(r)_{\geq 0}$ be a maximal element of $\mathcal{S}$. Then there exist three elements*

$$u_1, u_2, u_3 \in I$$

and a reduced vector $r' \in \mathbb{Z}^5(0)$ such that:

1. *the $\mathbb{Q}$-vector subspace of $\mathbb{Q}^5$ spanned by u_1, u_2, u_3, η has dimension 4;*
2. *the vector r' is orthogonal to this subspace;*
3. *$I(r)_{\geq 0} = I(r')_{\geq 0}$.*

Proof. Let

$$C := I(r)_{\geq 0} \cup \{\eta\}$$

and consider $\mathrm{Conv}(C) \subset \mathbb{Q}^5$. Choose a facet F of $\mathrm{Conv}(C)$ containing η. Let $r' \in \mathbb{Z}^5$ be a primitive integral normal vector to F, chosen so that

$$\mathrm{Conv}(C) \subset \{x \in \mathbb{Q}^5 \mid r' \cdot x \geq 0\}.$$

Since $\eta \in F$, we have $r' \cdot \eta = 0$. Because $\eta = (1,1,1,1,1)$, this gives $r' \in \mathbb{Z}^5(0)$.

The affine hyperplane $\mathrm{wt} = 5$ has dimension 4, and a facet in it has dimension 3. Hence we may choose three points $u_1, u_2, u_3 \in I \cap F$ such that u_1, u_2, u_3, η span a 4-dimensional vector subspace of $\mathbb{Q}^5$. By construction, r' is orthogonal to this subspace. Moreover,

$$I(r)_{\geq 0} \subset I(r')_{\geq 0}.$$

The maximality of $I(r)_{\geq 0}$ inside $\mathcal{S}$ therefore forces

$$I(r)_{\geq 0} = I(r')_{\geq 0}.$$

□

The preceding lemma gives a finite enumeration procedure.

Enumeration algorithm. Let $\mathcal{F}$ be the set of all ordered triples

$$\mathbf{u} = (u_1, u_2, u_3)$$

of pairwise distinct points of I. Fix a total order on $\mathcal{F}$. Start with $\mathcal{M}' = \emptyset$. For each $\mathbf{u} = (u_1, u_2, u_3) \in \mathcal{F}$, perform the following steps.

Step 1. Let $U \subset \mathbb{Q}^5$ be the $\mathbb{Q}$-subspace spanned by

$$u_1, u_2, u_3, \eta.$$

If $\dim_{\mathbb{Q}} U = 4$, take a reduced normal vector

$$r = (r_0, \ldots, r_4) \in \mathbb{Z}^5(0) \setminus \{0\}$$

of U, and go to Step 2. Otherwise, discard this triple.

Step 2. If
$$r_0 \geq r_1 \geq r_2 \geq r_3 \geq r_4$$
or
$$r_0 \leq r_1 \leq r_2 \leq r_3 \leq r_4,$$
go to Step 3. Otherwise, discard this triple. This step chooses a representative modulo the permutation action on the variables.

Step 3. If $r_0 \geq r_1 \geq r_2 \geq r_3 \geq r_4$, add $I(r)_{\geq 0}$ to $\mathcal{M}'$. If $r_0 \leq r_1 \leq r_2 \leq r_3 \leq r_4$, add $I(-r)_{\geq 0}$ to $\mathcal{M}'$.

Step 4. Delete from $\mathcal{M}'$ all elements that are not maximal with respect to inclusion.

After all triples have been processed, retain only those maximal supports $S \in \mathcal{M}'$ satisfying the barycenter condition
$$\eta \in \operatorname{Conv}(S).$$
These are precisely the maximal T-strictly semistable supports, up to permutation of the coordinates.

Remark 2.4. In the actual computation one can replace the direct ordered-triple enumeration by an orbit-pruned search under the Weyl group S_5. This does not change the mathematical output; it only removes repeated representatives during the finite search.

Proposition 2.5. *For* $(\dim \mathbb{C}^5, d) = (5, 5)$*, the maximal* T*-strictly semistable supports are, up to permutation of the coordinates, the thirty-eight supports*
$$S_k := I(r_k)_{\geq 0} \qquad (k = 1, \ldots, 38)$$
listed below. Each vector r_k *is reduced and written in the dominant chamber* $r_{k,0} \geq r_{k,1} \geq r_{k,2} \geq r_{k,3} \geq r_{k,4}$.

Proof. The finite enumeration above produces the following list after the barycenter condition $\eta \in \operatorname{Conv}(I(r)_{\geq 0})$ and the inclusion-maximality test are applied. Conversely, each displayed support is checked directly to be maximal among the supports of the form $I(r)_{\geq 0}$ and to satisfy $\eta \in \operatorname{Conv}(I(r)_{\geq 0})$. Hence each displayed support is strictly semistable with respect to T, and Lemma 2.3 shows that no other maximal support occurs, up to coordinate permutation. □

k	r_k	$\lvert S_k\rvert$
1	$(36, 1, -4, -9, -24)$	58
2	$(27, 2, -3, -8, -18)$	59
3	$(3, 0, 0, -1, -2)$	62
4	$(24, 9, -6, -11, -16)$	62
5	$(21, 6, -4, -9, -14)$	62
6	$(18, 3, -2, -7, -12)$	62
7	$(5, 1, 0, -2, -4)$	63
8	$(4, 2, -1, -2, -3)$	63
9	$(19, 4, -1, -6, -16)$	64
10	$(12, 2, -3, -3, -8)$	64
11	$(14, 4, -1, -6, -11)$	65
12	$(13, 8, -2, -7, -12)$	66
13	$(3, 2, 0, -2, -3)$	67
14	$(4, 1, 0, -1, -4)$	67

continued on the next page

k	r_k	$\lvert S_k\rvert$
15	$(9,4,-1,-6,-6)$	68
16	$(3,2,1,-2,-4)$	69
17	$(12,7,2,-8,-13)$	69
18	$(4,2,0,-1,-5)$	69
19	$(2,1,0,-1,-2)$	69
20	$(8,3,-2,-2,-7)$	69
21	$(16,11,6,-9,-24)$	70
22	$(16,6,1,-4,-19)$	70
23	$(4,-1,-1,-1,-1)$	70
24	$(14,9,4,-6,-21)$	71
25	$(11,6,1,-4,-14)$	71
26	$(24,9,4,-1,-36)$	73
27	$(18,8,3,-2,-27)$	73
28	$(12,7,2,-3,-18)$	73
29	$(6,1,1,-4,-4)$	73
30	$(6,6,1,-4,-9)$	73
31	$(7,2,2,-3,-8)$	74
32	$(3,3,-2,-2,-2)$	75
33	$(2,1,0,0,-3)$	76
34	$(8,3,3,-2,-12)$	76
35	$(4,4,-1,-1,-6)$	76
36	$(1,0,0,0,-1)$	80
37	$(2,2,2,-3,-3)$	81
38	$(1,1,1,1,-4)$	91

3 Boundary families for the full $\mathrm{SL}(5)$-action

In this section we pass from the maximal T-strictly semistable supports $S_k = I(r_k)_{\geq 0}$ to the boundary families for the full SL(5)-action. The construction is the quintic-threefold analogue of the passage from the torus data to the SL(7)-families in the cubic-fivefold case. The main difference is that, for quintic threefolds, no two distinct families Φ_k and Φ_ℓ contain one another.

We first record compact generic equations for the families. Write

$$r_k = (r_{k,0}, r_{k,1}, r_{k,2}, r_{k,3}, r_{k,4}), \qquad \rho_k = (r_{k,0}, r_{k,1}, r_{k,2}).$$

Definition 3.1 (Truncated general forms)**.** Let $\rho = (\rho_0, \rho_1, \rho_2) \in \mathbb{Z}^3$, $m \geq 0$, and $\beta \in \mathbb{Z}$. We denote by

$$\mathcal{P}_\rho^{m,\beta} \subset \mathbb{C}[x_0, x_1, x_2]_m$$

the vector space spanned by the monomials $x_0^{v_0} x_1^{v_1} x_2^{v_2}$ with

$$v_0 + v_1 + v_2 = m, \qquad \rho_0 v_0 + \rho_1 v_1 + \rho_2 v_2 \geq \beta.$$

For each k, a, b, m, β, the symbol

$$P_{k;a,b}^{m,\beta}$$

denotes an independent general element of $\mathcal{P}_{\rho_k}^{m,\beta}$. In particular, if $m = 0$, then $P_{k;a,b}^{0,\beta}$ is a general constant whenever the above inequality is satisfied.

With this notation, the monomial support condition

$$\mathrm{Supp}(f_k) = S_k = I(r_k)_{\geq 0}$$

is equivalent to the following forms.

Theorem 3.2 (Generic forms attached to the thirty-eight supports)**.** *For $k = 1, \dots, 38$, a general quintic with support $S_k = I(r_k)_{\geq 0}$ can be written as follows. All $P^{m,\beta}_{k;a,b}$'s appearing in the table are independent general elements of the spaces in Definition 3.1.*

k	generic form f_k
1	$f_1 = P^{5,0}_{1;0,0} + x_3 P^{4,9}_{1;1,0} + x_3^2 P^{3,18}_{1;2,0}$
	$+ x_3^3 P^{2,27}_{1;3,0} + x_3^4 P^{1,36}_{1;4,0}$
	$+ x_4 \big(P^{4,24}_{1;0,1} + x_3 P^{3,33}_{1;1,1} + x_3^2 P^{2,42}_{1;2,1}\big)$
	$+ x_4^2 \big(P^{3,48}_{1;0,2} + x_3 P^{2,57}_{1;1,2}\big)$
	$+ x_4^3 \big(P^{2,72}_{1;0,3}\big)$
2	$f_2 = P^{5,0}_{2;0,0} + x_3 P^{4,8}_{2;1,0} + x_3^2 P^{3,16}_{2;2,0}$
	$+ x_3^3 P^{2,24}_{2;3,0}$
	$+ x_4 \big(P^{4,18}_{2;0,1} + x_3 P^{3,26}_{2;1,1} + x_3^2 P^{2,34}_{2;2,1}\big)$
	$+ x_4^2 \big(P^{3,36}_{2;0,2} + x_3 P^{2,44}_{2;1,2}\big)$
	$+ x_4^3 \big(P^{2,54}_{2;0,3}\big)$
3	$f_3 = P^{5,0}_{3;0,0} + x_3 P^{4,1}_{3;1,0} + x_3^2 P^{3,2}_{3;2,0}$
	$+ x_3^3 P^{2,3}_{3;3,0}$
	$+ x_4 \big(P^{4,2}_{3;0,1} + x_3 P^{3,3}_{3;1,1} + x_3^2 P^{2,4}_{3;2,1}\big)$
	$+ x_4^2 \big(P^{3,4}_{3;0,2} + x_3 P^{2,5}_{3;1,2}\big)$
	$+ x_4^3 \big(P^{2,6}_{3;0,3}\big)$
4	$f_4 = P^{5,0}_{4;0,0} + x_3 P^{4,11}_{4;1,0} + x_3^2 P^{3,22}_{4;2,0}$
	$+ x_3^3 P^{2,33}_{4;3,0}$
	$+ x_4 \big(P^{4,16}_{4;0,1} + x_3 P^{3,27}_{4;1,1} + x_3^2 P^{2,38}_{4;2,1}\big)$
	$+ x_4^2 \big(P^{3,32}_{4;0,2} + x_3 P^{2,43}_{4;1,2}\big)$
	$+ x_4^3 \big(P^{2,48}_{4;0,3}\big)$
5	$f_5 = P^{5,0}_{5;0,0} + x_3 P^{4,9}_{5;1,0} + x_3^2 P^{3,18}_{5;2,0}$
	$+ x_3^3 P^{2,27}_{5;3,0}$
	$+ x_4 \big(P^{4,14}_{5;0,1} + x_3 P^{3,23}_{5;1,1} + x_3^2 P^{2,32}_{5;2,1}\big)$
	$+ x_4^2 \big(P^{3,28}_{5;0,2} + x_3 P^{2,37}_{5;1,2}\big)$
	$+ x_4^3 \big(P^{2,42}_{5;0,3}\big)$
6	$f_6 = P^{5,0}_{6;0,0} + x_3 P^{4,7}_{6;1,0} + x_3^2 P^{3,14}_{6;2,0}$
	$+ x_3^3 P^{2,21}_{6;3,0}$
	$+ x_4 \big(P^{4,12}_{6;0,1} + x_3 P^{3,19}_{6;1,1} + x_3^2 P^{2,26}_{6;2,1}\big)$
	$+ x_4^2 \big(P^{3,24}_{6;0,2} + x_3 P^{2,31}_{6;1,2}\big)$
	$+ x_4^3 \big(P^{2,36}_{6;0,3}\big)$
7	$f_7 = P^{5,0}_{7;0,0} + x_3 P^{4,2}_{7;1,0} + x_3^2 P^{3,4}_{7;2,0}$
	$+ x_3^3 P^{2,6}_{7;3,0}$
	$+ x_4 \big(P^{4,4}_{7;0,1} + x_3 P^{3,6}_{7;1,1} + x_3^2 P^{2,8}_{7;2,1}\big)$
	$+ x_4^2 \big(P^{3,8}_{7;0,2} + x_3 P^{2,10}_{7;1,2}\big)$

continued on the next page

k	generic form f_k
8	$f_8 = P^{5,0}_{8;0,0} + x_3 P^{4,2}_{8;1,0} + x_3^2 P^{3,4}_{8;2,0}$ $+ x_3^3 P^{2,6}_{8;3,0}$ $+ x_4\big(P^{4,3}_{8;0,1} + x_3 P^{3,5}_{8;1,1} + x_3^2 P^{2,7}_{8;2,1}\big)$ $+ x_4^2\big(P^{3,6}_{8;0,2} + x_3 P^{2,8}_{8;1,2}\big)$
9	$f_9 = P^{5,0}_{9;0,0} + x_3 P^{4,6}_{9;1,0} + x_3^2 P^{3,12}_{9;2,0}$ $+ x_3^3 P^{2,18}_{9;3,0}$ $+ x_4\big(P^{4,16}_{9;0,1} + x_3 P^{3,22}_{9;1,1} + x_3^2 P^{2,28}_{9;2,1}\big)$ $+ x_4^2\big(P^{3,32}_{9;0,2} + x_3 P^{2,38}_{9;1,2}\big)$
10	$f_{10} = P^{5,0}_{10;0,0} + x_3 P^{4,3}_{10;1,0} + x_3^2 P^{3,6}_{10;2,0}$ $+ x_3^3 P^{2,9}_{10;3,0} + x_3^4 P^{1,12}_{10;4,0}$ $+ x_4\big(P^{4,8}_{10;0,1} + x_3 P^{3,11}_{10;1,1} + x_3^2 P^{2,14}_{10;2,1}\big)$ $+ x_4^2\big(P^{3,16}_{10;0,2} + x_3 P^{2,19}_{10;1,2}\big)$ $+ x_4^3\big(P^{2,24}_{10;0,3}\big)$
11	$f_{11} = P^{5,0}_{11;0,0} + x_3 P^{4,6}_{11;1,0} + x_3^2 P^{3,12}_{11;2,0}$ $+ x_3^3 P^{2,18}_{11;3,0}$ $+ x_4\big(P^{4,11}_{11;0,1} + x_3 P^{3,17}_{11;1,1} + x_3^2 P^{2,23}_{11;2,1}\big)$ $+ x_4^2\big(P^{3,22}_{11;0,2} + x_3 P^{2,28}_{11;1,2}\big)$
12	$f_{12} = P^{5,0}_{12;0,0} + x_3 P^{4,7}_{12;1,0} + x_3^2 P^{3,14}_{12;2,0}$ $+ x_3^3 P^{2,21}_{12;3,0}$ $+ x_4\big(P^{4,12}_{12;0,1} + x_3 P^{3,19}_{12;1,1} + x_3^2 P^{2,26}_{12;2,1}\big)$ $+ x_4^2\big(P^{3,24}_{12;0,2}\big)$
13	$f_{13} = P^{5,0}_{13;0,0} + x_3 P^{4,2}_{13;1,0} + x_3^2 P^{3,4}_{13;2,0}$ $+ x_3^3 P^{2,6}_{13;3,0}$ $+ x_4\big(P^{4,3}_{13;0,1} + x_3 P^{3,5}_{13;1,1}\big)$ $+ x_4^2\big(P^{3,6}_{13;0,2}\big)$
14	$f_{14} = P^{5,0}_{14;0,0} + x_3 P^{4,1}_{14;1,0} + x_3^2 P^{3,2}_{14;2,0}$ $+ x_3^3 P^{2,3}_{14;3,0} + x_3^4 P^{1,4}_{14;4,0}$ $+ x_4\big(P^{4,4}_{14;0,1} + x_3 P^{3,5}_{14;1,1} + x_3^2 P^{2,6}_{14;2,1}\big)$ $+ x_4^2\big(P^{3,8}_{14;0,2}\big)$
15	$f_{15} = P^{5,0}_{15;0,0} + x_3 P^{4,6}_{15;1,0} + x_3^2 P^{3,12}_{15;2,0}$ $+ x_3^3 P^{2,18}_{15;3,0}$ $+ x_4\big(P^{4,6}_{15;0,1} + x_3 P^{3,12}_{15;1,1} + x_3^2 P^{2,18}_{15;2,1}\big)$ $+ x_4^2\big(P^{3,12}_{15;0,2} + x_3 P^{2,18}_{15;1,2}\big)$ $+ x_4^3\big(P^{2,18}_{15;0,3}\big)$
16	$f_{16} = P^{5,0}_{16;0,0} + x_3 P^{4,2}_{16;1,0} + x_3^2 P^{3,4}_{16;2,0}$ $+ x_3^3 P^{2,6}_{16;3,0}$ $+ x_4\big(P^{4,4}_{16;0,1} + x_3 P^{3,6}_{16;1,1}\big)$ $+ x_4^2\big(P^{3,8}_{16;0,2}\big)$

continued on the next page

k	generic form f_k
17	$f_{17} = P^{5,0}_{17;0,0} + x_3 P^{4,8}_{17;1,0} + x_3^2 P^{3,16}_{17;2,0} + x_3^3 P^{2,24}_{17;3,0} + x_4\big(P^{4,13}_{17;0,1} + x_3 P^{3,21}_{17;1,1}\big) + x_4^2\big(P^{3,26}_{17;0,2}\big)$
18	$f_{18} = P^{5,0}_{18;0,0} + x_3 P^{4,1}_{18;1,0} + x_3^2 P^{3,2}_{18;2,0} + x_3^3 P^{2,3}_{18;3,0} + x_3^4 P^{1,4}_{18;4,0} + x_4\big(P^{4,5}_{18;0,1} + x_3 P^{3,6}_{18;1,1} + x_3^2 P^{2,7}_{18;2,1}\big) + x_4^2\big(P^{3,10}_{18;0,2}\big)$
19	$f_{19} = P^{5,0}_{19;0,0} + x_3 P^{4,1}_{19;1,0} + x_3^2 P^{3,2}_{19;2,0} + x_3^3 P^{2,3}_{19;3,0} + x_4\big(P^{4,2}_{19;0,1} + x_3 P^{3,3}_{19;1,1} + x_3^2 P^{2,4}_{19;2,1}\big) + x_4^2\big(P^{3,4}_{19;0,2}\big)$
20	$f_{20} = P^{5,0}_{20;0,0} + x_3 P^{4,2}_{20;1,0} + x_3^2 P^{3,4}_{20;2,0} + x_3^3 P^{2,6}_{20;3,0} + x_3^4 P^{1,8}_{20;4,0} + x_4\big(P^{4,7}_{20;0,1} + x_3 P^{3,9}_{20;1,1} + x_3^2 P^{2,11}_{20;2,1}\big) + x_4^2\big(P^{3,14}_{20;0,2} + x_3 P^{2,16}_{20;1,2}\big)$
21	$f_{21} = P^{5,0}_{21;0,0} + x_3 P^{4,9}_{21;1,0} + x_3^2 P^{3,18}_{21;2,0} + x_3^3 P^{2,27}_{21;3,0} + x_4\big(P^{4,24}_{21;0,1} + x_3 P^{3,33}_{21;1,1}\big) + x_4^2\big(P^{3,48}_{21;0,2}\big)$
22	$f_{22} = P^{5,0}_{22;0,0} + x_3 P^{4,4}_{22;1,0} + x_3^2 P^{3,8}_{22;2,0} + x_3^3 P^{2,12}_{22;3,0} + x_3^4 P^{1,16}_{22;4,0} + x_4\big(P^{4,19}_{22;0,1} + x_3 P^{3,23}_{22;1,1} + x_3^2 P^{2,27}_{22;2,1}\big) + x_4^2\big(P^{3,38}_{22;0,2}\big)$
23	$f_{23} = P^{5,0}_{23;0,0} + x_3 P^{4,1}_{23;1,0} + x_3^2 P^{3,2}_{23;2,0} + x_3^3 P^{2,3}_{23;3,0} + x_3^4 P^{1,4}_{23;4,0} + x_4\big(P^{4,1}_{23;0,1} + x_3 P^{3,2}_{23;1,1} + x_3^2 P^{2,3}_{23;2,1} + x_3^3 P^{1,4}_{23;3,1}\big) + x_4^2\big(P^{3,2}_{23;0,2} + x_3 P^{2,3}_{23;1,2} + x_3^2 P^{1,4}_{23;2,2}\big) + x_4^3\big(P^{2,3}_{23;0,3} + x_3 P^{1,4}_{23;1,3}\big) + x_4^4\big(P^{1,4}_{23;0,4}\big)$
24	$f_{24} = P^{5,0}_{24;0,0} + x_3 P^{4,6}_{24;1,0} + x_3^2 P^{3,12}_{24;2,0} + x_3^3 P^{2,18}_{24;3,0} + x_4\big(P^{4,21}_{24;0,1} + x_3 P^{3,27}_{24;1,1}\big) + x_4^2\big(P^{3,42}_{24;0,2}\big)$
25	$f_{25} = P^{5,0}_{25;0,0} + x_3 P^{4,4}_{25;1,0} + x_3^2 P^{3,8}_{25;2,0} + x_3^3 P^{2,12}_{25;3,0} + x_4\big(P^{4,14}_{25;0,1} + x_3 P^{3,18}_{25;1,1} + x_3^2 P^{2,22}_{25;2,1}\big) + x_4^2\big(P^{3,28}_{25;0,2}\big)$

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k	generic form f_k
26	$f_{26} = P^{5,0}_{26;0,0} + x_3 P^{4,1}_{26;1,0} + x_3^2 P^{3,2}_{26;2,0}$ $+ x_3^3 P^{2,3}_{26;3,0} + x_3^4 P^{1,4}_{26;4,0}$ $+ x_4\big(P^{4,36}_{26;0,1} + x_3 P^{3,37}_{26;1,1} + x_3^2 P^{2,38}_{26;2,1}\big)$ $+ x_4^2\big(P^{3,72}_{26;0,2}\big)$
27	$f_{27} = P^{5,0}_{27;0,0} + x_3 P^{4,2}_{27;1,0} + x_3^2 P^{3,4}_{27;2,0}$ $+ x_3^3 P^{2,6}_{27;3,0} + x_3^4 P^{1,8}_{27;4,0}$ $+ x_4\big(P^{4,27}_{27;0,1} + x_3 P^{3,29}_{27;1,1} + x_3^2 P^{2,31}_{27;2,1}\big)$ $+ x_4^2\big(P^{3,54}_{27;0,2}\big)$
28	$f_{28} = P^{5,0}_{28;0,0} + x_3 P^{4,3}_{28;1,0} + x_3^2 P^{3,6}_{28;2,0}$ $+ x_3^3 P^{2,9}_{28;3,0} + x_3^4 P^{1,12}_{28;4,0}$ $+ x_4\big(P^{4,18}_{28;0,1} + x_3 P^{3,21}_{28;1,1} + x_3^2 P^{2,24}_{28;2,1}\big)$ $+ x_4^2\big(P^{3,36}_{28;0,2}\big)$
29	$f_{29} = P^{5,0}_{29;0,0} + x_3 P^{4,4}_{29;1,0} + x_3^2 P^{3,8}_{29;2,0}$ $+ x_3^3 P^{2,12}_{29;3,0}$ $+ x_4\big(P^{4,4}_{29;0,1} + x_3 P^{3,8}_{29;1,1} + x_3^2 P^{2,12}_{29;2,1}\big)$ $+ x_4^2\big(P^{3,8}_{29;0,2} + x_3 P^{2,12}_{29;1,2}\big)$ $+ x_4^3\big(P^{2,12}_{29;0,3}\big)$
30	$f_{30} = P^{5,0}_{30;0,0} + x_3 P^{4,4}_{30;1,0} + x_3^2 P^{3,8}_{30;2,0}$ $+ x_3^3 P^{2,12}_{30;3,0}$ $+ x_4\big(P^{4,9}_{30;0,1} + x_3 P^{3,13}_{30;1,1}\big)$ $+ x_4^2\big(P^{3,18}_{30;0,2}\big)$
31	$f_{31} = P^{5,0}_{31;0,0} + x_3 P^{4,3}_{31;1,0} + x_3^2 P^{3,6}_{31;2,0}$ $+ x_3^3 P^{2,9}_{31;3,0}$ $+ x_4\big(P^{4,8}_{31;0,1} + x_3 P^{3,11}_{31;1,1} + x_3^2 P^{2,14}_{31;2,1}\big)$ $+ x_4^2\big(P^{3,16}_{31;0,2}\big)$
32	$f_{32} = P^{5,0}_{32;0,0} + x_3 P^{4,2}_{32;1,0} + x_3^2 P^{3,4}_{32;2,0}$ $+ x_3^3 P^{2,6}_{32;3,0}$ $+ x_4\big(P^{4,2}_{32;0,1} + x_3 P^{3,4}_{32;1,1} + x_3^2 P^{2,6}_{32;2,1}\big)$ $+ x_4^2\big(P^{3,4}_{32;0,2} + x_3 P^{2,6}_{32;1,2}\big)$ $+ x_4^3\big(P^{2,6}_{32;0,3}\big)$
33	$f_{33} = P^{5,0}_{33;0,0} + x_3 P^{4,0}_{33;1,0} + x_3^2 P^{3,0}_{33;2,0}$ $+ x_3^3 P^{2,0}_{33;3,0} + x_3^4 P^{1,0}_{33;4,0} + x_3^5 P^{0,0}_{33;5,0}$ $+ x_4\big(P^{4,3}_{33;0,1} + x_3 P^{3,3}_{33;1,1} + x_3^2 P^{2,3}_{33;2,1}\big)$ $+ x_4^2\big(P^{3,6}_{33;0,2}\big)$
34	$f_{34} = P^{5,0}_{34;0,0} + x_3 P^{4,2}_{34;1,0} + x_3^2 P^{3,4}_{34;2,0}$ $+ x_3^3 P^{2,6}_{34;3,0} + x_3^4 P^{1,8}_{34;4,0}$ $+ x_4\big(P^{4,12}_{34;0,1} + x_3 P^{3,14}_{34;1,1} + x_3^2 P^{2,16}_{34;2,1}\big)$ $+ x_4^2\big(P^{3,24}_{34;0,2}\big)$

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k	generic form f_k
35	$f_{35} = P^{5,0}_{35;0,0} + x_3 P^{4,1}_{35;1,0} + x_3^2 P^{3,2}_{35;2,0}$ $+ x_3^3 P^{2,3}_{35;3,0} + x_3^4 P^{1,4}_{35;4,0}$ $+ x_4\big(P^{4,6}_{35;0,1} + x_3 P^{3,7}_{35;1,1} + x_3^2 P^{2,8}_{35;2,1}\big)$ $+ x_4^2\big(P^{3,12}_{35;0,2}\big)$
36	$f_{36} = P^{5,0}_{36;0,0} + x_3 P^{4,0}_{36;1,0} + x_3^2 P^{3,0}_{36;2,0}$ $+ x_3^3 P^{2,0}_{36;3,0} + x_3^4 P^{1,0}_{36;4,0} + x_3^5 P^{0,0}_{36;5,0}$ $+ x_4\big(P^{4,1}_{36;0,1} + x_3 P^{3,1}_{36;1,1} + x_3^2 P^{2,1}_{36;2,1}$ $+ x_3^3 P^{1,1}_{36;3,1}\big)$ $+ x_4^2\big(P^{3,2}_{36;0,2} + x_3 P^{2,2}_{36;1,2}\big)$
37	$f_{37} = P^{5,0}_{37;0,0} + x_3 P^{4,3}_{37;1,0} + x_3^2 P^{3,6}_{37;2,0}$ $+ x_4\big(P^{4,3}_{37;0,1} + x_3 P^{3,6}_{37;1,1}\big)$ $+ x_4^2\big(P^{3,6}_{37;0,2}\big)$
38	$f_{38} = P^{5,0}_{38;0,0} + x_3 P^{4,-1}_{38;1,0} + x_3^2 P^{3,-2}_{38;2,0}$ $+ x_3^3 P^{2,-3}_{38;3,0} + x_3^4 P^{1,-4}_{38;4,0} + x_3^5 P^{0,-5}_{38;5,0}$ $+ x_4\big(P^{4,4}_{38;0,1} + x_3 P^{3,3}_{38;1,1} + x_3^2 P^{2,2}_{38;2,1}$ $+ x_3^3 P^{1,1}_{38;3,1} + x_3^4 P^{0,0}_{38;4,1}\big)$

Equivalently, the above table is the expanded form of the uniform formula

$$f_k = \sum_{\substack{a,b\geq 0\\ a+b\leq 5}} x_3^a x_4^b \, P_{k;a,b}^{5-a-b,\,-ar_{k,3}-br_{k,4}},$$

with the convention that the summand is omitted when $\mathcal{P}_{\rho_k}^{5-a-b,\,-ar_{k,3}-br_{k,4}} = 0$.

The boundary families Φ_k

Let

$$G := \mathrm{SL}(5), \qquad W = \mathrm{Sym}^5\,\mathbb{C}^5,$$

and let

$$\mathcal{M}^{\mathrm{GIT}} := \mathbb{P}(W)^{ss}/\!/G$$

be the GIT quotient. We denote by

$$\pi : \mathbb{P}(W)^{ss} \longrightarrow \mathcal{M}^{\mathrm{GIT}}$$

the good quotient map. Let

$$\Sigma^{ss} := \mathbb{P}(W)^{ss} \setminus \mathbb{P}(W)^{s}$$

be the strictly semistable locus, and define the GIT boundary by

$$\partial\mathcal{M}^{\mathrm{GIT}} := \pi(\Sigma^{ss}) \subset \mathcal{M}^{\mathrm{GIT}}.$$

For each maximal strictly semistable support S_k, set

$$V_k := \mathrm{Span}_{\mathbb{C}}\{x^u \mid u \in S_k\} \subset W.$$

We first define the corresponding prequotient family by

$$\widetilde{\Phi}_k := \overline{G\cdot\mathbb{P}(V_k)} \cap \Sigma^{ss} \subset \mathbb{P}(W)^{ss},$$

where the closure is taken in $\mathbb{P}(W)$. The k-th boundary family in the GIT quotient is then defined by

$$\Phi_k := \pi(\widetilde{\Phi}_k) \subset \partial\mathcal{M}^{\mathrm{GIT}}.$$

Thus $\widetilde{\Phi}_k$ denotes the G-saturated family before passing to the quotient, whereas Φ_k denotes the corresponding subvariety of the GIT quotient.

If f_k is a general quintic with support S_k, then $[f_k]$ is a general support-family representative used to construct $\widetilde{\Phi}_k$, and the associated moduli point lies in Φ_k. In what follows, all dimensions of Φ_k are dimensions inside the quotient $\mathcal{M}^{\mathrm{GIT}}$, not dimensions of $\widetilde{\Phi}_k$ inside $\mathbb{P}(W)^{ss}$.

We record the basic topological properties of these families. Since $S_k = I(r_k)_{\geq 0}$, every point of $\mathbb{P}(V_k)$ is non-stable for the G-action. Hence

$$\overline{G \cdot \mathbb{P}(V_k)} \subset \mathbb{P}(W) \setminus \mathbb{P}(W)^s.$$

Consequently,

$$\widetilde{\Phi}_k = \overline{G \cdot \mathbb{P}(V_k)} \cap \mathbb{P}(W)^{ss}.$$

The variety G is irreducible and $\mathbb{P}(V_k)$ is irreducible, so $\overline{G \cdot \mathbb{P}(V_k)}$ is irreducible. Since S_k is strictly semistable, the intersection with $\mathbb{P}(W)^{ss}$ is nonempty; therefore $\widetilde{\Phi}_k$ is irreducible. Moreover, $\widetilde{\Phi}_k$ is closed in $\mathbb{P}(W)^{ss}$. By the good quotient property, its image

$$\Phi_k = \pi(\widetilde{\Phi}_k)$$

is a closed irreducible subvariety of $\partial\mathcal{M}^{\mathrm{GIT}}$, equipped with the reduced induced subvariety structure.

Theorem 3.3 (Pairwise non-inclusion in the GIT quotient)**.** *There is no inclusion among the quotient-side boundary families Φ_k, $k = 1, \ldots, 38$; namely,*

$$\Phi_k \not\subset \Phi_\ell \qquad (k \neq \ell).$$

Equivalently,

$$\widetilde{\Phi}_k \not\subset \pi^{-1}(\Phi_\ell) \qquad (k \neq \ell).$$

In particular, the thirty-eight subvarieties

$$\Phi_1, \ldots, \Phi_{38} \subset \mathcal{M}^{\mathrm{GIT}}$$

are pairwise distinct and pairwise non-contained.

Proof. The verification is summarized in Section 6. ☐

Theorem 3.4 (Boundary components in the GIT quotient)**.** *For the natural* $\mathrm{SL}(5)$*-action on* $\mathbb{P}(W) = \mathbb{P}(\mathrm{Sym}^5 \mathbb{C}^5)$*, the strictly semistable locus is covered by the prequotient families:*

$$\Sigma^{ss} = \widetilde{\Phi}_1 \cup \cdots \cup \widetilde{\Phi}_{38}.$$

Consequently, the GIT boundary is

$$\partial\mathcal{M}^{\mathrm{GIT}} = \Phi_1 \cup \cdots \cup \Phi_{38}.$$

Moreover, the thirty-eight subvarieties

$$\Phi_1, \ldots, \Phi_{38}$$

are exactly the irreducible components of $\partial\mathcal{M}^{\mathrm{GIT}}$.

Proof. Let $[F] \in \Sigma^{ss}$. By the Hilbert–Mumford numerical criterion, there exists $g \in G$ such that the support of $g \cdot F$ is contained in a maximal T-strictly semistable support. The enumeration in Section 2 gives exactly the thirty-eight maximal supports

$$S_1, \ldots, S_{38}.$$

Hence, for some k, we have

$$[g \cdot F] \in \mathbb{P}(V_k).$$

Since $[F]$ is strictly semistable and semistability is G-invariant, it follows that

$$[F] \in G \cdot \mathbb{P}(V_k) \cap \Sigma^{ss} \subset \widetilde{\Phi}_k.$$

Therefore

$$\Sigma^{ss} = \widetilde{\Phi}_1 \cup \cdots \cup \widetilde{\Phi}_{38}.$$

Applying the quotient map π gives

$$\partial \mathcal{M}^{\mathrm{GIT}} = \pi(\Sigma^{ss}) = \Phi_1 \cup \cdots \cup \Phi_{38}.$$

As observed above, each Φ_k is a closed irreducible subvariety of $\partial \mathcal{M}^{\mathrm{GIT}}$. By Theorem 3.3, no Φ_k is contained in another Φ_ℓ. A finite union of closed irreducible subvarieties with no member contained in another is the irreducible decomposition of the union. Hence

$$\Phi_1, \ldots, \Phi_{38}$$

are precisely the irreducible components of $\partial \mathcal{M}^{\mathrm{GIT}}$. □

4 Closed-orbit normal forms for quintic threefolds

The purpose of this section is to pass from the boundary families constructed in Section 3 to explicit polystable representatives. For each maximal support

$$S_k = I(r_k)_{\geq 0},$$

we start with a general quintic f_k supported on S_k and take the one-parameter-subgroup limit

$$\phi_k := \lim_{t \to 0} \lambda_{r_k}(t) \cdot f_k.$$

This limit lies in the fixed subspace W^{H_k}, where

$$H_k := \lambda_{r_k}(\mathbb{G}_m).$$

By Luna's centralizer reduction, the closedness of the resulting SL(5)-orbit is reduced to a closed-orbit problem for the centralizer $C_G(H_k)$ acting on W^{H_k}.

The table below records the closed-orbit normal forms ϕ_k^{nf} obtained in this way. In the toric-centralizer cases we use the convex-hull criterion, abbreviated as C-H; in the non-toric centralizer cases we use the Casimiro–Florentino criterion, abbreviated as C-F. The column $\dim \Phi_k$ records the quotient-side closed-orbit family dimension. These normal forms serve as the reference representatives for the singular-locus computations in Section 5, while the computational framework underlying the verification is recalled in Section 8. The complete case-by-case implementation will appear in the forthcoming full-length paper.

Let us put

$$B_\lambda(u, v) := uv(u - v)(u - \lambda v), \qquad B_{\lambda,\mu}(u, v) := v(u - v)(u - \lambda v)(u - \mu v).$$

Here B_d, C_d, Q_d, L_d, M_d denote general forms of the indicated degree, and $Q_{2,2}$ denotes a bihomogeneous form of bidegree $(2,2)$. All parameters are understood to lie in the relevant nonempty Zariski-open generic locus.

Table 3: Closed-orbit normal forms for the 38 quintic-threefold boundary families.

k	Test	Normal form	Parameters	Dim Φ_k
1	C-H	$\phi_1^{\mathrm{nf}} = x_1^4x_2 + x_0x_3^4 + x_0x_2^3x_4 + x_0x_1x_2x_3x_4 + \alpha x_0^2x_4^3$	$\alpha \in \mathbb{C}^\times$	1
2	C-H	$\phi_2^{\mathrm{nf}} = x_1^3x_2^2 + x_1^4x_3 + x_0x_2x_3^3 + x_0x_2^3x_4 + \alpha x_0x_1x_2x_3x_4 + \beta x_0^2x_4^3$	$\alpha, \beta \in \mathbb{C}^\times$	2
3	C-F	$\phi_3^{\mathrm{nf}} = B_5(x_1,x_2) + x_0x_2x_3^3 + x_0(x_1^2+x_2^2)x_3x_4 + x_0^2x_4^3$	B_5: general binary quintic	6
4	C-H	$\phi_4^{\mathrm{nf}} = x_1^2x_2^3 + x_1^3x_3x_4 + x_0x_2^4 + x_0x_1x_3^3 + \alpha x_0x_1x_2x_3x_4 + \beta x_0^2x_4^3$	$\alpha, \beta \in \mathbb{C}^\times$	2
5	C-H	$\phi_5^{\mathrm{nf}} = x_1^2x_2^3 + x_1^3x_3^2 + x_1^3x_2x_4 + x_0x_2^3x_3 + \alpha x_0x_1x_3^3 + \beta x_0x_1x_2x_3x_4 + \gamma x_0^2x_4^3$	$\alpha, \beta, \gamma \in \mathbb{C}^\times$	3
6	C-H	$\phi_6^{\mathrm{nf}} = x_1^2x_2^3 + x_1^3x_2x_3 + x_1^4x_4 + x_0x_2^2x_3^2 + \alpha x_0x_2^3x_4 + \beta x_0x_1x_3^3 + \gamma x_0x_1x_2x_3x_4 + \delta x_0x_1^2x_4^2 + \epsilon x_0^2x_4^3$	$\alpha, \beta, \gamma, \delta,$ $\epsilon \in \mathbb{C}^\times$	5
7	C-H	$\phi_7^{\mathrm{nf}} = x_2^5 + x_1^2x_2^2x_3 + x_1^4x_4 + x_0x_1x_3^3 + \alpha x_0x_1x_2x_3x_4 + \beta x_0^2x_3x_4^2$	$\alpha, \beta \in \mathbb{C}^\times$	2
8	C-H	$\phi_8^{\mathrm{nf}} = x_1^2x_2^2x_3 + x_1^3x_4^2 + x_0x_2^4 + x_0x_1x_3^3 + \alpha x_0x_1x_2x_3x_4 + \beta x_0^2x_3x_4^2$	$\alpha, \beta \in \mathbb{C}^\times$	2
9	C-H	$\phi_9^{\mathrm{nf}} = x_1x_2^4 + x_1^2x_2^2x_3 + \alpha x_1^3x_3^2 + x_1^4x_4 + x_0x_2x_3^3 + \beta x_0x_2^3x_4 + \gamma x_0x_1x_2x_3x_4 + \delta x_0^2x_3x_4^2$	$\alpha, \beta, \gamma, \delta \in \mathbb{C}^\times$	4
10	C-F	$\phi_{10}^{\mathrm{nf}} = x_1^3Q_{\mathbf{q}}(x_2,x_3) + x_1^4x_4 + x_0B_\lambda(x_2,x_3) + x_0x_1x_4R_{\mathbf{r}}(x_2,x_3) + \beta x_0x_1^2x_4^2 + x_0^2x_4^3$	$\lambda \in \mathbb{C} \setminus \{0,1\}$; $\mathbf{q}, \mathbf{r} \in \mathbb{C}^3$, $\beta \in \mathbb{C}$	8
11	C-H	$\phi_{11}^{\mathrm{nf}} = x_1x_2^4 + \alpha x_1^2x_2^2x_3 + \beta x_1^3x_3^2 + x_1^3x_2x_4 + \gamma x_0x_2^2x_3^2 + x_0x_2^3x_4 + \delta x_0x_1x_3^3 + \epsilon x_0x_1x_2x_3x_4 + \zeta x_0x_1^2x_4^2 + x_0^2x_3x_4^2$	$\alpha, \beta, \gamma, \delta,$ $\epsilon, \zeta \in \mathbb{C}^\times$	6
12	C-H	$\phi_{12}^{\mathrm{nf}} = x_1x_2^4 + \alpha x_1^2x_2x_3^2 + \beta x_1^2x_2^2x_4 + x_1^3x_4^2 + x_0x_2^3x_3 + \gamma x_0x_1x_3^3 + \delta x_0x_1x_2x_3x_4 + \epsilon x_0^2x_3^2x_4 + x_0^2x_2x_4^2$	$\alpha, \beta, \gamma, \delta,$ $\epsilon \in \mathbb{C}^\times$	5
13	C-H	$\phi_{13}^{\mathrm{nf}} = x_2^5 + \alpha x_1x_2^3x_3 + \beta x_1^2x_2x_3^2 + x_1^3x_4^2 + x_0x_2^3x_4 + \gamma x_0x_1x_2x_3x_4 + x_0^2x_3^3 + \delta x_0^2x_2x_4^2$	$\alpha, \beta, \gamma, \delta \in \mathbb{C}^\times$	4
14	C-H	$\phi_{14}^{\mathrm{nf}} = x_2^5 + \alpha x_1x_2^3x_3 + \beta x_1^2x_2x_3^2 + x_1^4x_4 + x_0x_3^4 + \gamma x_0x_2^3x_4 + \delta x_0x_1x_2x_3x_4 + x_0^2x_2x_4^2$	$\alpha, \beta, \gamma, \delta \in \mathbb{C}^\times$	4
15	C-F	$\phi_{15}^{\mathrm{nf}} = x_1x_2^4 + x_1^2x_2^2x_3 + x_1^3Q(x_3,x_4) + x_0x_2^3x_4 + x_0x_1x_2R(x_3,x_4) + x_0^2B(x_3,x_4)$	$Q, R \in \mathrm{Sym}^2\langle x_3, x_4\rangle$; $B \in \mathrm{Sym}^3\langle x_3, x_4\rangle$	9
16	C-H	$\phi_{16}^{\mathrm{nf}} = x_2^4x_4 + x_1x_2^2x_3^2 + x_1^3x_3x_4 + x_0x_1x_2x_3x_4 + \alpha x_0^2x_3^3 + \beta x_0^2x_1x_4^2$	$\alpha, \beta \in \mathbb{C}^\times$	2

continued on the next page

k	Test	Normal form	Parameters	Dim Φ_k
17	C-H	$\phi^{\mathrm{nf}}_{17} = x_2^4x_3 + x_1x_2^3x_4 + \alpha x_1^2x_2x_3^2 + x_1^3x_3x_4 + \beta x_0x_2^2x_3^2 + \gamma x_0x_1x_2x_3x_4 + \delta x_0x_1^2x_4^2 + x_0^2x_3^3 + \epsilon x_0^2x_2x_4^2$	$\alpha,\beta,\gamma,\delta,\epsilon \in \mathbb{C}^\times$	5
18	C-H	$\phi^{\mathrm{nf}}_{18} = x_2^5 + x_1x_2^2x_3^2 + x_1^3x_3x_4 + x_0x_3^4 + \alpha x_0x_1x_2x_3x_4 + \beta x_0^2x_1x_4^2$	$\alpha,\beta \in \mathbb{C}^\times$	2
19	C-H	$\phi^{\mathrm{nf}}_{19} = x_2^5 + \alpha x_1x_2^3x_3 + \beta x_1^2x_2x_3^2 + \gamma x_1^2x_2^2x_4 + x_1^3x_3x_4 + \delta x_0x_2^2x_3^2 + \epsilon x_0x_2^3x_4 + x_0x_1x_3^3 + \zeta x_0x_1x_2x_3x_4 + \xi x_0x_1^2x_4^2 + x_0^2x_3^2x_4 + \theta x_0^2x_2x_4^2$	$\alpha,\beta,\gamma,\delta,\epsilon,\zeta,\xi,\theta \in \mathbb{C}^\times$	8
20	C-F	$\phi^{\mathrm{nf}}_{20} = x_0B_{\lambda,\mu}(x_2,x_3) + x_1^2C_{\mathbf{c}}(x_2,x_3) + x_1^3x_2x_4 + x_0x_1x_4Q_{\mathbf{q}}(x_2,x_3) + x_0x_1^2x_4^2 + x_0^2x_4^2M_{\mathbf{m}}(x_2,x_3)$	λ,μ distinct in $\mathbb{C}\setminus\{0,1\}$; $\mathbf{c}\in\mathbb{C}^4$, $\mathbf{q}\in\mathbb{C}^3$, $\mathbf{m}\in\mathbb{C}^2$	11
21	C-H	$\phi^{\mathrm{nf}}_{21} = x_2^3x_3^2 + x_2^4x_4 + x_1^3x_3x_4 + x_0x_1x_3^3 + \alpha x_0x_1x_2x_3x_4 + \beta x_0^3x_4^2$	$\alpha,\beta \in \mathbb{C}^\times$	2
22	C-H	$\phi^{\mathrm{nf}}_{22} = x_2^4x_3 + x_1x_2^2x_3^2 + \alpha x_1^2x_3^3 + x_1^3x_2x_4 + x_0x_3^4 + \beta x_0x_2^3x_4 + \gamma x_0x_1x_2x_3x_4 + \delta x_0^2x_1x_4^2$	$\alpha,\beta,\gamma,\delta \in \mathbb{C}^\times$	4
23	C-F	$\phi^{\mathrm{nf}}_{23} = x_0B_4(x_1,x_2,x_3,x_4)$	B_4: smooth quartic surface in $\mathbb{P}^3$	19
24	C-H	$\phi^{\mathrm{nf}}_{24} = x_2^3x_3^2 + x_1x_2^3x_4 + \alpha x_1^2x_3^3 + x_1^3x_3x_4 + x_0x_2x_3^3 + \beta x_0x_1x_2x_3x_4 + \gamma x_0^3x_4^2$	$\alpha,\beta,\gamma \in \mathbb{C}^\times$	3
25	C-H	$\phi^{\mathrm{nf}}_{25} = x_2^4x_3 + \alpha x_1x_2^2x_3^2 + \beta x_1^2x_3^3 + x_1^2x_2^2x_4 + x_1^3x_3x_4 + x_0x_2x_3^3 + \gamma x_0x_2^3x_4 + \delta x_0x_1x_2x_3x_4 + \epsilon x_0^2x_3^2x_4 + \zeta x_0^2x_1x_4^2$	$\alpha,\beta,\gamma,\delta,\epsilon,\zeta \in \mathbb{C}^\times$	6
26	C-H	$\phi^{\mathrm{nf}}_{26} = x_2x_3^4 + x_1^4x_4 + x_0x_2^3x_4 + \alpha x_0x_1x_2x_3x_4 + x_0^3x_4^2$	$\alpha \in \mathbb{C}^\times$	1
27	C-H	$\phi^{\mathrm{nf}}_{27} = x_2^2x_3^3 + x_1x_3^4 + x_1^3x_2x_4 + x_0x_2^3x_4 + \alpha x_0x_1x_2x_3x_4 + \beta x_0^3x_4^2$	$\alpha,\beta \in \mathbb{C}^\times$	2
28	C-H	$\phi^{\mathrm{nf}}_{28} = x_2^3x_3^2 + x_1x_2x_3^3 + x_1^2x_2^2x_4 + \beta x_1^3x_3x_4 + x_0x_3^4 + \alpha x_0x_2^3x_4 + \gamma x_0x_1x_2x_3x_4 + \delta x_0^2x_3^2x_4 + \epsilon x_0^3x_4^2$	$\alpha,\beta,\gamma,\delta,\epsilon \in \mathbb{C}^\times$	5
29	C-F	$\phi^{\mathrm{nf}}_{29} = x_3A_4(x_1,x_2) + x_4C_4(x_1,x_2) + x_0Q_{2,2}(x_1,x_2;x_3,x_4) + x_0^2x_3x_4(x_3-x_4)$	A_4, C_4: binary quartics; $Q_{2,2}$ of bidegree $(2,2)$	15
30	C-F	$\phi^{\mathrm{nf}}_{30} = x_2^4x_3 + x_2^2x_3^2L_1(x_0,x_1) + x_2^3x_4L_2(x_0,x_1) + x_3^3Q_1(x_0,x_1) + x_2x_3x_4Q_2(x_0,x_1) + x_0x_1(x_0-x_1)x_4^2$	L_1, L_2: binary linear; Q_1, Q_2: binary quadratic	9
31	C-F	$\phi^{\mathrm{nf}}_{31} = x_4B_\lambda(x_1,x_2) + x_3^2C_3(x_1,x_2) + x_0x_3^3L_1(x_1,x_2) + x_0x_3x_4Q_2(x_1,x_2) + x_0^2x_3^2x_4 + x_0^2x_4^2M_1(x_1,x_2)$	$\lambda\in\mathbb{C}\setminus\{0,1\}$; C_3, L_1, Q_2, M_1 of degrees $3,1,2,1$	11
32	C-F	$\phi^{\mathrm{nf}}_{32} = x_0^2C_0(x_2,x_3,x_4) + x_0x_1C_1(x_2,x_3,x_4) + x_1^2C_2(x_2,x_3,x_4)$	C_0, C_1, C_2: ternary cubics	18

continued on the next page

k	Test	Normal form	Parameters	Dim Φ_k
33	C-F	$\phi_{33}^{\mathrm{nf}} = B_5(x_2, x_3) + x_1^3 x_2 x_4 + x_0 x_1 x_4 (x_2^2 + x_3^2) + x_0^3 x_4^2$	B_5: general binary quintic	6
34	C-F	$\phi_{34}^{\mathrm{nf}} = x_4 B_\lambda(x_1, x_2) + x_3^3 Q_2(x_1, x_2) + x_0 x_3^4 + x_0 x_3 x_4 R_2(x_1, x_2) + \beta x_0^2 x_3^2 x_4 + x_0^3 x_4^2$	$\lambda \in \mathbb{C} \setminus \{0, 1\}$; Q_2, R_2: binary quadratic; $\beta \in \mathbb{C}^\times$	8
35	C-F	$\phi_{35}^{\mathrm{nf}} = x_0 A_4(x_2, x_3) + x_1 C_4(x_2, x_3) + x_4 Q_{2,2}(x_0, x_1; x_2, x_3) + x_4^2 x_0 x_1 (x_0 - x_1)$	A_4, C_4: binary quartics; $Q_{2,2}$ of bidegree $(2, 2)$	15
36	C-F	$\phi_{36}^{\mathrm{nf}} = B_5(x_1, x_2, x_3) + x_0 x_4 C_3(x_1, x_2, x_3) + x_0^2 x_3 x_4^2$	B_5, C_3: ternary forms of degrees $5, 3$	24
37	C-F	$\phi_{37}^{\mathrm{nf}} = x_3^2 C_0(x_0, x_1, x_2) + x_3 x_4 C_1(x_0, x_1, x_2) + x_4^2 C_2(x_0, x_1, x_2)$	C_0, C_1, C_2: ternary cubics	18
38	C-F	$\phi_{38}^{\mathrm{nf}} = x_4 B_4(x_0, x_1, x_2, x_3)$	B_4: smooth quartic surface in $\mathbb{P}^3$	19

5 Singular loci and extremal quintic threefold singularities

In this section we determine the singular loci of the general closed-orbit representatives constructed in Section 4. For $k = 1, \ldots, 38$, let

$$\phi_k^{\mathrm{nf}} \in \mathrm{Sym}^5 \mathbb{C}^5$$

be the normal form listed in Table 3, and set

$$X_k := V(\phi_k^{\mathrm{nf}}) \subset \mathbb{P}^4.$$

We compute the saturated Jacobian scheme

$$\mathrm{Sing}(X_k) = V\big(J(\phi_k^{\mathrm{nf}})\big), \qquad J(\phi) := \left(\frac{\partial \phi}{\partial x_0}, \ldots, \frac{\partial \phi}{\partial x_4}\right) : (x_0, \ldots, x_4)^\infty.$$

All statements in this section are made for general parameters, equivalently on a fixed nonempty Zariski-open subset of the corresponding normal-form parameter space.

The singular loci that occur on the boundary have a uniform structure. Their positive-dimensional parts are explicit low-degree configurations: lines and line arrangements, smooth conics, cuspidal plane curves, planes, and smooth quartic surfaces. In several cases these positive-dimensional components are accompanied by isolated singular points. For a general boundary representative, all such isolated singularities are quasi-homogeneous and fall into exactly eleven analytic types.

We first fix notation for these isolated analytic types. Each type below is an isolated hypersurface singularity in four local variables. Moreover, in every case the defining equation is weighted homogeneous of degree D with weights (w_1, w_2, w_3, w_4) satisfying

$$w_1 + w_2 + w_3 + w_4 = D.$$

Thus, by the standard formula for weighted homogeneous isolated hypersurface singularities, its local minimal exponent is

$$\widetilde{\alpha}_0 = \frac{w_1 + w_2 + w_3 + w_4}{D} = 1.$$

This value is precisely the critical threshold

$$\frac{n+1}{d} = \frac{4+1}{5} = 1$$

appearing in Park's stability criterion for quintic hypersurfaces in $\mathbb{P}^4$. For this reason we regard the eleven isolated analytic types listed below as the extremal quintic threefold singularities.

The complete singular-locus computation for the thirty-eight boundary families is summarized after these definitions. The table records the geometry of the positive-dimensional part, the relevant incidence relations, and the isolated analytic type whenever an isolated point occurs.

Definition 5.1. Let $B_4(X_1, X_2, X_3, X_4)$ be a homogeneous quartic whose projectivization $V(B_4) \subset \mathbb{P}^3$ is smooth. We denote by

$$\mathrm{QH}(1,1,1,1;4)_{81}(B_4)$$

the isolated hypersurface singularity analytically equivalent to

$$B_4(X_1, X_2, X_3, X_4) = 0.$$

It is quasi-homogeneous of weighted degree 4 with respect to the weights

$$(w_{X_1}, w_{X_2}, w_{X_3}, w_{X_4}) = (1,1,1,1).$$

Its Milnor number is $\mu = 81$, and its local minimal exponent is

$$\widetilde{\alpha} = 1.$$

We abbreviate this singularity by

$$\mathbf{S}^{\mathrm{I}}_{81}(B_4).$$

Definition 5.2. We denote by

$$\mathrm{QH}(7,8,9,12;36)_{87}(\tau)$$

the isolated hypersurface singularity analytically equivalent to

$$X_1^4 X_2 + X_3^4 + X_2^3 X_4 + \tau X_1 X_2 X_3 X_4 + X_4^3 = 0,$$

where

$$\tau \in \mathbb{C}^\times, \qquad \tau^4 \neq 256.$$

It is quasi-homogeneous of weighted degree 36 with respect to the weights

$$(w_{X_1}, w_{X_2}, w_{X_3}, w_{X_4}) = (7,8,9,12).$$

Its Milnor number is $\mu = 87$, and its local minimal exponent is

$$\widetilde{\alpha} = 1.$$

We abbreviate this singularity by

$$\mathbf{S}^{\mathrm{I}}_{87}(\tau).$$

Definition 5.3. We denote by

$$\mathrm{QH}(5,6,7,9;27)_{88}(\alpha, \beta)$$

the isolated hypersurface singularity analytically equivalent to

$$X_1^3 X_2^2 + X_1^4 X_3 + X_2 X_3^3 + X_2^3 X_4 + \alpha X_1 X_2 X_3 X_4 + \beta X_4^3 = 0,$$

where

$$\alpha, \beta \in \mathbb{C}^\times,$$

and

$$\alpha^6 - 2\alpha^5 + \alpha^4 + 54\alpha^3\beta - 378\alpha^2\beta + 576\alpha\beta + 729\beta^2 - 256\beta \neq 0.$$

It is quasi-homogeneous of weighted degree 27 with respect to the weights

$$(w_{X_1}, w_{X_2}, w_{X_3}, w_{X_4}) = (5, 6, 7, 9).$$

Its Milnor number is $\mu = 88$, and its local minimal exponent is

$$\widetilde{\alpha} = 1.$$

We abbreviate this singularity by

$$\mathbf{S}^{\mathrm{I}}_{88}(\alpha, \beta).$$

Definition 5.4. Let $B_5(X_1, X_2)$ be a binary quintic for which the hypersurface germ below is isolated at the origin. We denote by

$$\mathrm{QH}(3, 3, 4, 5; 15)_{88}(B_5)$$

the isolated hypersurface singularity analytically equivalent to

$$B_5(X_1, X_2) + X_2X_3^3 + (X_1^2 + X_2^2)X_3X_4 + X_4^3 = 0.$$

It is quasi-homogeneous of weighted degree 15 with respect to the weights

$$(w_{X_1}, w_{X_2}, w_{X_3}, w_{X_4}) = (3, 3, 4, 5).$$

Its Milnor number is $\mu = 88$, and its local minimal exponent is

$$\widetilde{\alpha} = 1.$$

We abbreviate this singularity by

$$\mathbf{S}^{\mathrm{II}}_{88}(B_5).$$

Definition 5.5. Let B_4 be a square-free binary quartic and let Q_2, R_2 be binary quadratic forms. Let

$$\mathcal{R}^{\mathrm{I}}_{90}(B_4, Q_2, R_2, \beta)$$

denote the weighted discriminant condition for the weighted-homogeneous germ below to be isolated at the origin. We denote by

$$\mathrm{QH}(2, 3, 3, 4; 12)_{90}(B_4, Q_2, R_2, \beta)$$

the isolated hypersurface singularity analytically equivalent to

$$B_4(X_2, X_3) + X_1^3Q_2(X_2, X_3) + X_1^4X_4 + X_1X_4R_2(X_2, X_3) + \beta X_1^2X_4^2 + X_4^3 = 0,$$

where

$$\beta \in \mathbb{C}, \qquad \mathcal{R}^{\mathrm{I}}_{90}(B_4, Q_2, R_2, \beta) \neq 0.$$

It is quasi-homogeneous of weighted degree 12 with respect to the weights

$$(w_{X_1}, w_{X_2}, w_{X_3}, w_{X_4}) = (2, 3, 3, 4).$$

Its Milnor number is $\mu = 90$, and its local minimal exponent is

$$\widetilde{\alpha} = 1.$$

We abbreviate this singularity by

$$\mathbf{S}^{\mathrm{I}}_{90}(B_4, Q_2, R_2, \beta).$$

Definition 5.6. Let
$$\mathcal{R}_{91}(\alpha,\beta,\gamma,\delta,\epsilon)$$
denote the bad-parameter equation for which the weighted-homogeneous germ below is not isolated at the origin. We denote by

$$\mathrm{QH}(3,4,5,6;18)_{91}(\alpha,\beta,\gamma,\delta,\epsilon)$$

the isolated hypersurface singularity analytically equivalent to

$$X_1^2X_2^3+X_1^3X_2X_3+X_1^4X_4+X_2^2X_3^2+\alpha X_2^3X_4+\beta X_1X_3^3+\gamma X_1X_2X_3X_4+\delta X_1^2X_4^2+\epsilon X_4^3=0,$$

where
$$\alpha,\beta,\gamma,\delta,\epsilon\in\mathbb{C}^\times,\qquad \mathcal{R}_{91}(\alpha,\beta,\gamma,\delta,\epsilon)\neq 0.$$
It is quasi-homogeneous of weighted degree 18 with respect to the weights

$$(w_{X_1},w_{X_2},w_{X_3},w_{X_4})=(3,4,5,6).$$

Its Milnor number is $\mu=91$, and its local minimal exponent is

$$\widetilde{\alpha}=1.$$

We abbreviate this singularity by
$$\mathbf{S}^{\mathrm{I}}_{91}(\alpha,\beta,\gamma,\delta,\epsilon).$$

Definition 5.7. Let
$$\mathcal{R}_{96}(\alpha,\beta)$$
denote the bad-parameter equation for which the weighted-homogeneous germ below is not isolated at the origin. We denote by

$$\mathrm{QH}(4,5,7,9;25)_{96}(\alpha,\beta)$$

the isolated hypersurface singularity analytically equivalent to

$$X_2^5+X_1^2X_2^2X_3+X_1^4X_4+X_1X_3^3+\alpha X_1X_2X_3X_4+\beta X_3X_4^2=0,$$

where
$$\alpha,\beta\in\mathbb{C}^\times,\qquad \mathcal{R}_{96}(\alpha,\beta)\neq 0.$$
It is quasi-homogeneous of weighted degree 25 with respect to the weights

$$(w_{X_1},w_{X_2},w_{X_3},w_{X_4})=(4,5,7,9).$$

Its Milnor number is $\mu=96$, and its local minimal exponent is

$$\widetilde{\alpha}=1.$$

We abbreviate this singularity by
$$\mathbf{S}^{\mathrm{I}}_{96}(\alpha,\beta).$$

Definition 5.8. Let
$$\mathcal{R}^{\mathrm{II}}_{96}(\alpha,\beta,\gamma,\delta)$$
denote the bad-parameter equation for which the weighted-homogeneous germ below is not isolated at the origin. We denote by

$$\mathrm{QH}(3,4,5,7;19)_{96}(\alpha,\beta,\gamma,\delta)$$

the isolated hypersurface singularity analytically equivalent to

$$X_1X_2^4 + X_1^2X_2^2X_3 + \alpha X_1^3X_3^2 + X_1^4X_4 + X_2X_3^3 + \beta X_2^3X_4 \\ + \gamma X_1X_2X_3X_4 + \delta X_3X_4^2 = 0,$$

where

$$\alpha, \beta, \gamma, \delta \in \mathbb{C}^\times, \qquad \mathcal{R}_{96}^{\mathrm{II}}(\alpha, \beta, \gamma, \delta) \neq 0.$$

It is quasi-homogeneous of weighted degree 19 with respect to the weights

$$(w_{X_1}, w_{X_2}, w_{X_3}, w_{X_4}) = (3, 4, 5, 7).$$

Its Milnor number is $\mu = 96$, and its local minimal exponent is

$$\widetilde{\alpha} = 1.$$

We abbreviate this singularity by

$$\mathbf{S}_{96}^{\mathrm{II}}(\alpha, \beta, \gamma, \delta).$$

Definition 5.9. Let B_5 be a ternary quintic, let C_3 be a ternary cubic, and let L_1 be a nonzero linear form in (X_1, X_2, X_3). Let

$$\mathcal{R}_{96}^{\mathrm{III}}(B_5, C_3, L_1)$$

denote the bad-parameter equation for which the weighted-homogeneous germ below is not isolated at the origin. We denote by

$$\mathrm{QH}(1, 1, 1, 2; 5)_{96}(B_5, C_3, L_1)$$

the isolated hypersurface singularity analytically equivalent to

$$B_5(X_1, X_2, X_3) + X_4C_3(X_1, X_2, X_3) + X_4^2L_1(X_1, X_2, X_3) = 0,$$

where

$$\mathcal{R}_{96}^{\mathrm{III}}(B_5, C_3, L_1) \neq 0.$$

It is quasi-homogeneous of weighted degree 5 with respect to the weights

$$(w_{X_1}, w_{X_2}, w_{X_3}, w_{X_4}) = (1, 1, 1, 2).$$

Its Milnor number is $\mu = 96$, and its local minimal exponent is

$$\widetilde{\alpha} = 1.$$

We abbreviate this singularity by

$$\mathbf{S}_{96}^{\mathrm{III}}(B_5, C_3, L_1).$$

Definition 5.10. Let A_4, C_4 be binary quartics in (X_1, X_2), let $Q_{2,2}$ be a bihomogeneous form of bidegree $(2, 2)$ in (X_1, X_2) and (X_3, X_4), and let B_3 be a binary cubic in (X_3, X_4). Let

$$\mathcal{R}_{100}^{\mathrm{I}}(A_4, C_4, Q_{2,2}, B_3)$$

denote the bad-parameter equation for which the weighted-homogeneous germ below is not isolated at the origin. We denote by

$$\mathrm{QH}(1, 1, 2, 2; 6)_{100}(A_4, C_4, Q_{2,2}, B_3)$$

the isolated hypersurface singularity analytically equivalent to

$$X_3A_4(X_1, X_2) + X_4C_4(X_1, X_2) + Q_{2,2}(X_1, X_2; X_3, X_4) + B_3(X_3, X_4) = 0,$$

where

$$\mathcal{R}^{\mathrm{I}}_{100}(A_4, C_4, Q_{2,2}, B_3) \neq 0.$$

It is quasi-homogeneous of weighted degree 6 with respect to the weights

$$(w_{X_1}, w_{X_2}, w_{X_3}, w_{X_4}) = (1, 1, 2, 2).$$

Its Milnor number is $\mu = 100$, and its local minimal exponent is

$$\widetilde{\alpha} = 1.$$

We abbreviate this singularity by

$$\mathbf{S}^{\mathrm{I}}_{100}(A_4, C_4, Q_{2,2}, B_3).$$

Definition 5.11. Let

$$\mathcal{R}^{\mathrm{I}}_{102}(\alpha, \beta, \gamma, \delta)$$

denote the bad-parameter equation for which the weighted-homogeneous germ below is not isolated at the origin. We denote by

$$\mathrm{QH}(3, 4, 5, 8; 20)_{102}(\alpha, \beta, \gamma, \delta)$$

the isolated hypersurface singularity analytically equivalent to

$$\begin{aligned} X_2^5 + \alpha X_1 X_2^3 X_3 + \beta X_1^2 X_2 X_3^2 + X_1^4 X_4 + X_3^4 + \gamma X_2^3 X_4 \\ + \delta X_1 X_2 X_3 X_4 + X_2 X_4^2 = 0, \end{aligned}$$

where

$$\alpha, \beta, \gamma, \delta \in \mathbb{C}^{\times}, \qquad \mathcal{R}^{\mathrm{I}}_{102}(\alpha, \beta, \gamma, \delta) \neq 0.$$

It is quasi-homogeneous of weighted degree 20 with respect to the weights

$$(w_{X_1}, w_{X_2}, w_{X_3}, w_{X_4}) = (3, 4, 5, 8).$$

Its Milnor number is $\mu = 102$, and its local minimal exponent is

$$\widetilde{\alpha} = 1.$$

We abbreviate this singularity by

$$\mathbf{S}^{\mathrm{I}}_{102}(\alpha, \beta, \gamma, \delta).$$

Isolated singular points for quintic threefolds

The table is ordered by increasing Milnor number μ. All entries are for the same generic open loci used in the case-by-case computations. We use

$$P_0 = (1 : 0 : 0 : 0 : 0), \qquad P_\infty = (0 : 0 : 0 : 0 : 1).$$

In the "occurrences" column, the notation $k : P$ means that the isolated point P occurs in Case k. The local minimal exponent is $\widetilde{\alpha} = 1$ in every row.

Table 4: Isolated analytic singularity types, ordered by Milnor number.

μ	Analytic type	Occurrences	QH data	Local normal form
81	$\mathbf{S}^{\mathrm{I}}_{81}(B_4)$	$23 : P_0$ $38 : P_\infty$	$\mathrm{QH}(1,1,1,1;4)_{81}$ weights $(1,1,1,1)$	$B_4(X_1,X_2,X_3,X_4)=0,$ $V(B_4)\subset \mathbf{P}^3$ smooth.
87	$\mathbf{S}^{\mathrm{I}}_{87}(\tau)$	$1 : P_0$ $26 : P_\infty$	$\mathrm{QH}(7,8,9,12;36)_{87}$ weights $(7,8,9,12)$	$X_1^4X_2+X_3^4+X_2^3X_4$ $+\tau X_1X_2X_3X_4+X_4^3=0,$ $\tau\in\mathbb{C}^\times,\ \tau^4\neq 256.$
88	$\mathbf{S}^{\mathrm{I}}_{88}(\alpha,\beta)$	$2 : P_0$ $27 : P_\infty$	$\mathrm{QH}(5,6,7,9;27)_{88}$ weights $(5,6,7,9)$	$X_1^3X_2^2+X_1^4X_3+X_2X_3^3+X_2^3X_4$ $+\alpha X_1X_2X_3X_4+\beta X_4^3=0,$ $\alpha,\beta\in\mathbb{C}^\times,\ D^{\mathrm{I}}_{88}(\alpha,\beta)\neq 0.$
88	$\mathbf{S}^{\mathrm{II}}_{88}(B_5)$	$3 : P_0$ $33 : P_\infty$	$\mathrm{QH}(3,3,4,5;15)_{88}$ weights $(3,3,4,5)$	$B_5(X_1,X_2)+X_2X_3^3$ $+(X_1^2+X_2^2)X_3X_4+X_4^3=0,$ B_5 general binary quintic.
90	$\mathbf{S}^{\mathrm{I}}_{90}$ (B_4,Q_2,R_2,β)	$10 : P_0$ $34 : P_\infty$	$\mathrm{QH}(2,3,3,4;12)_{90}$ weights $(2,3,3,4)$	$B_4(X_2,X_3)+X_1^3Q_2(X_2,X_3)+X_1^4X_4$ $+X_1X_4R_2(X_2,X_3)+\beta X_1^2X_4^2+X_4^3=0,$ $\mathcal{R}^{\mathrm{I}}_{90}(B_4,Q_2,R_2,\beta)\neq 0.$
91	$\mathbf{S}^{\mathrm{I}}_{91}$ $(\alpha,\beta,\gamma,\delta,\epsilon)$	$6 : P_0$ $28 : P_\infty$	$\mathrm{QH}(3,4,5,6;18)_{91}$ weights $(3,4,5,6)$	$X_1^2X_2^3+X_1^3X_2X_3+X_1^4X_4+X_2^2X_3^2$ $+\alpha X_2^3X_4+\beta X_1X_3^3+\gamma X_1X_2X_3X_4$ $+\delta X_1^2X_4^2+\epsilon X_4^3=0,$ $\mathcal{R}_{91}(\alpha,\beta,\gamma,\delta,\epsilon)\neq 0.$
96	$\mathbf{S}^{\mathrm{I}}_{96}(\alpha,\beta)$	$7 : P_0$ $18 : P_\infty$	$\mathrm{QH}(4,5,7,9;25)_{96}$ weights $(4,5,7,9)$	$X_2^5+X_1^2X_2^2X_3+X_1^4X_4+X_1X_3^3$ $+\alpha X_1X_2X_3X_4+\beta X_3X_4^2=0,$ $\mathcal{R}_{96}(\alpha,\beta)\neq 0.$
96	$\mathbf{S}^{\mathrm{II}}_{96}$ $(\alpha,\beta,\gamma,\delta)$	$9 : P_0$ $22 : P_\infty$	$\mathrm{QH}(3,4,5,7;19)_{96}$ weights $(3,4,5,7)$	$X_1X_2^4+X_1^2X_2^2X_3+\alpha X_1^3X_3^2+X_1^4X_4$ $+X_2X_3^3+\beta X_2^3X_4+\gamma X_1X_2X_3X_4$ $+\delta X_3X_4^2=0,$ $\mathcal{R}^{\mathrm{II}}_{96}(\alpha,\beta,\gamma,\delta)\neq 0.$
96	$\mathbf{S}^{\mathrm{III}}_{96}$ (B_5,C_3,L_1)	$36 : P_0$ $36 : P_\infty$	$\mathrm{QH}(1,1,1,2;5)_{96}$ weights $(1,1,1,2)$	$B_5(X_1,X_2,X_3)+X_4C_3(X_1,X_2,X_3)$ $+X_4^2L_1(X_1,X_2,X_3)=0,$ $\mathcal{R}^{\mathrm{III}}_{96}(B_5,C_3,L_1)\neq 0.$
100	$\mathbf{S}^{\mathrm{I}}_{100}$ $(A_4,C_4,Q_{2,2},B_3)$	$29 : P_0$ $35 : P_\infty$	$\mathrm{QH}(1,1,2,2;6)_{100}$ weights $(1,1,2,2)$	$X_3A_4(X_1,X_2)+X_4C_4(X_1,X_2)$ $+Q_{2,2}(X_1,X_2;X_3,X_4)+B_3(X_3,X_4)=0,$ $\mathcal{R}^{\mathrm{I}}_{100}(A_4,C_4,Q_{2,2},B_3)\neq 0.$
102	$\mathbf{S}^{\mathrm{I}}_{102}$ $(\alpha,\beta,\gamma,\delta)$	$14 : P_0$ $14 : P_\infty$	$\mathrm{QH}(3,4,5,8;20)_{102}$ weights $(3,4,5,8)$	$X_2^5+\alpha X_1X_2^3X_3+\beta X_1^2X_2X_3^2$ $+X_1^4X_4+X_3^4+\gamma X_2^3X_4$ $+\delta X_1X_2X_3X_4+X_2X_4^2=0,$ $\mathcal{R}^{\mathrm{I}}_{102}(\alpha,\beta,\gamma,\delta)\neq 0.$

Singular loci for quintic threefolds

All rows are for general parameters in the nonempty Zariski-open locus specified in the case-by-case computations. We use

$$P_0=(1:0:0:0:0),\qquad P_\infty=(0:0:0:0:1).$$

For the line arrangements,

$$B_\lambda(u,v):=uv(u-v)(u-\lambda v),\qquad B_{\lambda,\mu}(u,v):=v(u-v)(u-\lambda v)(u-\mu v).$$

The column “isolated type” records only isolated singular points; special points lying on positive-dimensional components are described in the geometry column. In Case 29, $B_3^{(29)} = x_3x_4(x_3-x_4)$.

Table 5: Singular loci of the 38 quintic-threefold boundary families.

k Singular locus	Geometry / incidences	Isolated type
1 $\mathrm{Sing}(X_1) = C \cup \{P_0\}$ $C = V(x_0, x_1, x_3^4 + x_2^3x_4)$	rational plane quartic; cusp at P_∞	P_0 : $\mathbf{S}^{\mathrm{I}}_{87}(\tau)$
2 $\mathrm{Sing}(X_2) = L \cup C \cup \{P_0\}$ $L = V(x_0, x_1, x_2)$ $C = V(x_0, x_1, x_3^3 + x_2^2x_4)$	L line; C cuspidal cubic; $L \cap C = \{P_\infty\}$	P_0 : $\mathbf{S}^{\mathrm{I}}_{88}(\alpha, \beta)$
3 $\mathrm{Sing}(X_3) = L \cup \{P_0\}$ $L = V(x_0, x_1, x_2)$	L line; special point P_∞	P_0 : $\mathbf{S}^{\mathrm{II}}_{88}(B_5)$
4 $\mathrm{Sing}(X_4) = N \cup L$ $N = V(x_2, x_3, x_4)$ $L = V(x_0, x_1, x_2)$	two disjoint lines	$\varnothing$
5 $\mathrm{Sing}(X_5) = N \cup L \cup M$ $N = V(x_2, x_3, x_4)$ $L = V(x_0, x_1, x_2)$ $M = V(x_0, x_1, x_3)$	three lines; $L \cap M = \{P_\infty\}$; $N \cap L = N \cap M = \varnothing$	$\varnothing$
6 $\mathrm{Sing}(X_6) = L \cup C \cup \{P_0\}$ $L = V(x_0, x_1, x_2)$ $C = V(x_0, x_1, x_3^2 + \alpha x_2x_4)$	L line; C smooth conic; $L \cap C = \{P_\infty\}$	P_0 : $\mathbf{S}^{\mathrm{I}}_{91}$ $(\alpha, \beta, \gamma, \delta, \epsilon)$
7 $\mathrm{Sing}(X_7) = L \cup \{P_0\}$ $L = V(x_0, x_1, x_2)$	L line; special point P_∞	P_0 : $\mathbf{S}^{\mathrm{I}}_{96}(\alpha, \beta)$
8 $\mathrm{Sing}(X_8) = N \cup L$ $N = V(x_2, x_3, x_4)$ $L = V(x_0, x_1, x_2)$	two disjoint lines	$\varnothing$
9 $\mathrm{Sing}(X_9) = L \cup \{P_0\}$ $L = V(x_0, x_1, x_2)$	L line; special point P_∞	P_0 : $\mathbf{S}^{\mathrm{II}}_{96}(\alpha, \beta, \gamma, \delta)$
10 $\mathrm{Sing}(X_{10}) = C_\lambda \cup \{P_0\}$ $C_\lambda = V(x_0, x_1, B_\lambda(x_2, x_3))$	$C_\lambda = L_0 \cup L_\infty \cup L_1 \cup L_\lambda$; four lines through P_∞	P_0 : $\mathbf{S}^{\mathrm{I}}_{90}$ $(B_\lambda, Q_{\mathbf{q}}, R_{\mathbf{r}}, \beta)$
11 $\mathrm{Sing}(X_{11}) = N \cup L$ $N = V(x_2, x_3, x_4)$ $L = V(x_0, x_1, x_2)$	two disjoint lines	$\varnothing$
12 $\mathrm{Sing}(X_{12}) = N \cup L$ $N = V(x_2, x_3, x_4)$ $L = V(x_0, x_1, x_2)$	two disjoint lines	$\varnothing$
13 $\mathrm{Sing}(X_{13}) = N \cup L$ $N = V(x_2, x_3, x_4)$ $L = V(x_0, x_1, x_2)$	two disjoint lines	$\varnothing$
14 $\mathrm{Sing}(X_{14}) = \{P_0, P_\infty\}$	two isolated points	P_0, P_∞ : $\mathbf{S}^{\mathrm{I}}_{102}(\alpha, \beta, \gamma, \delta)$
15 $\mathrm{Sing}(X_{15}) = N \cup L$ $N = V(x_2, x_3, x_4)$ $L = V(x_0, x_1, x_2)$	two disjoint lines	$\varnothing$

continued on the next page

k Singular locus	Geometry / incidences	Isolated type
16 $\mathrm{Sing}(X_{16}) = N \cup L$ $N = V(x_2, x_3, x_4)$ $L = V(x_0, x_1, x_2)$	two disjoint lines	$\varnothing$
17 $\mathrm{Sing}(X_{17}) = N \cup L$ $N = V(x_2, x_3, x_4)$ $L = V(x_0, x_1, x_2)$	two disjoint lines	$\varnothing$
18 $\mathrm{Sing}(X_{18}) = N \cup \{P_\infty\}$ $N = V(x_2, x_3, x_4)$	N line; special point $P_0 \in N$	P_∞ : $\mathbf{S}^{\mathrm{I}}_{96}(\alpha, \beta)$
19 $\mathrm{Sing}(X_{19}) = N \cup L$ $N = V(x_2, x_3, x_4)$ $L = V(x_0, x_1, x_2)$	two disjoint lines	$\varnothing$
20 $\mathrm{Sing}(X_{20}) = N \cup C_{\lambda,\mu}$ $N = V(x_2, x_3, x_4)$ $C_{\lambda,\mu} = V(x_0, x_1, B_{\lambda,\mu}(x_2, x_3))$	N line; $C_{\lambda,\mu}$ four lines through P_∞; disjoint from N	$\varnothing$
21 $\mathrm{Sing}(X_{21}) = N \cup L$ $N = V(x_2, x_3, x_4)$ $L = V(x_0, x_1, x_2)$	two disjoint lines	$\varnothing$
22 $\mathrm{Sing}(X_{22}) = N \cup \{P_\infty\}$ $N = V(x_2, x_3, x_4)$	N line	P_∞ : $\mathbf{S}^{\mathrm{II}}_{96}$ $(\alpha, \beta, \gamma, \delta)$
23 $\mathrm{Sing}(X_{23}) = S \cup \{P_0\}$ $S = V(x_0, B_4)$	S smooth quartic surface	P_0 : $\mathbf{S}^{\mathrm{I}}_{81}(B_4)$
24 $\mathrm{Sing}(X_{24}) = N \cup M \cup L$ $N = V(x_2, x_3, x_4)$ $M = V(x_1, x_3, x_4)$ $L = V(x_0, x_1, x_2)$	three lines; $N \cap M = \{P_0\}$; L disjoint from $N \cup M$	$\varnothing$
25 $\mathrm{Sing}(X_{25}) = N \cup L$ $N = V(x_2, x_3, x_4)$ $L = V(x_0, x_1, x_2)$	two disjoint lines	$\varnothing$
26 $\mathrm{Sing}(X_{26}) = C \cup \{P_\infty\}$ $C = V(x_3, x_4, x_1^4 + x_0 x_2^3)$	C irreducible plane quartic	P_∞ : $\mathbf{S}^{\mathrm{I}}_{87}(\alpha)$
27 $\mathrm{Sing}(X_{27}) = N \cup C \cup \{P_\infty\}$ $N = V(x_2, x_3, x_4)$ $C = V(x_3, x_4, x_1^3 + x_0 x_2^2)$	N line; C cuspidal plane cubic; cusp P_0	P_∞ : $\mathbf{S}^{\mathrm{I}}_{88}(\alpha, \beta)$
28 $\mathrm{Sing}(X_{28}) = N \cup C \cup \{P_\infty\}$ $N = V(x_2, x_3, x_4)$ $C = V(x_3, x_4, x_1^2 + \alpha x_0 x_2)$	N line; C smooth conic	P_∞ : $\mathbf{S}^{\mathrm{I}}_{91}$ $(\alpha, \beta, \gamma, \delta, \epsilon)$
29 $\mathrm{Sing}(X_{29}) = L \cup \{P_0\}$ $L = V(x_0, x_1, x_2)$	L line	P_0 : $\mathbf{S}^{\mathrm{I}}_{100}$ $(A_4, C_4, Q_{2,2}, B_3^{(29)})$
30 $\mathrm{Sing}(X_{30}) = N \cup L$ $N = V(x_2, x_3, x_4)$ $L = V(x_0, x_1, x_2)$	two disjoint lines	$\varnothing$
31 $\mathrm{Sing}(X_{31}) = C_\lambda \cup L$ $C_\lambda = V(x_3, x_4, B_\lambda(x_1, x_2))$ $L = V(x_0, x_1, x_2)$	C_λ four lines through P_0; L line; $C_\lambda \cap L = \varnothing$	$\varnothing$

continued on the next page

k Singular locus	Geometry / incidences	Isolated type
32 $\mathrm{Sing}(X_{32}) = N \cup \Pi$ $N = V(x_2, x_3, x_4)$ $\Pi = V(x_0, x_1)$	N line; $\Pi \simeq \mathbb{P}^2$; $N \cap \Pi = \varnothing$	$\varnothing$
33 $\mathrm{Sing}(X_{33}) = N \cup \{P_\infty\}$ $N = V(x_2, x_3, x_4)$	N line; special point $P_0 \in N$	P_∞ : $\mathbf{S}^{\mathrm{II}}_{88}(\widehat{B}_5)$
34 $\mathrm{Sing}(X_{34}) = C_\lambda \cup \{P_\infty\}$ $C_\lambda = V(x_3, x_4, B_\lambda(x_1, x_2))$	C_λ four lines through P_0	P_∞ : $\mathbf{S}^{\mathrm{I}}_{90}(B_\lambda, Q_2, R_2, \beta)$
35 $\mathrm{Sing}(X_{35}) = N \cup \{P_\infty\}$ $N = V(x_2, x_3, x_4)$	N line	P_∞ : $\mathbf{S}^{\mathrm{I}}_{100}$ $(A_4, C_4, \widehat{Q}_{2,2}, B_3)$
36 $\mathrm{Sing}(X_{36}) = \{P_0, P_\infty\}$	two isolated points	P_0, P_∞ : $\mathbf{S}^{\mathrm{III}}_{96}(B_5, C_3, X_3)$
37 $\mathrm{Sing}(X_{37}) = \Pi \cup L$ $\Pi = V(x_3, x_4)$ $L = V(x_0, x_1, x_2)$	$\Pi \simeq \mathbb{P}^2$; L line; $\Pi \cap L = \varnothing$	$\varnothing$
38 $\mathrm{Sing}(X_{38}) = S \cup \{P_\infty\}$ $S = V(x_4, B_4)$	S smooth quartic surface	P_∞ : $\mathbf{S}^{\mathrm{I}}_{81}(B_4)$

6 Pairwise non-inclusion of the boundary families

In this announcement we record the structure and the numerical output of the non-inclusion verification. The certified version of the computation, including exact specializations and reproducibility scripts, will be given in the forthcoming full-length paper.

The goal of this section is to summarize the verification of the non-inclusion statement for the thirty-eight boundary families constructed in Section 3. We use the following convention throughout this section. A candidate inclusion means a possible specialization relation

$$\Phi_k \subset \overline{\Phi_\ell} \qquad (k \neq \ell),$$

so that a general point of Φ_k would arise as a specialization of a general point of Φ_ℓ. This is slightly stronger than what is needed for Theorem 3.3: if no such closure containment exists, then a fortiori no ordinary containment $\Phi_k \subset \Phi_\ell$ exists.

There are

$$38 \cdot 37 = 1406$$

ordered pairs (k, ℓ), $k \neq \ell$, to exclude. Rather than treating these pairs one by one, we apply a sequence of monotone necessary conditions. Each filter is designed so that, if $\Phi_k \subset \overline{\Phi_\ell}$ were true, then the invariant of the special family Φ_k would have to be compatible with the corresponding invariant of the more general family Φ_ℓ. A violation of this compatibility rules out the ordered pair. The filters become progressively finer:

filter	invariant used	number of candidates left
0	all ordered pairs	1406
1	dimension	759
2	apolar Hilbert functions and Betti tables	450
3	Hilbert functions of singular schemes	22
4	singular one-cycles and Hessian ranks	4
5	generic connected stabilizer tori	0.

The point of presenting the argument in this filtered form is that each step has a clear geometric meaning: dimension gives the coarsest obstruction, apolar algebras record the projective form algebraically, singular schemes record the geometry of the hypersurface, singular one-cycles retain the curve-theoretic part of that geometry, and stabilizer tori distinguish the final few highly special cases.

6.1 Filter 1: dimension

The first obstruction is purely dimensional. If one irreducible family lies in the closure of another, its dimension cannot be larger. This filter is coarse, but it removes all ordered pairs in which the proposed special family is already too large to occur as a specialization of the proposed general family.

For each boundary family Φ_k, let

$$d_k := \dim \Phi_k.$$

From the closed-orbit family dimension table, the values of d_k are as follows:

d	$\{ k \mid d_k = d \}$
1	$\{1, 26\}$
2	$\{2, 4, 7, 8, 16, 18, 21, 27\}$
3	$\{5, 24\}$
4	$\{9, 13, 14, 22\}$
5	$\{6, 12, 17, 28\}$
6	$\{3, 11, 25, 33\}$
8	$\{10, 19, 34\}$
9	$\{15, 30\}$
11	$\{20, 31\}$
15	$\{29, 35\}$
18	$\{32, 37\}$
19	$\{23, 38\}$
24	$\{36\}$.

If there is a candidate inclusion

$$\Phi_k \subset \overline{\Phi_\ell},$$

then necessarily

$$d_k \le d_\ell.$$

Therefore the dimension filter only rules out the ordered pairs (k, ℓ) satisfying

$$d_k > d_\ell.$$

Equivalently, define

$$\mathcal{R}_1 := \{ (k, \ell) \mid 1 \le k, \ell \le 38,\ k \ne \ell,\ d_k > d_\ell \}.$$

For every $(k, \ell) \in \mathcal{R}_1$, we conclude

$$\Phi_k \not\subset \Phi_\ell.$$

This filter gives

$$|\mathcal{R}_1| = 647, \qquad 38 \cdot 37 - |\mathcal{R}_1| = 759.$$

When $d_k = d_\ell$, Filter 1 makes no conclusion.

For each k, put

$$D_1(k) := \{ \ell \mid (k, \ell) \in \mathcal{R}_1 \}.$$

The complete list of pairs excluded by Filter 1 is as follows.

Table 6: Ordered pairs (k, ℓ) ruled out by Filter 1.

k	$D_1(k)$
1	$\varnothing$
2	$\{1, 26\}$
3	$\{1, 2, 4, 5, 6, 7, 8, 9, 12, 13, 14, 16, 17, 18, 21, 22, 24, 26, 27, 28\}$
4	$\{1, 26\}$
5	$\{1, 2, 4, 7, 8, 16, 18, 21, 26, 27\}$
6	$\{1, 2, 4, 5, 7, 8, 9, 13, 14, 16, 18, 21, 22, 24, 26, 27\}$
7	$\{1, 26\}$
8	$\{1, 26\}$
9	$\{1, 2, 4, 5, 7, 8, 16, 18, 21, 24, 26, 27\}$
10	$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 16, 17, 18, 21, 22, 24, 25, 26, 27, 28, 33\}$
11	$\{1, 2, 4, 5, 6, 7, 8, 9, 12, 13, 14, 16, 17, 18, 21, 22, 24, 26, 27, 28\}$
12	$\{1, 2, 4, 5, 7, 8, 9, 13, 14, 16, 18, 21, 22, 24, 26, 27\}$
13	$\{1, 2, 4, 5, 7, 8, 16, 18, 21, 24, 26, 27\}$
14	$\{1, 2, 4, 5, 7, 8, 16, 18, 21, 24, 26, 27\}$
15	$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16, 17, 18, 19, 21, 22, 24, 25, 26, 27, 28, 33, 34\}$
16	$\{1, 26\}$
17	$\{1, 2, 4, 5, 7, 8, 9, 13, 14, 16, 18, 21, 22, 24, 26, 27\}$
18	$\{1, 26\}$
19	$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 16, 17, 18, 21, 22, 24, 25, 26, 27, 28, 33\}$
20	$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 21, 22, 24, 25, 26, 27, 28, 30, 33, 34\}$
21	$\{1, 26\}$
22	$\{1, 2, 4, 5, 7, 8, 16, 18, 21, 24, 26, 27\}$
23	$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 37\}$
24	$\{1, 2, 4, 7, 8, 16, 18, 21, 26, 27\}$
25	$\{1, 2, 4, 5, 6, 7, 8, 9, 12, 13, 14, 16, 17, 18, 21, 22, 24, 26, 27, 28\}$
26	$\varnothing$
27	$\{1, 26\}$
28	$\{1, 2, 4, 5, 7, 8, 9, 13, 14, 16, 18, 21, 22, 24, 26, 27\}$
29	$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 25, 26, 27, 28, 30, 31, 33, 34\}$
30	$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16, 17, 18, 19, 21, 22, 24, 25, 26, 27, 28, 33, 34\}$
31	$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 21, 22, 24, 25, 26, 27, 28, 30, 33, 34\}$
32	$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 25, 26, 27, 28, 29, 30, 31, 33, 34, 35\}$
33	$\{1, 2, 4, 5, 6, 7, 8, 9, 12, 13, 14, 16, 17, 18, 21, 22, 24, 26, 27, 28\}$
34	$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 16, 17, 18, 21, 22, 24, 25, 26, 27, 28, 33\}$
35	$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 25, 26, 27, 28, 30, 31, 33, 34\}$
36	$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 37, 38\}$
37	$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 25, 26, 27, 28, 29, 30, 31, 33, 34, 35\}$
38	$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 37\}$

6.2 Filter 2: apolar Hilbert functions and Betti numbers

Dimension leaves many possible pairs, so we next use an invariant of the form itself: the apolar algebra. The Hilbert function of the apolar algebra is controlled by the ranks of the catalecticant maps, and these ranks cannot increase under specialization. Once the Hilbert function is fixed, the graded Betti numbers of the apolar algebra are upper semicontinuous. Thus apolar data provides a coordinate-free algebraic obstruction to a candidate inclusion.

Let

$$R = \mathbb{C}[x_0, \ldots, x_4], \qquad S = \mathbb{C}[\partial_0, \ldots, \partial_4],$$

where S acts on R by differentiation. For a quintic form $f \in R_5$, set

$$A_f := S/\operatorname{Ann}(f).$$

For each boundary family Φ_k, we take a general form f_k with support contained in $I(r_k)_{\geq 0}$, and compute the apolar algebra

$$A_k := A_{f_k}.$$

The computation was carried out by random sampling. For every k, the trials were repeated until a block of four consecutive trials gave the same Hilbert function, the same length, and the same graded Betti table of A_k. The common output in such a stable block was then adopted as the apolar invariant of the general member of Φ_k.

The adopted Hilbert functions are

$$H_{A_k}(t) = (1, 5, 15, 15, 5, 1) \qquad (1 \leq k \leq 37),$$

and

$$H_{A_{38}}(t) = (1, 5, 14, 14, 5, 1).$$

Equivalently,

$$\deg A_k = 42 \quad (1 \leq k \leq 37), \qquad \deg A_{38} = 40.$$

The adopted Betti tables fall into the following types.

For $m \geq 0$, define

$$B_m := \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline \text{total} & 1 & 20 & 35+m & 35+m & 20 & 1 \\ 0 & 1 & . & . & . & . & . \\ 1 & . & . & . & . & . & . \\ 2 & . & 20 & 35 & m & . & . \\ 3 & . & . & m & 35 & 20 & . \\ 4 & . & . & . & . & . & . \\ 5 & . & . & . & . & . & 1 \end{array}.$$

We also use the two special types

$$C := \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline \text{total} & 1 & 21 & 40 & 40 & 21 & 1 \\ 0 & 1 & . & . & . & . & . \\ 1 & . & . & . & . & . & . \\ 2 & . & 20 & 36 & 4 & 1 & . \\ 3 & . & 1 & 4 & 36 & 20 & . \\ 4 & . & . & . & . & . & . \\ 5 & . & . & . & . & . & 1 \end{array},$$

and

$$D := \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline \text{total} & 1 & 17 & 30 & 30 & 17 & 1 \\ 0 & 1 & . & . & . & . & . \\ 1 & . & 1 & . & . & . & . \\ 2 & . & 16 & 30 & . & . & . \\ 3 & . & . & . & 30 & 16 & . \\ 4 & . & . & . & . & 1 & . \\ 5 & . & . & . & . & . & 1 \end{array}.$$

The adopted classification is

Betti type	k
B_0	$\{19\}$
B_1	$\{17, 31, 35\}$
B_2	$\{12, 13, 14, 20, 22, 25, 29, 30\}$
B_3	$\{11, 15, 16, 18, 32, 37\}$
B_4	$\{9\}$
B_6	$\{2, 4, 5, 6, 7, 8, 10, 36\}$
B_8	$\{1, 3\}$
B_{14}	$\{23\}$
C	$\{21, 24, 26, 27, 28, 33, 34\}$
D	$\{38\}$.

We use the following semicontinuity test. Suppose that a candidate inclusion

$$\Phi_k \subset \overline{\Phi_\ell}$$

holds. Then the apolar Hilbert function cannot increase under the corresponding specialization; hence a necessary condition is

$$H_{A_k}(q) \leq H_{A_\ell}(q) \qquad \text{for all } q.$$

Moreover, after restricting to a fixed Hilbert-function stratum, graded Betti numbers are upper semicontinuous. Therefore a necessary condition is

$$\beta_{i,j}(A_k) \geq \beta_{i,j}(A_\ell) \qquad \text{for all } i, j.$$

Consequently, after Filter 1, we exclude an ordered pair (k, ℓ) if either

$$H_{A_k}(q) > H_{A_\ell}(q) \quad \text{for some } q,$$

or, when $H_{A_k} = H_{A_\ell}$,

$$\beta_{i,j}(A_k) < \beta_{i,j}(A_\ell) \quad \text{for some } i, j.$$

Let

$$\mathcal{P}_1 := \{\, (k, \ell) \mid 1 \leq k, \ell \leq 38,\ k \neq \ell,\ d_k \leq d_\ell \,\}$$

be the set of ordered pairs surviving Filter 1. Define

$$\mathcal{R}_2^{\mathrm{HF}} := \{(k, \ell) \in \mathcal{P}_1 \mid \exists q,\ H_{A_k}(q) > H_{A_\ell}(q)\},$$

and

$$\mathcal{R}_2^{\mathrm{Betti}} := \{(k, \ell) \in \mathcal{P}_1 \mid H_{A_k} = H_{A_\ell} \text{ and } \exists i, j,\ \beta_{i,j}(A_k) < \beta_{i,j}(A_\ell)\}.$$

Then

$$\mathcal{R}_2 := \mathcal{R}_2^{\mathrm{HF}} \cup \mathcal{R}_2^{\mathrm{Betti}}$$

is the set of ordered pairs excluded by Filter 2.

The number of pairs excluded by this filter is

$$|\mathcal{R}_2^{\mathrm{Betti}}| = 273, \qquad |\mathcal{R}_2^{\mathrm{HF}}| = 36,$$

and hence

$$|\mathcal{R}_2| = 309.$$

Since Filter 1 leaves 759 ordered pairs, Filter 1 and Filter 2 together leave

$$759 - 309 = 450$$

ordered pairs.

For each k, put

$$D_2(k) := \{ \ell \mid (k, \ell) \in \mathcal{R}_2 \}.$$

The complete list of ordered pairs excluded by Filter 2 is as follows.

Table 7: Ordered pairs (k, ℓ) ruled out by Filter 2.

k	$D_2(k)$
1	$\{21, 23, 24, 26, 27, 28, 33, 34, 38\}$
2	$\{3, 21, 23, 24, 27, 28, 33, 34, 38\}$
3	$\{23, 33, 34, 38\}$
4	$\{3, 21, 23, 24, 27, 28, 33, 34, 38\}$
5	$\{3, 23, 24, 28, 33, 34, 38\}$
6	$\{3, 23, 28, 33, 34, 38\}$
7	$\{3, 21, 23, 24, 27, 28, 33, 34, 38\}$
8	$\{3, 21, 23, 24, 27, 28, 33, 34, 38\}$
9	$\{3, 6, 10, 23, 28, 33, 34, 36, 38\}$
10	$\{23, 34, 38\}$
11	$\{3, 10, 23, 33, 34, 36, 38\}$
12	$\{3, 6, 10, 11, 15, 23, 28, 32, 33, 34, 36, 37, 38\}$
13	$\{3, 6, 9, 10, 11, 15, 23, 28, 32, 33, 34, 36, 37, 38\}$
14	$\{3, 6, 9, 10, 11, 15, 23, 28, 32, 33, 34, 36, 37, 38\}$
15	$\{23, 36, 38\}$
16	$\{2, 3, 4, 5, 6, 7, 8, 9, 10, 21, 23, 24, 27, 28, 33, 34, 36, 38\}$
17	$\{3, 6, 10, 11, 12, 15, 20, 23, 25, 28, 29, 30, 32, 33, 34, 36, 37, 38\}$
18	$\{2, 3, 4, 5, 6, 7, 8, 9, 10, 21, 23, 24, 27, 28, 33, 34, 36, 38\}$
19	$\{10, 15, 20, 23, 29, 30, 31, 32, 34, 35, 36, 37, 38\}$
20	$\{23, 32, 36, 37, 38\}$
21	$\{2, 3, 4, 5, 6, 7, 8, 10, 23, 36, 38\}$
22	$\{3, 6, 9, 10, 11, 15, 23, 28, 32, 33, 34, 36, 37, 38\}$
23	$\{38\}$
24	$\{3, 5, 6, 10, 23, 36, 38\}$
25	$\{3, 10, 11, 15, 23, 32, 33, 34, 36, 37, 38\}$
26	$\{1, 2, 3, 4, 5, 6, 7, 8, 10, 23, 36, 38\}$
27	$\{2, 3, 4, 5, 6, 7, 8, 10, 23, 36, 38\}$
28	$\{3, 6, 10, 23, 36, 38\}$
29	$\{23, 32, 36, 37, 38\}$
30	$\{15, 23, 32, 36, 37, 38\}$
31	$\{20, 23, 29, 32, 36, 37, 38\}$
32	$\{23, 36, 38\}$
33	$\{3, 10, 23, 36, 38\}$
34	$\{10, 23, 36, 38\}$
35	$\{23, 29, 32, 36, 37, 38\}$
36	$\varnothing$
37	$\{23, 36, 38\}$
38	$\varnothing$

6.3 Filter 3: Hilbert functions of singular loci

Apolar data still does not see enough of the geometry of the hypersurface. The next filter uses the singular scheme itself. If a family of hypersurfaces specializes, the singular scheme can acquire new components or embedded structure, but its Hilbert function in each degree is constrained by semicontinuity. Hence the Hilbert function of the saturated Jacobian scheme gives a sharper geometric test for non-inclusion.

Let

$$S = \mathbb{C}[x_0, \ldots, x_4], \qquad \mathfrak{m} = (x_0, \ldots, x_4).$$

For a quintic form $f \in S_5$, define the saturated Jacobian ideal

$$J(f) := \left(\frac{\partial f}{\partial x_0}, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4} \right) : \mathfrak{m}^{\infty}.$$

We regard

$$Z(f) := \operatorname{Proj}(S/J(f)) \subset \mathbb{P}^4$$

as the singular scheme of the hypersurface $X_f = V(f)$.

For each boundary family Φ_k, we choose a general form

$$f_k \in \langle x^u \mid u \in I(r_k)_{\geq 0} \rangle$$

by random sampling and compute the Hilbert function

$$h_k(q) := \dim_{\mathbb{C}}(S/J(f_k))_q, \qquad 0 \leq q \leq 30.$$

The computation was carried out with the saturated Jacobian ideal. For every k, the random trials were repeated until four consecutive trials gave the same Hilbert function $h_k(0), \ldots, h_k(30)$, the same dimension $\dim(S/J(f_k))$, and the same degree $\deg(S/J(f_k))$. The common output in such a stable block was then adopted as the singular-locus Hilbert function of a general member of Φ_k.

The semicontinuity principle used here is as follows. If

$$\Phi_k \subset \overline{\Phi_\ell},$$

then a general member of Φ_k is a specialization of a general member of Φ_ℓ. Since the singular scheme can only jump up under specialization, we must have

$$h_k(q) \geq h_\ell(q) \qquad \text{for all } q.$$

Therefore, if there exists q such that

$$h_k(q) < h_\ell(q),$$

then we conclude that

$$\Phi_k \not\subset \Phi_\ell.$$

Let $\mathcal{P}_{12}$ be the set of ordered pairs surviving Filters 1 and 2. Thus

$$|\mathcal{P}_{12}| = 450.$$

Define

$$\mathcal{R}_3 := \{(k, \ell) \in \mathcal{P}_{12} \mid \exists q,\ 0 \leq q \leq 30,\ h_k(q) < h_\ell(q)\}.$$

For every $(k, \ell) \in \mathcal{R}_3$, Filter 3 gives

$$\Phi_k \not\subset \Phi_\ell.$$

The computation gives

$$|\mathcal{R}_3| = 428.$$

Hence the number of ordered pairs surviving Filters 1, 2, and 3 is

$$|\mathcal{P}_{12}| - |\mathcal{R}_3| = 450 - 428 = 22.$$

Equivalently, starting from all ordered pairs (k, ℓ), $k \neq \ell$, we have

$$38 \cdot 37 = 1406$$

and

$$1406 \xrightarrow{\text{Filter 1}} 759 \xrightarrow{\text{Filter 2}} 450 \xrightarrow{\text{Filter 3}} 22.$$

The remaining ordered pairs are not asserted to give inclusions; they are only the pairs not ruled out by Filters 1–3.

$$\mathcal{P}_{123} = \mathcal{P}_{12} \setminus \mathcal{R}_3.$$

Explicitly,

$$\begin{aligned}\mathcal{P}_{123} = \{&(1,4),(1,5),(1,8),(1,15),(1,29),\\ &(2,5),(2,6),(2,8),(2,15),(2,29),\\ &(4,15),\\ &(5,15),(5,29),\\ &(6,29),\\ &(7,16),\\ &(8,15),(8,29),\\ &(10,29),\\ &(18,22),\\ &(21,27),\\ &(24,33),\\ &(28,33)\}.\end{aligned}$$

For convenience, we also record the same list source-wise.

Table 8: Ordered pairs surviving Filters 1–3.

k	$\{\ell \mid (k,\ell)$ survives Filters 1–3$\}$
1	$\{4,5,8,15,29\}$
2	$\{5,6,8,15,29\}$
4	$\{15\}$
5	$\{15,29\}$
6	$\{29\}$
7	$\{16\}$
8	$\{15,29\}$
10	$\{29\}$
18	$\{22\}$
21	$\{27\}$
24	$\{33\}$
28	$\{33\}$

6.4 Filter 4: singular one-cycle capacity and Hessian rank

After the singular-scheme Hilbert-function test, only twenty-two ordered pairs remain. At this stage the coarse Hilbert function is not enough; we must look at how the one-dimensional part of the singular scheme could specialize. The curve part of the singular scheme defines an effective singular one-cycle. A candidate inclusion forces the singular one-cycle of the general family to limit to an effective subcycle of the singular one-cycle of the special family. We combine this cycle-capacity constraint with the semicontinuity of the Hessian rank: along a specialization, the generic transverse Hessian rank on a matched component cannot increase.

We now refine the singular-locus Hilbert-function filter by using the curve part of the singular scheme of the closed-orbit representative.

Let

$$S = \mathbb{C}[x_0,\ldots,x_4], \qquad \mathfrak{m} = (x_0,\ldots,x_4).$$

For the closed-orbit normal form ϕ_k^{nf}, set

$$X_k^{\mathrm{nf}} := V(\phi_k^{\mathrm{nf}}) \subset \mathbb{P}^4$$

and define the saturated Jacobian ideal

$$J_k := \left(\frac{\partial \phi_k^{\mathrm{nf}}}{\partial x_0}, \dots, \frac{\partial \phi_k^{\mathrm{nf}}}{\partial x_4} \right) : \mathfrak{m}^{\infty}.$$

Let

$$\mathrm{Sing}(X_k^{\mathrm{nf}}) = \mathrm{Proj}(S/J_k).$$

We take a primary decomposition

$$J_k = \bigcap_i Q_{k,i}, \qquad P_{k,i} := \sqrt{Q_{k,i}}.$$

For every component with

$$\dim \mathrm{Proj}(S/P_{k,i}) = 1,$$

we record the following data:

$$d_{k,i} := \deg(S/P_{k,i}),$$
$$m_{k,i} := \frac{\deg(S/Q_{k,i})}{\deg(S/P_{k,i})},$$

and

$$\rho_{k,i} := \mathrm{rank}\big(\mathrm{Hess}(\phi_k^{\mathrm{nf}})\big) \quad \text{at a general point of } V(P_{k,i}).$$

Thus the curve part of the singular scheme gives an effective singular one-cycle

$$\Gamma_k = \sum_i m_{k,i} C_{k,i}, \qquad C_{k,i} := \mathrm{Proj}(S/P_{k,i}).$$

For convenience, we use the notation

$$C_{d,m}^{(\rho)}$$

for a curve component of degree d, singular-cycle multiplicity m, and generic Hessian rank ρ.

The cases appearing in the pairs surviving Filters 1–3 have the following curve packages:

k	Γ_k
1	$C^{(2)}_{4,3}$
2	$C^{(2)}_{1,3} + C^{(2)}_{3,2}$
4	$C^{(2)}_{1,2} + C^{(2)}_{1,10}$
5	$C^{(3)}_{1,1} + C^{(3)}_{1,1} + C^{(2)}_{1,8}$
6	$C^{(2)}_{1,6} + C^{(3)}_{2,1}$
7	$C^{(2)}_{1,4}$
8	$C^{(1)}_{1,4} + C^{(2)}_{1,9}$
10	$C^{(2)}_{1,2} + C^{(2)}_{1,2} + C^{(2)}_{1,2} + C^{(2)}_{1,2}$
15	$C^{(2)}_{1,3} + C^{(1)}_{1,9}$
16	$C^{(2)}_{1,9} + C^{(1)}_{1,4}$
18	$C^{(2)}_{1,4}$
21	$C^{(2)}_{1,10} + C^{(2)}_{1,2}$
22	$C^{(2)}_{1,2}$
24	$C^{(2)}_{1,8} + C^{(3)}_{1,1} + C^{(3)}_{1,1}$
27	$C^{(2)}_{1,3} + C^{(2)}_{3,2}$
28	$C^{(2)}_{1,6} + C^{(3)}_{2,1}$
29	$C^{(1)}_{1,8}$
33	$C^{(2)}_{1,4}$.

Suppose that

$$\Phi_k \subset \overline{\Phi_\ell}.$$

Then a general closed-orbit representative of Φ_k is a specialization of a general closed-orbit representative of Φ_ℓ. Hence the curve part of the singular scheme of X^{nf}_ℓ must be able to specialize to an effective subcycle of the curve part of $\mathrm{Sing}(X^{\mathrm{nf}}_k)$.

We use the following two necessary conditions.

First, a line component in the singular one-cycle of X^{nf}_ℓ must specialize to a line component in the singular one-cycle of X^{nf}_k. Moreover, the total multiplicity assigned to a line component of X^{nf}_k cannot exceed its multiplicity in Γ_k. We call this the singular one-cycle capacity condition.

Second, along a matched component, Hessian rank cannot increase under specialization. Therefore, if a component of X^{nf}_ℓ with generic Hessian rank ρ_ℓ specializes to a component of X^{nf}_k with generic Hessian rank ρ_k, then necessarily

$$\rho_k \leq \rho_\ell.$$

If this inequality fails for every possible capacity-compatible matching, then we conclude

$$\Phi_k \not\subset \Phi_\ell.$$

Let $\mathcal{P}_{123}$ be the set of ordered pairs surviving Filters 1–3. From Filter 3 we have

$$|\mathcal{P}_{123}| = 22.$$

Define $\mathcal{R}^{\mathrm{cap}}_4$ to be the set of pairs in $\mathcal{P}_{123}$ excluded by the singular one-cycle capacity condition, and define $\mathcal{R}^{\mathrm{rank}}_4$ to be the set of remaining pairs excluded by the Hessian-rank condition. Then

$$\mathcal{R}_4 := \mathcal{R}^{\mathrm{cap}}_4 \cup \mathcal{R}^{\mathrm{rank}}_4.$$

The capacity condition excludes the following 15 pairs:

$$\begin{aligned}\mathcal{R}_4^{\mathrm{cap}} = \{&(1,4),(1,5),(1,8),(1,15),(1,29),\\ &(2,5),(2,6),(2,8),(2,15),(2,29),\\ &(4,15),\\ &(5,15),\\ &(6,29),\\ &(7,16),\\ &(10,29)\}.\end{aligned}$$

The Hessian-rank condition further excludes the following 3 pairs:

$$\mathcal{R}_4^{\mathrm{rank}} = \{(5,29),(8,15),(8,29)\}.$$

For example, in the pair $(5,29)$, the line component of $\ell = 29$ has type

$$C_{1,8}^{(1)}.$$

The only line component of $k = 5$ with sufficient multiplicity has type

$$C_{1,8}^{(2)}.$$

Thus the matching would force Hessian rank to increase from 1 to 2 under specialization, which is impossible. The same argument excludes $(8,15)$ and $(8,29)$.

Consequently,

$$|\mathcal{R}_4^{\mathrm{cap}}| = 15, \qquad |\mathcal{R}_4^{\mathrm{rank}}| = 3,$$

and hence

$$|\mathcal{R}_4| = 18.$$

Since

$$|\mathcal{P}_{123}| = 22,$$

the number of ordered pairs surviving Filters 1–4 is

$$|\mathcal{P}_{123}| - |\mathcal{R}_4| = 22 - 18 = 4.$$

Thus the cumulative count is

$$38 \cdot 37 = 1406$$

and

$$1406 \xrightarrow{\text{Filter 1}} 759 \xrightarrow{\text{Filter 2}} 450 \xrightarrow{\text{Filter 3}} 22 \xrightarrow{\text{Filter 4}} 4.$$

Equivalently, Filters 1–4 exclude

$$1406 - 4 = 1402$$

ordered pairs.

For reference, the complete list of ordered pairs excluded by Filter 4 is

$$\begin{aligned}\mathcal{R}_4 = \{&(1,4),(1,5),(1,8),(1,15),(1,29),\\ &(2,5),(2,6),(2,8),(2,15),(2,29),\\ &(4,15),\\ &(5,15),(5,29),\\ &(6,29),\\ &(7,16),\\ &(8,15),(8,29),\\ &(10,29)\}.\end{aligned}$$

The ordered pairs surviving Filters 1–4 are therefore

$$\mathcal{P}_{1234} = \{(18, 22), (21, 27), (24, 33), (28, 33)\}.$$

For convenience, we also record the same list source-wise.

Table 9: Ordered pairs surviving Filters 1–4.

k	$\{\ell \mid (k, \ell)$ survives Filters 1–4$\}$
18	$\{22\}$
21	$\{27\}$
24	$\{33\}$
28	$\{33\}$

6.5 Filter 5: generic stabilizer tori

The first four filters reduce the problem to four ordered pairs. These pairs are not separated by the preceding numerical and singular-locus data, so we use a GIT-theoretic invariant: the generic connected stabilizer. Stabilizers can only increase under specialization. Therefore, if both the special and the general families have a one-dimensional generic stabilizer torus, a candidate inclusion forces these tori to be conjugate in SL(5). For diagonal tori this is equivalent to equality of the corresponding weight multisets, up to simultaneous sign.

Let

$$G = \mathrm{SL}(5), \qquad W = \mathrm{Sym}^5\mathbb{C}^5.$$

For a boundary family Φ_k, let

$$r_k = (r_{k,0}, \dots, r_{k,4})$$

be the corresponding one-parameter subgroup weight vector, and put

$$H_k := \lambda_{r_k}(\mathbb{G}_m) \subset G, \qquad \lambda_{r_k}(t) = \mathrm{diag}(t^{r_{k,0}}, \dots, t^{r_{k,4}}).$$

After Filters 1–4, the only remaining ordered pairs are

$$\mathcal{P}_{1234} = \{(18, 22), (21, 27), (24, 33), (28, 33)\}.$$

Thus only the following indices are relevant:

$$K = \{18, 21, 22, 24, 27, 28, 33\}.$$

We use the following general stabilizer obstruction. Suppose

$$\Phi_k \subset \overline{\Phi_\ell}.$$

Then a general point of Φ_k is a specialization of a general point of Φ_ℓ. Hence the connected stabilizer can only increase under this specialization. Therefore, if the general connected stabilizers of both Φ_k and Φ_ℓ are one-dimensional tori, then they must be conjugate in G. In particular, if

$$\mathrm{Stab}^\circ_G(\Phi_k^{\mathrm{gen}}) = H_k, \qquad \mathrm{Stab}^\circ_G(\Phi_\ell^{\mathrm{gen}}) = H_\ell,$$

then a necessary condition for

$$\Phi_k \subset \overline{\Phi_\ell}$$

is

$$H_k \sim_G H_\ell.$$

For diagonal one-dimensional subtori of SL(5), conjugacy is equivalent to equality of the weight multisets, after primitive normalization and up to simultaneous sign. Since

$$\lambda_r(\mathbb{G}_m) = \lambda_{-r}(\mathbb{G}_m),$$

we allow simultaneous sign change in comparing weight multisets.

We first record the generic stabilizer check for the seven relevant families. Let

$$W^{H_k} := \{F \in W \mid h \cdot F = F \text{ for all } h \in H_k\}.$$

Equivalently, W^{H_k} is spanned by the degree-five monomials of r_k-weight zero. The table below records the dimension check on the normal-form slice, together with the infinitesimal stabilizer computation.

k	r_k	$\#I(r_k)_{=0}$	$\dim \mathbb{P}(W^{H_k})$	$\dim \Phi_k$	$\dim C_G(H_k)/H_k$	$\dim \mathrm{Stab}^\circ_G(\Phi_k^{\mathrm{gen}})$
18	$(4, 2, 0, -1, -5)$	6	5	2	3	1
22	$(16, 6, 1, -4, -19)$	8	7	4	3	1
21	$(16, 11, 6, -9, -24)$	6	5	2	3	1
27	$(18, 8, 3, -2, -27)$	6	5	2	3	1
24	$(14, 9, 4, -6, -21)$	7	6	3	3	1
28	$(12, 7, 2, -3, -18)$	9	8	5	3	1
33	$(2, 1, 0, 0, -3)$	12	11	6	5	1

For each $k \in K$, the normal-form slice gives

$$\dim \mathbb{P}(W^{H_k}) - \dim \Phi_k = \dim C_G(H_k)/H_k.$$

Thus the residual stabilizer for the action of $C_G(H_k)/H_k$ on the normal-form slice is finite at a general point. The infinitesimal stabilizer calculation gives a one-dimensional connected stabilizer in G, and this one-dimensional stabilizer contains H_k. Hence

$$\mathrm{Stab}^\circ_G(\Phi_k^{\mathrm{gen}}) = H_k \qquad (k \in K).$$

We now compare the diagonal torus weights. Write

$$M_k$$

for the sorted weight multiset of H_k. The relevant multisets are

k	M_k
18	$\{-5, -1, 0, 2, 4\}$
22	$\{-19, -4, 1, 6, 16\}$
21	$\{-24, -9, 6, 11, 16\}$
27	$\{-27, -2, 3, 8, 18\}$
24	$\{-21, -6, 4, 9, 14\}$
28	$\{-18, -3, 2, 7, 12\}$
33	$\{-3, 0, 0, 1, 2\}.$

The four remaining ordered pairs are checked as follows:

(k, ℓ)	M_k	M_ℓ	comparison
$(18, 22)$	$\{-5, -1, 0, 2, 4\}$	$\{-19, -4, 1, 6, 16\}$	$M_{18} \neq \pm M_{22}$
$(21, 27)$	$\{-24, -9, 6, 11, 16\}$	$\{-27, -2, 3, 8, 18\}$	$M_{21} \neq \pm M_{27}$
$(24, 33)$	$\{-21, -6, 4, 9, 14\}$	$\{-3, 0, 0, 1, 2\}$	$M_{24} \neq \pm M_{33}$
$(28, 33)$	$\{-18, -3, 2, 7, 12\}$	$\{-3, 0, 0, 1, 2\}$	$M_{28} \neq \pm M_{33}.$

Therefore, in each of the four cases, the generic connected stabilizer tori are not conjugate in $G = \mathrm{SL}(5)$. Hence none of the four inclusions can hold. Thus Filter 5 excludes

$$\mathcal{R}_5 = \{(18, 22), (21, 27), (24, 33), (28, 33)\}.$$

Since

$$\mathcal{P}_{1234} = \{(18, 22), (21, 27), (24, 33), (28, 33)\},$$

we have

$$\mathcal{R}_5 = \mathcal{P}_{1234}.$$

Consequently,

$$\mathcal{P}_{12345} := \mathcal{P}_{1234} \setminus \mathcal{R}_5 = \varnothing.$$

The cumulative count is therefore

$$38 \cdot 37 = 1406$$

and

$$1406 \xrightarrow{\text{Filter 1}} 759 \xrightarrow{\text{Filter 2}} 450 \xrightarrow{\text{Filter 3}} 22 \xrightarrow{\text{Filter 4}} 4 \xrightarrow{\text{Filter 5}} 0.$$

Thus Filters 1–5 exclude all ordered pairs

$$(k, \ell), \qquad 1 \le k, \ell \le 38, \quad k \ne \ell.$$

Equivalently, for every distinct pair $k \ne \ell$,

$$\Phi_k \not\subset \Phi_\ell.$$

For reference, the list of ordered pairs surviving Filters 1–5 is $\varnothing$.

Conclusion

Outline of the verification of Theorem 3.3. The five filters above eliminate every ordered pair (k, ℓ), $1 \le k, \ell \le 38$, $k \ne \ell$. Equivalently, there is no candidate closure containment

$$\Phi_k \subset \overline{\Phi_\ell} \qquad (k \ne \ell).$$

In particular, there is no ordinary containment $\Phi_k \subset \Phi_\ell$ for $k \ne \ell$. Hence the thirty-eight boundary families $\Phi_1, \dots, \Phi_{38}$ are pairwise non-contained. □

7 Adjacency

In this section we study the codimension-one adjacency relation among the 38 boundary components of the GIT compactification of quintic threefolds. The computation is carried out by the same slice-matching method as in the cubic fivefold case, using the maximal states $r_1, \dots, r_{38}$ obtained in the classification of the strictly semistable locus.

For each k, let f_k be a general quintic with support $I(r_k)_{\ge 0}$, and let Φ_k denote the corresponding closed boundary component. For a codimension-one slice $W \subset I(r_i)_{=0}$, we say that there is a directed slice-matching from Φ_i to Φ_j if, after replacing r_j by $\pm r_j$ when necessary, the slice W is realized by

$$W = I(\pm r_j)_{\ge 0} \cap I(r_i)_{=0}.$$

Definition 7.1. For $1 \le i, j \le 38$, we write $\Phi_i \rightsquigarrow \Phi_j$ if there is a directed slice-matching from Φ_i to Φ_j in codimension one. We say that Φ_i and Φ_j are *codimension-one adjacent*, and write $\Phi_i \sim \Phi_j$, if both $\Phi_i \rightsquigarrow \Phi_j$ and $\Phi_j \rightsquigarrow \Phi_i$ occur.

Thus the codimension-one adjacency graph is the mutual graph associated with the directed slice-matching relation. One-sided arrows are recorded separately, but they are not counted as codimension-one wall adjacencies.

Theorem 7.2. *The codimension-one adjacency graph of the* 38 *boundary components has* 38 *vertices and* 184 *edges. It is connected, has no isolated vertices, and its diameter is* 4.

Proof. The statement follows from the output of the codimension-one slice-matching computation. The mutual graph is obtained by retaining exactly those pairs (i, j) for which both directed arrows $\Phi_i \rightsquigarrow \Phi_j$ and $\Phi_j \rightsquigarrow \Phi_i$ occur. Counting the resulting undirected edges yields 184 edges on 38 vertices. A direct inspection of the neighbour sets shows that every vertex has positive degree, hence there are no isolated vertices, and a standard graph-theoretic check gives connectedness and diameter 4. $\square$

The complete list of neighbour sets is recorded in Table 10.

Table 10: Neighbour sets for the codimension-one adjacency graph of the boundary components.

k	$N(k)$
1	$\{2, 14, 18, 22, 26, 27\}$
2	$\{1, 3, 9, 10, 14, 22, 24, 26, 27\}$
3	$\{2, 4, 5, 6, 10, 32\}$
4	$\{3, 5, 6, 8, 18, 21, 24, 32\}$
5	$\{3, 4, 6, 10, 15, 18, 21, 22, 24, 27, 32\}$
6	$\{3, 4, 5, 10, 11, 19, 23, 29, 30, 32\}$
7	$\{8, 9, 18, 21, 24, 26\}$
8	$\{4, 7\}$
9	$\{2, 7, 11, 14, 15, 22, 24, 26, 27, 36\}$
10	$\{2, 3, 5, 6, 15, 20, 23, 29, 35\}$
11	$\{6, 9, 15, 19, 20, 23, 29, 30, 36\}$
12	$\{13, 15, 19, 31, 32, 35\}$
13	$\{12, 14, 15, 17, 30\}$
14	$\{1, 2, 9, 13, 22, 26, 27\}$
15	$\{5, 9, 10, 11, 12, 13, 19, 20, 25, 28, 29, 30, 32, 35, 36, 38\}$
16	$\{18, 21\}$
17	$\{13, 19, 20, 29, 30, 37\}$
18	$\{1, 4, 5, 7, 16, 22\}$
19	$\{6, 11, 12, 15, 17, 20, 23, 25, 28, 29, 30, 31, 32, 35, 37, 38\}$
20	$\{10, 11, 15, 17, 19, 23, 29, 31, 32, 35, 36, 37\}$
21	$\{4, 5, 7, 16, 24, 28, 33, 37\}$
22	$\{1, 2, 5, 9, 14, 18, 25, 27, 30, 36\}$
23	$\{6, 10, 11, 19, 20, 29, 30, 32, 35, 36, 37, 38\}$
24	$\{2, 4, 5, 7, 9, 21, 28, 30, 33, 34, 37\}$
25	$\{15, 19, 22, 28, 30, 31, 35, 36, 38\}$
26	$\{1, 2, 7, 9, 14, 27\}$
27	$\{1, 2, 5, 9, 14, 22, 26, 33, 34\}$
28	$\{15, 19, 21, 24, 25, 33, 34, 35, 37, 38\}$
29	$\{6, 10, 11, 15, 17, 19, 20, 23, 30, 31, 32, 34, 35, 36, 37, 38\}$
30	$\{6, 11, 13, 15, 17, 19, 22, 23, 24, 25, 29, 31, 34, 35, 36, 37\}$
31	$\{12, 19, 20, 25, 29, 30, 32, 34, 35, 36, 37, 38\}$
32	$\{3, 4, 5, 6, 12, 15, 19, 20, 23, 29, 31, 35, 36, 37, 38\}$
33	$\{21, 24, 27, 28, 34, 37\}$
34	$\{24, 27, 28, 29, 30, 31, 33, 35, 38\}$
35	$\{10, 12, 15, 19, 20, 23, 25, 28, 29, 30, 31, 32, 34, 36, 37, 38\}$
36	$\{9, 11, 15, 20, 22, 23, 25, 29, 30, 31, 32, 35, 37, 38\}$
37	$\{17, 19, 20, 21, 23, 24, 28, 29, 30, 31, 32, 33, 35, 36, 38\}$
38	$\{15, 19, 23, 25, 28, 29, 31, 32, 34, 35, 36, 37\}$

For convenience, we record the vertices of highest and lowest degree:

$$\deg(\Phi_{15}) = \deg(\Phi_{19}) = \deg(\Phi_{29}) = \deg(\Phi_{30}) = \deg(\Phi_{35}) = 16,$$

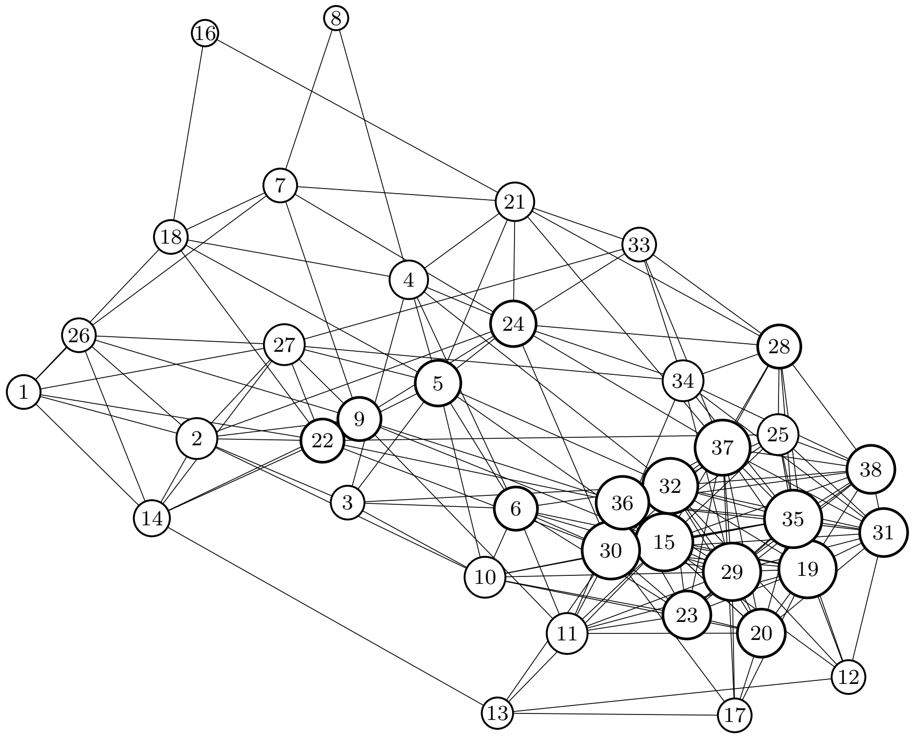

Figure 1: Codimension-one wall-adjacency graph of the boundary components $\Phi_1, \ldots, \Phi_{38}$ for quintic threefolds. Two vertices are joined exactly when the corresponding components are mutually adjacent. The size of each vertex is proportional to its degree in the wall-adjacency graph.

whereas

$$\deg(\Phi_8) = \deg(\Phi_{16}) = 2.$$

Hence the codimension-one wall-adjacency graph has a dense core centered around

$$\Phi_{15},\ \Phi_{19},\ \Phi_{29},\ \Phi_{30},\ \Phi_{35},$$

while Φ_8 and Φ_{16} lie near the periphery.

Remark 7.3. The same computation also produces 244 one-sided arrows. These are useful as auxiliary incidence data, but in the present paper we do not count them as codimension-one adjacencies.

8 Outline of the computational verification

This section is the computational core of the paper. The preceding Sections 1–7 contain the complete statements needed for most applications: the enumeration of the maximal strictly semistable supports, the construction of the thirty-eight boundary families, the closed-orbit normal forms, the singular-locus and minimal-exponent tables, the pairwise non-inclusion theorem, and the codimension-one adjacency graph. Thus a reader whose main purpose is to use the classification and its consequences may read Sections 1–7 independently of the present section.

The purpose of this section is to record the computational framework used in the verification of the announced results. The complete case-by-case implementation of this framework will appear in the forthcoming full-length paper. For each $k = 1, \ldots, 38$, starting from the witness vector r_k, we compute the associated one-parameter-subgroup limit

$$\phi_k := \lim_{t \to 0} \lambda_{r_k}(t) \cdot f_k.$$

We then describe the stabilizing torus

$$H_k := \lambda_{r_k}(\mathbb{G}_m),$$

the centralizer $C_G(H_k)$, and the fixed subspace W^{H_k}. The closedness of the resulting orbit is checked by Luna's centralizer reduction: in the toric centralizer cases we use the convex-hull criterion, while in the non-toric centralizer cases we use the Casimiro–Florentino criterion. This produces the closed-orbit normal forms ϕ_k^{nf} recorded in Table 3, together with the corresponding parameter counts.

We next compute the singular locus of

$$X_k := V(\phi_k^{\mathrm{nf}}) \subset \mathbb{P}^4$$

by solving the saturated Jacobian equations. These computations identify the positive-dimensional components of $\mathrm{Sing}(X_k)$, the isolated singular points, and the genericity conditions imposed on the parameters of the normal forms. At each relevant point of the singular locus, we then choose local coordinates and reduce the local equation to an explicit quasi-homogeneous or split normal form. These local calculations give the minimal exponents used in Section 5 and justify the global equality

$$\widetilde{\alpha}(X_k) = 1 = \frac{4+1}{5}$$

for the general closed-orbit representative in every boundary component.

All computations below are carried out on the same nonempty Zariski-open loci of parameters used in the preceding tables. Conditions such as the non-vanishing of a discriminant or resultant simply exclude proper closed subsets where additional singular points appear or where the analytic type changes. In this way, the case-by-case calculations below provide the detailed computational foundation for the results stated in Sections 1–7, while those earlier sections remain a concise reference for the final classification.

Closed-Orbit Criterion and the Toric/Non-Toric Dichotomy

The closed-orbit verification is always made after passing to the one-parameter-subgroup limit. Let

$$\lambda : \mathbb{G}_m \longrightarrow G = \mathrm{SL}(5)$$

be the chosen one-parameter subgroup, let

$$H := \lambda(\mathbb{G}_m), \qquad W^H := \{F \in W \mid h \cdot F = F \text{ for all } h \in H\},$$

and let

$$\varphi = \lim_{t \to 0} \lambda(t) \cdot f \in W^H$$

be the weight-zero limit. Luna's centralizer reduction replaces the closed-orbit question for the original G-action by the residual closed-orbit question for the centralizer

$$C_G(H)$$

acting on the fixed vector space W^H. In the computations below, H acts trivially on W^H, so the effective residual group is $C_G(H)/H$.

There are two different closed-orbit tests, depending on the structure of $C_G(H)$.

Toric centralizer. If the H-weights on $\langle x_0, \dots, x_4 \rangle$ are pairwise distinct, then

$$C_G(H) = T$$

is the diagonal torus. In this case the effective group is the torus T/H, and the closed-orbit test is the convex-hull criterion. For a vector

$$\varphi = \sum a_u x^u$$

with nonzero coefficients on the displayed monomials, the relevant characters are the images of

$$u - \eta, \qquad \eta = (1,1,1,1,1),$$

in the real character space of T/H. The torus orbit is closed if and only if the origin lies in the relative interior of the convex hull of these characters. In practice we certify this by writing a positive affine relation among the exponent vectors and checking that the same vectors have the expected affine rank. This is the method used in the toric-centralizer cases below.

Non-toric centralizer. If some H-weight has multiplicity greater than one, then $C_G(H)$ contains a non-abelian block, for example a GL(2)-block. In this situation the convex-hull criterion for a torus is not the closed-orbit criterion for the full residual group $C_G(H)$. It only checks diagonal one-parameter subgroups after a fixed choice of basis and therefore does not by itself control one-parameter subgroups obtained by conjugating inside the non-abelian blocks.

For non-toric centralizers we use the Casimiro–Florentino criterion. For a reductive group R acting on an affine variety and a point x, let $Y(R)$ be the set of one-parameter subgroups of R, and set

$$\Lambda_x := \{\lambda \in Y(R) \mid \lim_{t \to 0} \lambda(t) \cdot x \text{ exists}\}.$$

For $\lambda \in Y(R)$, put

$$P(\lambda) := \{g \in R \mid \lim_{t \to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists}\}.$$

The criterion says that x is polystable if and only if Λ_x is symmetric: for every $\lambda \in \Lambda_x$, there is $\lambda' \in \Lambda_x$ such that

$$P(\lambda) \cap P(\lambda')$$

is a Levi subgroup of both parabolic subgroups.

The computations use the following concrete form of this criterion. After conjugating inside the non-abelian blocks of $C_G(H)$, every one-parameter subgroup can be written diagonally. For such a subgroup, write $w(m)$ for the weight of an occurring monomial m, and write

$$S = 0$$

for the determinant-one condition on $C_G(H)$. We exhibit a positive linear identity

$$\sum_j c_j w(m_j) = CS, \qquad c_j > 0, \qquad C > 0,$$

using monomials m_j that occur with nonzero coefficient in φ. If $\lambda \in \Lambda_\varphi$, then all the weights $w(m_j)$ are nonnegative. Since $S = 0$, the positive identity forces all of these weights to be zero. Solving the resulting linear equations gives

$$\Lambda_\varphi = \{\mu(t^m) \mid m \in \mathbb{Z}\}$$

for a single one-parameter subgroup μ. This set is symmetric, because $\mu(t^m)$ and $\mu(t^{-m})$ define opposite parabolic subgroups and their intersection is the associated Levi subgroup. Hence φ is polystable for the residual $C_G(H)$-action. This is the method used in the non-toric-centralizer cases in the full case-by-case verification.

This paper is an announcement of our results. The detailed proofs and the complete case-by-case computations will appear in a forthcoming full-length paper.

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Yasutaka Shibata
Email: yasutaka@shibata-math.com