

The GIT Boundary of Quintic Threefolds

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Abstract

We determine the quotient-side strictly semistable boundary for the natural $G = \mathrm{SL}(5)$ -action on $\mathbb{P}(\mathrm{Sym}^5 \mathbb{C}^5)$, proving that it has exactly twenty-one irreducible components, of dimensions between 1 and 24. Using the Hilbert–Mumford criterion and exact integer linear algebra, we enumerate, up to coordinate permutation, thirty-eight maximal strictly T -semistable monomial supports for the diagonal maximal torus T and prove that their G -saturation cover the strictly semistable locus. Zero-weight reduction shows that reversing a supporting half-space does not change its quotient image; the thirty-eight supports therefore form seventeen pairs and four fixed cases.

For each support we construct a polystable weight-zero normal form and compute the corresponding quotient dimension. Exact rank certificates identify the generic connected projective stabilizers of the resulting twenty-one families as pairwise nonconjugate one-dimensional tori. Luna’s étale slice theorem then excludes containments between distinct families. Lakhani’s four first-level families give the four largest components; the other seventeen are additional components.

For a general closed-orbit representative in every component, we determine the saturated Jacobian scheme and local analytic stratification. The positive-dimensional singular loci are explicit configurations of lines, smooth conics, cuspidal plane curves, planes, and smooth quartic surfaces. The isolated singularities form exactly eleven weighted-homogeneous local families whose normalized weights occur in Yonemura’s list. Every general representative has global minimal exponent $\tilde{\alpha}(X) = 1 = (4 + 1)/5$, the critical value $(n + 1)/d$ for quintic hypersurfaces in $\mathbb{P}^4$.

1 Introduction

The problem and the complete boundary catalog

Let

$$V := H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1)) \simeq \mathbb{C}^5, \quad W := \mathrm{Sym}^5 V, \quad G := \mathrm{SL}(V) \simeq \mathrm{SL}(5).$$

A nonzero form $f \in W$ defines a quintic threefold

$$X_f := V(f) \subset \mathbb{P}(V^\vee) \simeq \mathbb{P}^4,$$

and a smooth such hypersurface is a Calabi–Yau threefold. The natural projective GIT quotient is

$$\mathcal{M}^{\mathrm{GIT}} := \mathbb{P}(W)^{ss} // G.$$

We denote the good quotient map by

$$\pi : \mathbb{P}(W)^{ss} \longrightarrow \mathcal{M}^{\mathrm{GIT}}$$

and write

$$\Sigma^{ss} := \mathbb{P}(W)^{ss} \setminus \mathbb{P}(W)^s.$$

The object classified in this paper is the quotient of the strictly semistable locus,

$$\partial \mathcal{M}^{\mathrm{GIT}} := \pi(\Sigma^{ss}).$$

Thus the term *GIT boundary* always means $\pi(\mathbb{P}(W)^{ss} \setminus \mathbb{P}(W)^s)$; it does not mean the entire complement of the smooth-quintic locus in $\mathcal{M}^{\text{GIT}}$. For later reference, if $r = (r_0, \dots, r_4) \in \mathbb{Z}^5$ satisfies $\sum_{i=0}^4 r_i = 0$, we write $\lambda_r(t) := \text{diag}(t^{r_0}, \dots, t^{r_4})$, $t \in \mathbb{G}_m$, for the corresponding one-parameter subgroup of the diagonal maximal torus, and put $H_r := \lambda_r(\mathbb{G}_m)$.

The purpose of the present introduction is to record the complete output of the paper before explaining the proofs. The twenty-one explicit normal-form families are given in Table 1; the component classification is listed in Table 2; the singularities of a general closed orbit in every component are listed in Table 3; and the defining equations, weights, and occurrences of the isolated weighted-homogeneous local families are displayed in Table 4. These four catalogs use the same component index $a = 1, \dots, 21$, so that the equation, quotient-theoretic invariants, and singularity data of each component can be read across the introduction. Throughout the catalogs, *general* means after restriction to a fixed nonempty Zariski-open subset of the parameter space in the corresponding row; all orbit-theoretic and singularity-theoretic assertions are made on that open subset.

Theorem A (Complete classification of the strictly semistable boundary). *For the natural $\text{SL}(5)$ -action on $\mathbb{P}(\text{Sym}^5 \mathbb{C}^5)$, the following statements hold.*

1. *Up to permutation of the coordinates, there are exactly thirty-eight maximal strictly semistable monomial supports for the diagonal maximal torus. Their G -saturation cover Σ^{ss} .*
2. *The thirty-eight support-indexed quotient families are identified by seventeen two-cycles and four fixed points under reversal of the supporting half-space. The resulting twenty-one closed irreducible families are precisely the irreducible components*

$$\partial \mathcal{M}^{\text{GIT}} = \Theta_1 \cup \dots \cup \Theta_{21}.$$

There is no equality or containment between two distinct members of this list.

3. *For every $a = 1, \dots, 21$, a general point of Θ_a is represented by a member of the explicit polystable weight-zero normal-form family F_a displayed in Table 1. The parameters are restricted to the indicated nonempty Zariski-open generic loci; no uniqueness of the displayed parameterizations is asserted. Its quotient dimension and generic connected projective stabilizer are*

$$\dim \Theta_a = d_a, \quad \text{Stab}_G([F_a])^\circ = H_a := H_{r_a} = \lambda_{r_a}(\mathbb{G}_m),$$

where the complete list of Θ_a , r_a , and d_a is given in Table 2. The twenty-one tori H_a are pairwise nonconjugate in G .

4. *For $X_a := V(F_a) \subset \mathbb{P}^4$, the support and local stratification of the saturated Jacobian scheme, including its generic transverse structure and all isolated singular points, are summarized in Tables 3 and 4. Exactly eleven isolated weighted-homogeneous local normal-form families occur. Their defining equations, integral weight systems, Milnor numbers, and component occurrences are displayed in Table 4; the corresponding ordered normalized weights all occur in Yonemura's Table 2.2 [Yon90]. No other isolated local family occurs for a general boundary representative.*
5. *Every general closed-orbit representative has global minimal exponent*

$$\tilde{\alpha}(X_a) = 1 = \frac{4+1}{5}, \quad a = 1, \dots, 21.$$

Tables 1–4 form part of the statement of the theorem.

Logical architecture. The quotient-side component count is established in three distinct stages. Proposition 3.1 first gives a cover by thirty-eight support-indexed quotient images. Zero-weight reduction and reversal of the supporting half-space, formalized in Corollary 4.2 and Proposition 4.4, reduce this to the twenty-one candidate images of Corollary 4.5. Corollary 5.2 proves that those candidates are nonempty, closed, and irreducible. Theorem 6.1 then uses the generic stabilizers and Luna’s slice theorem to prove pairwise non-inclusion, so the candidates are exactly the irreducible components. Appendices A and B provide, respectively, the support-by-support normal-form constructions and the component-by-component singularity and minimal-exponent calculations.

The twenty-one normal-form families

For the component numbering in Theorem A, we choose the support-case representatives

$$k(a) = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 19, 20, 23, 29, 32, 36$$

in that order and set

$$F_a := \phi_{k(a)}^{\text{nf}}, \quad X_a := V(F_a) \subset \mathbb{P}^4.$$

Thus $k(a) = a$ for $1 \leq a \leq 15$, whereas the component index and the chosen oriented-support label differ for $a \geq 16$. The column $k(a)$ should therefore be retained when passing between the twenty-one-row catalog and the thirty-eight support-by-support calculations. The following table reproduces the complete equations rather than merely referring to the case-by-case construction later in the paper. It is meant to be directly usable: a reader can recover a general closed-orbit equation for any boundary component without consulting the thirty-eight-case table. The displayed families are normal forms in the sense of explicit polystable representatives after a change of projective coordinates; the parameterizations are not claimed to be unique.

We use

$$B_\lambda(u, v) := uv(u - v)(u - \lambda v), \quad B_{\lambda, \mu}(u, v) := v(u - v)(u - \lambda v)(u - \mu v).$$

Symbols such as $A_d, B_d, C_d, D_d, Q_d, L_d, M_d$ denote general forms of the indicated degree in the displayed variables, and $Q_{2,2}$ denotes a bihomogeneous form of bidegree $(2, 2)$. A bold coefficient vector parametrizes the binary form whose degree is forced by homogeneity. All parameters are understood to lie in the relevant nonempty Zariski-open locus, in addition to the explicit conditions in the last column.

Table 1: The twenty-one polystable weight-zero normal-form families. A general point of Θ_a is represented by a member of the family F_a in row a . The label $k(a)$ refers to the corresponding case among the thirty-eight oriented support cases. All displayed parameters are restricted to the relevant nonempty Zariski-open generic locus. This table forms part of Theorem A.

Θ_a	k	Normal-form family F_a	Parameters and genericity
Θ_1	1	$F_1 = x_1^4 x_2 + x_0 x_3^4 + x_0 x_2^3 x_4$ $+ x_0 x_1 x_2 x_3 x_4 + \alpha x_0^2 x_4^3$	$\alpha \in \mathbb{C}^\times$
Θ_2	2	$F_2 = x_1^3 x_2^2 + x_1^4 x_3 + x_0 x_2 x_3^3 + x_0 x_2^3 x_4$ $+ \alpha x_0 x_1 x_2 x_3 x_4 + \beta x_0^2 x_4^3$	$\alpha, \beta \in \mathbb{C}^\times$
Θ_3	3	$F_3 = B_5(x_1, x_2) + x_0 x_2 x_3^3$ $+ x_0(x_1^2 + x_2^2)x_3 x_4 + x_0^2 x_4^3$	B_5 : general binary quintic
Θ_4	4	$F_4 = x_1^2 x_2^3 + x_1^3 x_3 x_4 + x_0 x_2^4 + x_0 x_1 x_3^3$ $+ \alpha x_0 x_1 x_2 x_3 x_4 + \beta x_0^2 x_4^3$	$\alpha, \beta \in \mathbb{C}^\times$

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Θ_a	k	Normal-form family F_a	Parameters and genericity
Θ_5	5	$F_5 = x_1^2 x_2^3 + x_1^3 x_2^2 + x_1^3 x_2 x_4 + x_0 x_2^3 x_3$ $+ \alpha x_0 x_1 x_3^3 + \beta x_0 x_1 x_2 x_3 x_4 + \gamma x_0^2 x_4^3$	$\alpha, \beta, \gamma \in \mathbb{C}^\times$
Θ_6	6	$F_6 = x_1^2 x_2^3 + x_1^3 x_2 x_3 + x_1^4 x_4 + x_0 x_2^2 x_3^2$ $+ \alpha x_0 x_2^3 x_4 + \beta x_0 x_1 x_3^3 + \gamma x_0 x_1 x_2 x_3 x_4$ $+ \delta x_0 x_1^2 x_4^2 + \epsilon x_0^2 x_4^3$	$\alpha, \beta, \gamma, \delta,$ $\epsilon \in \mathbb{C}^\times$
Θ_7	7	$F_7 = x_2^5 + x_1^2 x_2^2 x_3 + x_1^4 x_4 + x_0 x_1 x_3^3$ $+ \alpha x_0 x_1 x_2 x_3 x_4 + \beta x_0^2 x_3 x_4^2$	$\alpha, \beta \in \mathbb{C}^\times$
Θ_8	8	$F_8 = x_1^2 x_2^2 x_3 + x_1^3 x_4^2 + x_0 x_2^4 + x_0 x_1 x_3^3$ $+ \alpha x_0 x_1 x_2 x_3 x_4 + \beta x_0^2 x_3 x_4^2$	$\alpha, \beta \in \mathbb{C}^\times$
Θ_9	9	$F_9 = x_1 x_2^4 + x_1^2 x_2^2 x_3 + \alpha x_1^3 x_3^2 + x_1^4 x_4$ $+ x_0 x_2 x_3^3 + \beta x_0 x_2^3 x_4 + \gamma x_0 x_1 x_2 x_3 x_4$ $+ \delta x_0^2 x_3 x_4^2$	$\alpha, \beta, \gamma, \delta \in \mathbb{C}^\times$
Θ_{10}	10	$F_{10} = x_1^3 Q_{\mathbf{q}}(x_2, x_3) + x_1^4 x_4 + x_0 B_{\lambda}(x_2, x_3)$ $+ x_0 x_1 x_4 R_{\mathbf{r}}(x_2, x_3) + \beta x_0 x_1^2 x_4^2 + x_0^2 x_4^3$	$\lambda \in \mathbb{C} \setminus \{0, 1\};$ $\mathbf{q}, \mathbf{r} \in \mathbb{C}^3, \beta \in \mathbb{C}$
Θ_{11}	11	$F_{11} = x_1 x_2^4 + \alpha x_1^2 x_2^2 x_3 + \beta x_1^3 x_3^2 + x_1^3 x_2 x_4$ $+ \gamma x_0 x_2^2 x_3^2 + x_0 x_2^3 x_4 + \delta x_0 x_1 x_3^3$ $+ \epsilon x_0 x_1 x_2 x_3 x_4 + \zeta x_0 x_1^2 x_4^2 + x_0^2 x_3 x_4^2$	$\alpha, \beta, \gamma, \delta,$ $\epsilon, \zeta \in \mathbb{C}^\times$
Θ_{12}	12	$F_{12} = x_1 x_2^4 + \alpha x_1^2 x_2 x_3^2 + \beta x_1^2 x_2^2 x_4 + x_1^3 x_4^2$ $+ x_0 x_2^3 x_3 + \gamma x_0 x_1 x_3^3 + \delta x_0 x_1 x_2 x_3 x_4$ $+ \epsilon x_0^2 x_3^2 x_4 + x_0^2 x_2 x_4^2$	$\alpha, \beta, \gamma, \delta,$ $\epsilon \in \mathbb{C}^\times$
Θ_{13}	13	$F_{13} = x_2^5 + \alpha x_1 x_2^3 x_3 + \beta x_1^2 x_2 x_3^2 + x_1^3 x_4^2$ $+ x_0 x_2^3 x_4 + \gamma x_0 x_1 x_2 x_3 x_4 + x_0^2 x_3^3$ $+ \delta x_0^2 x_2 x_4^2$	$\alpha, \beta, \gamma, \delta \in \mathbb{C}^\times$
Θ_{14}	14	$F_{14} = x_2^5 + \alpha x_1 x_2^3 x_3 + \beta x_1^2 x_2 x_3^2 + x_1^4 x_4 + x_0 x_3^4$ $+ \gamma x_0 x_2^3 x_4 + \delta x_0 x_1 x_2 x_3 x_4 + x_0^2 x_2 x_4^2$	$\alpha, \beta, \gamma, \delta \in \mathbb{C}^\times$
Θ_{15}	15	$F_{15} = x_1 x_2^4 + x_1^2 x_2^2 x_3$ $+ x_1^3 (\gamma x_3^2 + \beta x_3 x_4 + x_4^2) + x_0 x_3^3 x_4$ $+ x_0 x_1 x_2 R(x_3, x_4) + x_0^2 B(x_3, x_4)$	$\gamma, \beta \in \mathbb{C};$ $R \in \text{Sym}^2 \langle x_3, x_4 \rangle;$ $B \in \text{Sym}^3 \langle x_3, x_4 \rangle$
Θ_{16}	19	$F_{16} = x_2^5 + \alpha x_1 x_2^3 x_3 + \beta x_1^2 x_2 x_3^2 + \gamma x_1^2 x_2^2 x_4$ $+ x_1^3 x_3 x_4 + \delta x_0 x_2^2 x_3^2 + \epsilon x_0 x_2^3 x_4 + x_0 x_1 x_3^3$ $+ \zeta x_0 x_1 x_2 x_3 x_4 + \xi x_0 x_1^2 x_4^2 + x_0^2 x_3^2 x_4$ $+ \theta x_0^2 x_2 x_4^2$	$\alpha, \beta, \gamma, \delta,$ $\epsilon, \zeta, \xi, \theta \in \mathbb{C}^\times$
Θ_{17}	20	$F_{17} = x_0 B_{\lambda, \mu}(x_2, x_3) + x_1^2 C_{\mathbf{c}}(x_2, x_3) + x_1^3 x_2 x_4$ $+ x_0 x_1 x_4 Q_{\mathbf{q}}(x_2, x_3) + x_0 x_1^2 x_4^2$ $+ x_0^2 x_4^2 M_{\mathbf{m}}(x_2, x_3)$	λ, μ distinct in $\mathbb{C} \setminus \{0, 1\};$ $\mathbf{c} \in \mathbb{C}^4, \mathbf{q} \in \mathbb{C}^3,$ $\mathbf{m} \in \mathbb{C}^2$
Θ_{18}	23	$F_{18} = x_0 B_4(x_1, x_2, x_3, x_4)$	B_4 : smooth quartic surface in $\mathbb{P}^3$
Θ_{19}	29	$F_{19} = x_3 A_4(x_1, x_2) + x_4 C_4(x_1, x_2)$ $+ x_0 Q_{2,2}(x_1, x_2; x_3, x_4) + x_0^2 x_3 x_4 (x_3 - x_4)$	A_4, C_4 : binary quartics; $Q_{2,2}$ of bidegree $(2, 2)$
Θ_{20}	32	$F_{20} = x_0^2 C_0(x_2, x_3, x_4) + x_0 x_1 C_1(x_2, x_3, x_4)$ $+ x_1^2 C_2(x_2, x_3, x_4)$	C_0, C_1, C_2 : ternary cubics
Θ_{21}	36	$F_{21} = B_5(x_1, x_2, x_3) + x_0 x_4 C_3(x_1, x_2, x_3)$ $+ x_0^2 x_3 x_4^2$	B_5, C_3 : ternary forms of degrees 5, 3

The twenty-one irreducible components at a glance

For each oriented support case k , let Φ_k denote the image in $\mathcal{M}^{\text{GIT}}$ of the G -saturated family determined by the k -th maximal support; the formal definition is given in Section 3. The normal-form table chooses one of the thirty-eight support cases for each component and denotes its label by $k(a)$. The vector r_a in the following table is primitive and defines both the supporting hyperplane and the one-dimensional torus

$$H_a = H_{r_a} = \lambda_{r_a}(\mathbb{G}_m).$$

The equality in the second column records the complete orbit of the original support label under reversal of the supporting half-space. In particular, the table simultaneously records the reduction from thirty-eight oriented supports to twenty-one quotient components, the chosen closed-orbit case, the generic connected stabilizer, and the quotient dimension.

Table 2: The complete quotient-side component catalog. For a general closed-orbit representative in row a , one has $\text{Stab}_G([F_a])^\circ = \lambda_{r_a}(\mathbb{G}_m)$.

a	Quotient component Θ_a	$k(a)$	Supporting normal r_a	d_a
1	$\Theta_1 = \Phi_1 = \Phi_{26}$	1	$(36, 1, -4, -9, -24)$	1
2	$\Theta_2 = \Phi_2 = \Phi_{27}$	2	$(27, 2, -3, -8, -18)$	2
3	$\Theta_3 = \Phi_3 = \Phi_{33}$	3	$(3, 0, 0, -1, -2)$	6
4	$\Theta_4 = \Phi_4 = \Phi_{21}$	4	$(24, 9, -6, -11, -16)$	2
5	$\Theta_5 = \Phi_5 = \Phi_{24}$	5	$(21, 6, -4, -9, -14)$	3
6	$\Theta_6 = \Phi_6 = \Phi_{28}$	6	$(18, 3, -2, -7, -12)$	5
7	$\Theta_7 = \Phi_7 = \Phi_{18}$	7	$(5, 1, 0, -2, -4)$	2
8	$\Theta_8 = \Phi_8 = \Phi_{16}$	8	$(4, 2, -1, -2, -3)$	2
9	$\Theta_9 = \Phi_9 = \Phi_{22}$	9	$(19, 4, -1, -6, -16)$	4
10	$\Theta_{10} = \Phi_{10} = \Phi_{34}$	10	$(12, 2, -3, -3, -8)$	8
11	$\Theta_{11} = \Phi_{11} = \Phi_{25}$	11	$(14, 4, -1, -6, -11)$	6
12	$\Theta_{12} = \Phi_{12} = \Phi_{17}$	12	$(13, 8, -2, -7, -12)$	5
13	$\Theta_{13} = \Phi_{13}$	13	$(3, 2, 0, -2, -3)$	4
14	$\Theta_{14} = \Phi_{14}$	14	$(4, 1, 0, -1, -4)$	4
15	$\Theta_{15} = \Phi_{15} = \Phi_{30}$	15	$(9, 4, -1, -6, -6)$	9
16	$\Theta_{16} = \Phi_{19}$	19	$(2, 1, 0, -1, -2)$	8
17	$\Theta_{17} = \Phi_{20} = \Phi_{31}$	20	$(8, 3, -2, -2, -7)$	11
18	$\Theta_{18} = \Phi_{23} = \Phi_{38}$	23	$(4, -1, -1, -1, -1)$	19
19	$\Theta_{19} = \Phi_{29} = \Phi_{35}$	29	$(6, 1, 1, -4, -4)$	15
20	$\Theta_{20} = \Phi_{32} = \Phi_{37}$	32	$(3, 3, -2, -2, -2)$	18
21	$\Theta_{21} = \Phi_{36}$	36	$(1, 0, 0, 0, -1)$	24

The unsigned permutation signatures of the twenty-one displayed vectors are pairwise distinct. Hence the tori H_a are pairwise nonconjugate. The largest component is Θ_{21} , of dimension 24; the next largest are Θ_{18} , Θ_{20} , and Θ_{19} , of dimensions 19, 18, and 15, respectively. The dimensions are part of the classification, but the proof that the rows are distinct irreducible components uses stabilizers rather than dimension comparisons.

The singularity catalog

We now summarize the geometry of a general X_a . To align this catalog with the component-indexed singular-locus table (Table 10), set

$$P_0 := (1 : 0 : 0 : 0 : 0), \quad P_\infty := (0 : 0 : 0 : 0 : 1).$$

Throughout the introduction, we use the fixed coordinate labels P_0 and P_∞ in place of case-local labels such as P and Q ; the latter are retained when convenient in the case-by-case analysis of Appendix B.

In Table 3, the symbols A_n and D_n refer to the generic transverse hypersurface singularity along the indicated smooth stratum. Their indices also give the generic transverse Jacobian length. More generally, by the *generic transverse Jacobian length* we mean the length obtained from the full Jacobian ideal after localizing at the generic point of the stratum and passing to a transverse local ring. It need not equal the Milnor number of a transverse polynomial obtained by freezing the stratum parameter; this distinction is relevant in row 19. The phrase “special” marks a nonisolated point at which the generic transverse model changes; every such point is treated explicitly in Appendix B and has local minimal exponent one. For the few non-ADE strata, the table records an analytic model or the generic Jacobian length. A dash in the last column means that there is no isolated singular point.

Table 3: Generic singularity catalog for the twenty-one boundary components.
In every row, $\tilde{\alpha}(X_a) = 1$.

a	Reduced singular support of X_a	Generic transverse structure and special strata	Isolated local family
1	rational plane quartic C with a $(3, 4)$ -cusp at P_∞ ; isolated point P_0	A_3 on $C \setminus \{P_\infty\}$; P_∞ special	$P_0 : \mathbf{S}_{87}^{Y52}$
2	line L and cuspidal plane cubic C , $L \cap C = \{P_\infty\}$; isolated point P_0	A_3 on $L \setminus \{P_\infty\}$; A_2 on $C \setminus \{P_\infty\}$; P_∞ special	$P_0 : \mathbf{S}_{88}^{Y84}$
3	line L with distinguished point P_∞ ; isolated point P_0	A_4 on $L \setminus \{P_\infty\}$; P_∞ special	$P_0 : \mathbf{S}_{88}^{Y15}$
4	two disjoint lines $M \sqcup L$	A_2 on the general point of M ; A_{10} on the general point of L ; special points $P_0 \in M$ and $P_\infty \in L$	—
5	three lines $N \sqcup (L \cup M)$, $L \cap M = \{P_\infty\}$	A_1 on general points of N and M ; A_8 on $L \setminus \{P_\infty\}$; distinguished $P_0 \in N$ and intersection P_∞ special	—
6	line L and smooth conic C , $L \cap C = \{P_\infty\}$; isolated point P_0	A_6 on $L \setminus \{P_\infty\}$; A_1 on $C \setminus \{P_\infty\}$; P_∞ special	$P_0 : \mathbf{S}_{91}^{Y53}$
7	line L with distinguished point P_∞ ; isolated point P_0	A_4 on $L \setminus \{P_\infty\}$; P_∞ special	$P_0 : \mathbf{S}_{96}^{Y86}$
8	two disjoint lines $N \sqcup L$	on general N : suspension of an ordinary triple plane-curve point (Jacobian length 4); on general L : A_9 ; special points $P_0 \in N$, $P_\infty \in L$	—
9	line L with distinguished point P_∞ ; isolated point P_0	A_2 on $L \setminus \{P_\infty\}$; P_∞ special	$P_0 : \mathbf{S}_{96}^{Y94}$
10	four lines C_λ through the common vertex P_∞ ; isolated point P_0	A_2 away from the common vertex; P_∞ special	$P_0 : \mathbf{S}_{90}^{Y2}$
11	two disjoint lines $N \sqcup L$	A_1 on general N ; A_5 on general L ; special points $P_0 \in N$, $P_\infty \in L$	—
12	two disjoint lines $N \sqcup L$	D_5 on general N ; A_6 on general L ; special points $P_0 \in N$, $P_\infty \in L$	—
13	two disjoint lines $N \sqcup L$	D_6 on the general point of each line; special points $P_0 \in N$, $P_\infty \in L$	—
14	two isolated points P_0 and P_∞	—	$P_0, P_\infty : \mathbf{S}_{102}^{Y62}$
15	two disjoint lines $N \sqcup L$	A_3 on $N \setminus \{P_0\}$; P_0 special; along L : relative weighted cone of normal type $(3, 2, 1; 6)$, Jacobian length 9	—
16	two disjoint lines $N \sqcup L$	A_4 on the general point of each line; special points $P_0 \in N$, $P_\infty \in L$	—
17	line N disjoint from four lines $C_{\lambda, \mu}$ through the common vertex P_∞	A_2 on general N ; A_1 on each of the four other lines away from P_∞ ; distinguished $P_0 \in N$ and vertex P_∞ special	—
18	smooth quartic surface S ; isolated point P_0	simple normal crossing along S	$P_0 : \mathbf{S}_{81}^{Y1}$

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a	Reduced singular support of X_a	Generic transverse structure and special strata	Isolated local family
19	line L ; isolated point P_0	generic transverse model $U^2 + G_4(V, W)$, where G_4 is square-free binary quartic; generic Jacobian length 8	$P_0 : \mathbf{S}_{100}^{Y3}$
20	disjoint union of a line N and a plane Π	on $\Pi \setminus \Delta_\Pi$: $U^2 + V^2$; on the smooth discriminant curve Δ_Π : $U^2 + zV^2$; along N : a cubic-cone family $C_t(y_2, y_3, y_4) = 0$, Jacobian length 7	—
21	two isolated points P_0 and P_∞	—	$P_0, P_\infty : \mathbf{S}_{96}^{Y21}$

The notation in the last column is decoded completely in the following table, which also displays a defining equation for each local normal-form family. If a family is written as $\mathbf{S}_\mu^{YN}$, then μ is its Milnor number and N is the row of Yonemura's Table 2.2 [Yon90] containing its ordered normalized weight. The precise nonempty Zariski-open genericity conditions on the displayed parameters are given in Definitions 8.1–8.11. Every displayed integral weight system $(w_1, w_2, w_3, w_4; D)$ satisfies

$$w_1 + w_2 + w_3 + w_4 = D,$$

so every isolated germ in the table has local minimal exponent one.

Table 4: The eleven isolated weighted-homogeneous local normal-form families. The second column displays a defining equation for each family. All parameters are restricted to the nonempty Zariski-open generic loci specified in Definitions 8.1–8.11. The integral weights determine the ordered normalized weights used to identify the corresponding rows of Yonemura's Table 2.2 [Yon90]. This table forms part of Theorem A.

Local family	Local normal form	Integral weights ($w; D$)	μ	Yon. No.	Component / point
$\mathbf{S}_{81}^{Y1}(B_4)$	$B_4(X_1, X_2, X_3, X_4) = 0$	(1, 1, 1, 1; 4)	81	1	$\Theta_{18} : P_0$
$\mathbf{S}_{87}^{Y52}(\tau)$	$X_1^4 X_2 + X_3^4 + X_2^3 X_4$ $+ \tau X_1 X_2 X_3 X_4 + X_4^3 = 0$	(7, 8, 9, 12; 36)	87	52	$\Theta_1 : P_0$
$\mathbf{S}_{88}^{Y84}(\alpha, \beta)$	$X_1^3 X_2^2 + X_1^4 X_3 + X_2 X_3^3 + X_2^3 X_4$ $+ \alpha X_1 X_2 X_3 X_4 + \beta X_4^3 = 0$	(5, 6, 7, 9; 27)	88	84	$\Theta_2 : P_0$
$\mathbf{S}_{88}^{Y15}(B_5)$	$B_5(X_1, X_2) + X_2 X_3^3$ $+ (X_1^2 + X_2^2) X_3 X_4 + X_4^3 = 0$	(3, 3, 4, 5; 15)	88	15	$\Theta_3 : P_0$
$\mathbf{S}_{90}^{Y2}(B_4, Q_2, R_2, \beta)$	$B_4(X_2, X_3) + X_1^3 Q_2(X_2, X_3) + X_1^4 X_4$ $+ X_1 X_4 R_2(X_2, X_3) + \beta X_1^2 X_4^2 + X_4^3 = 0$	(2, 3, 3, 4; 12)	90	2	$\Theta_{10} : P_0$
$\mathbf{S}_{91}^{Y53}(\alpha, \beta, \gamma, \delta, \epsilon)$	$X_1^2 X_2^3 + X_1^3 X_2 X_3 + X_1^4 X_4 + X_2^2 X_3^2$ $+ \alpha X_2^3 X_4 + \beta X_1 X_3^3 + \gamma X_1 X_2 X_3 X_4$ $+ \delta X_1^2 X_4^2 + \epsilon X_4^3 = 0$	(3, 4, 5, 6; 18)	91	53	$\Theta_6 : P_0$
$\mathbf{S}_{96}^{Y86}(\alpha, \beta)$	$X_2^5 + X_1^2 X_2^2 X_3 + X_1^4 X_4 + X_1 X_3^3$ $+ \alpha X_1 X_2 X_3 X_4 + \beta X_3 X_4^2 = 0$	(4, 5, 7, 9; 25)	96	86	$\Theta_7 : P_0$
$\mathbf{S}_{96}^{Y94}(\alpha, \beta, \gamma, \delta)$	$X_1 X_2^4 + X_1^2 X_2^2 X_3 + \alpha X_1^3 X_3^2 + X_1^4 X_4$ $+ X_2 X_3^3 + \beta X_2^3 X_4 + \gamma X_1 X_2 X_3 X_4$ $+ \delta X_3 X_4^2 = 0$	(3, 4, 5, 7; 19)	96	94	$\Theta_9 : P_0$
$\mathbf{S}_{96}^{Y21}(B_5, C_3, L_1)$	$B_5(X_1, X_2, X_3) + X_4 C_3(X_1, X_2, X_3)$ $+ X_4^2 L_1(X_1, X_2, X_3) = 0$	(1, 1, 1, 2; 5)	96	21	$\Theta_{21} : P_0,$ P_∞

continued on the next page

Local family	Local normal form	Integral weights ($w; D$)	μ	Yon. No.	Component / point
$\mathbf{S}_{100}^{Y3}$ ($A_4, C_4,$ $Q_{2,2}, B_3$)	$X_3 A_4(X_1, X_2) + X_4 C_4(X_1, X_2)$ $+ Q_{2,2}(X_1, X_2; X_3, X_4) + B_3(X_3, X_4) = 0$	(1, 1, 2, 2; 6)	100	3	$\Theta_{19} : P_0$
$\mathbf{S}_{102}^{Y62}$ ($\alpha, \beta, \gamma, \delta$)	$X_2^5 + \alpha X_1 X_2^3 X_3 + \beta X_1^2 X_2 X_3^2$ $+ X_1^4 X_4 + X_3^4 + \gamma X_2^3 X_4$ $+ \delta X_1 X_2 X_3 X_4 + X_2 X_4^2 = 0$	(3, 4, 5, 8; 20)	102	62	$\Theta_{14} : P_0,$ P_∞

Parameter notation. The arguments displayed in each equation specify the variables of the auxiliary forms. Thus $B_4(X_1, X_2, X_3, X_4)$ is a homogeneous quartic whose projectivization in $\mathbb{P}^3$ is smooth, whereas $B_4(X_2, X_3)$ is a square-free binary quartic. The symbols $B_5(X_1, X_2)$ and $B_5(X_1, X_2, X_3)$ denote, respectively, a binary and a ternary quintic. The forms Q_2, R_2 are binary quadratics in (X_2, X_3) ; C_3 and L_1 are, respectively, a ternary cubic and a nonzero ternary linear form in (X_1, X_2, X_3) ; A_4, C_4 are binary quartics in (X_1, X_2) ; $Q_{2,2}$ is bihomogeneous of bidegree (2, 2) in (X_1, X_2) and (X_3, X_4) ; and B_3 is a binary cubic in (X_3, X_4) . Greek letters denote complex scalar parameters. The precise smoothness, square-freeness, nonvanishing, isolatedness, and discriminant conditions defining the relevant nonempty Zariski-open loci are given in Definitions 8.1– 8.11.

The Yonemura assertion is deliberately an assertion about normalized weight systems. A row of Yonemura’s list [Yon90] gives a weight system and one particular nondegenerate realization. We do not claim that every parameter value in one of the families above is analytically equivalent to the polynomial displayed in that row. The term *Yonemura-type* means exactly that the normalized weight system occurs in Yonemura’s list. What is proved here is an occurrence-and-exhaustion statement on the quintic GIT boundary: the eleven families in Table 4 occur, and no other isolated weighted-homogeneous local family occurs for a general closed orbit in any boundary component.

Proof map for Theorem A

Theorem A is a synthesis of the structural results in the body and the case-by-case verifications in the appendices. Item (1) is Proposition 2.5 together with Proposition 3.1. Item (2) is proved in three stages. Corollary 4.2 and Proposition 4.4 identify the seventeen opposite-support pairs and the four fixed cases, while Corollary 4.5 reduces the thirty-eight-family cover to twenty-one candidate quotient images. Corollary 5.2 proves that the candidates are nonempty, closed, and irreducible, and Theorem 6.1 proves pairwise non-inclusion and hence the irreducible-component decomposition. The normal forms and quotient dimensions in item (3) are organized in Section 5 and proved support by support in Appendix A; its stabilizer assertions are Proposition 6.3 and Corollary 6.6. Item (4) is stated in Section 8 and proved by the component-by-component local calculations in Appendix B. Item (5) is proved in the same appendix. Thus Tables 1–4 are both a statement of the classification and a row-by-row index to its proofs.

Why thirty-eight supports give twenty-one components

Fix the diagonal maximal torus $T \subset G$, and let

$$I := \left\{ u = (u_0, \dots, u_4) \in \mathbb{Z}_{\geq 0}^5 \mid u_0 + \dots + u_4 = 5 \right\}$$

be the exponent set of quintic monomials. Its barycenter is

$$\eta = (1, 1, 1, 1, 1).$$

For a nonzero vector

$$r = (r_0, \dots, r_4) \in \mathbb{Z}^5, \quad \sum r_i = 0,$$

put

$$I(r)_{\geq 0} := \{u \in I \mid r \cdot u \geq 0\}.$$

The convex-geometric Hilbert–Mumford criterion identifies T -semistability with the condition that the convex hull of the support contain η , and strict T -semistability with the case in which η lies on a supporting hyperplane. Exact integer linear algebra, inclusion tests, and reduction by the Weyl group S_5 yield exactly thirty-eight maximal supports

$$S_k = I(r_k)_{\geq 0}, \quad k = 1, \dots, 38.$$

For

$$V_k := \text{Span}_{\mathbb{C}}\{x^u \mid u \in S_k\}, \quad Y_k := \overline{G \cdot \mathbb{P}(V_k)},$$

one has

$$\Sigma^{ss} = \bigcup_{k=1}^{38} (Y_k \cap \Sigma^{ss}), \quad \partial \mathcal{M}^{\text{GIT}} = \bigcup_{k=1}^{38} \Phi_k,$$

where

$$\Phi_k := \pi(Y_k \cap \Sigma^{ss}).$$

This is a support-indexed cover, not yet a component decomposition.

For an arbitrary r , write

$$W = \bigoplus_{m \in \mathbb{Z}} W_m(r), \quad V(r) := \bigoplus_{m \geq 0} W_m(r), \quad Z(r) := W_0(r) = W^{H_r}.$$

Set

$$Y(r) := \overline{G \cdot \mathbb{P}(V(r))} \subset \mathbb{P}(W), \quad \Phi(r) := \pi(Y(r) \cap \mathbb{P}(W)^{ss}).$$

The central quotient-theoretic statement of the paper is the supporting-hyperplane principle

$$\Phi(r) = \overline{\pi(\mathbb{P}(V(r)) \cap \mathbb{P}(W)^{ss})} = \overline{\pi(\mathbb{P}(Z(r)) \cap \mathbb{P}(W)^{ss})}.$$

Indeed, if $f = f_0 + f_{>0} \in V(r)$ is semistable, then the one-parameter-subgroup limit is $f_0 \in Z(r)$, the limit remains semistable, and

$$\pi([f]) = \pi([f_0]).$$

Consequently

$$\Phi(r) = \Phi(-r).$$

Thus the quotient remembers the supporting hyperplane but not the orientation of its normal.

Returning the opposite normal to the dominant chamber gives the involution

$$\iota(r_0, r_1, r_2, r_3, r_4) := (-r_4, -r_3, -r_2, -r_1, -r_0).$$

On the thirty-eight dominant normals it has seventeen two-cycles,

$$\begin{aligned} &(1, 26), (2, 27), (3, 33), (4, 21), (5, 24), (6, 28), \\ &(7, 18), (8, 16), (9, 22), (10, 34), (11, 25), (12, 17), \\ &(15, 30), (20, 31), (23, 38), (29, 35), (32, 37), \end{aligned}$$

and four fixed points 13, 14, 19, 36. Hence

$$38 = 17 \cdot 2 + 4$$

counts thirty-eight oriented maximal half-spaces but only twenty-one supporting hyperplanes up to coordinate permutation and reversal. This is exactly the identification displayed in the second column of Table 2.

Closed orbits, stabilizers, and non-inclusion

For a general form $f_k \in V_k$, the limit

$$\phi_k := \lim_{t \rightarrow 0} \lambda_{r_k}(t) \cdot f_k$$

is its weight-zero part and lies in W^{H_k} . Luna's centralizer reduction turns the closedness problem for the G -orbit of ϕ_k into a closed-orbit problem for

$$C_G(H_k)/H_k$$

on W^{H_k} . For torus centralizers we use a relative-interior criterion for the character polytope; for centralizers with nontrivial linear blocks we combine discriminant and resultant conditions with the Casimiro–Florentino criterion [CF12]. This produces an explicit polystable normal form ϕ_k^{nf} for every $k = 1, \dots, 38$, together with the quotient dimensions used in Table 2. The twenty-one quotient-side representatives selected from these forms are reproduced in Table 1.

After the supporting-hyperplane identifications, it remains to prove that the twenty-one closed irreducible families are not further identified and that none is contained in another. For a nonzero quintic F , consider

$$\mathcal{D}_F : \mathfrak{sl}_5 \oplus \mathbb{C} \longrightarrow W, \quad (A, c) \longmapsto A \cdot F - cF.$$

This is represented by a 126×25 matrix and satisfies

$$\dim \text{Stab}_G([F]) = 25 - \text{rank } \mathcal{D}_F.$$

The torus H_a is always contained in the stabilizer of the normal form in row a , so $\text{rank } \mathcal{D}_{F_a} \leq 24$. Exact integral specializations and nonzero 24×24 minors prove that the generic rank is exactly 24. Therefore

$$\text{Stab}_G([F_a])^\circ = H_a.$$

The twenty-one unsigned weight signatures are distinct, hence these tori are pairwise nonconjugate.

Finally, Luna's étale slice theorem gives the closure obstruction used in the component theorem: if $\Theta_a \subseteq \Theta_b$, then a conjugate of the generic connected stabilizer of Θ_b must be contained in the generic connected stabilizer of Θ_a . Since both are connected one-dimensional tori, this inclusion would be an equality, contradicting nonconjugacy. Thus all ordered containments between distinct rows of Table 2 are excluded simultaneously.

Singularities and the extremal minimal exponent

The adjective *extremal* refers here to the equality between the global minimal exponent and the critical value $(n+1)/d$: every general boundary representative attains that value, and none of its singular strata forces a smaller one.

For every row of the component catalog, the saturated Jacobian ideal

$$J(F_a) := \left(\frac{\partial F_a}{\partial x_0}, \dots, \frac{\partial F_a}{\partial x_4} \right) : (x_0, \dots, x_4)^\infty$$

is computed explicitly. This determines the scheme structure along the positive-dimensional singular components and gives the stratification summarized in Table 3. The local analysis uses the holomorphic splitting lemma, transverse A - and D -type models, Newton nondegeneracy, weighted blow-ups and log canonical thresholds, Thom–Sebastiani formulas, and Saito's rationality criterion.

At an isolated point, the local equation belongs to one of the eleven weighted-homogeneous families in Table 4. Since the sum of the weights equals the weighted degree, its local minimal

exponent is one. At a positive-dimensional stratum, the generic transverse models have local minimal exponent greater than or equal to one, and every distinguished point at which the transverse type changes has local minimal exponent exactly one. Consequently no singular stratum lowers the global value below one, while at least one stratum in every component realizes one. This proves

$$\tilde{\alpha}(X_a) = 1 \quad (a = 1, \dots, 21).$$

The value is the critical quantity

$$\frac{n+1}{d} = \frac{4+1}{5} = 1$$

for degree-five hypersurfaces in $\mathbb{P}^4$ in the minimal-exponent approach to GIT stability [MP20, Par25].

Relation to earlier work and organization of the paper

The GIT compactification of quintic threefolds was studied by Lakhani [Lak10]. His four first-level minimal-orbit families become, in the present quotient-side classification, the four components of largest dimension, namely $\Theta_{18}, \Theta_{19}, \Theta_{20}, \Theta_{21}$; the remaining seventeen components are not contained in their union. The precise dictionary, the comparison of the polystability arguments, and the separation between Lakhani's second-level degeneration loci and the new irreducible components are given in Section 7.

We use the standard GIT framework of [MFK94, Dol03, Kir84], Luna's results on closed orbits and étale slices [Lun73, Lun75], and the convex-geometric form of the Hilbert–Mumford criterion. The finite support enumeration is in the spirit of the effective procedures developed in [GMMS26].

Section 2 fixes the action convention, recalls the Hilbert–Mumford criterion, and classifies the thirty-eight maximal strictly T -semistable supports. Section 3 passes to the full $\mathrm{SL}(5)$ -action, defines the prequotient and quotient-side families, and proves the boundary covering. Section 4 proves zero-weight reduction and supporting-hyperplane invariance, reducing the thirty-eight oriented support labels to twenty-one candidate quotient images. Section 5 develops the general closed-orbit criteria and records one polystable representative for each of the twenty-one quotient families. Section 6 computes the generic connected stabilizers and applies Luna's slice theorem to prove pairwise non-inclusion and the component theorem. Section 7 then compares the resulting classification with Lakhani's. Section 8 states the component-indexed singularity catalog and the minimal-exponent conclusion.

Appendix A contains the thirty-eight support-by-support constructions of the polystable normal forms, while Appendix B contains the twenty-one component-by-component calculations of singular schemes, local analytic types, and minimal exponents.

2 Numerical criterion and maximal strictly semistable supports

2.1 Numerical criterion

Let

$$V := H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1)) = \langle x_0, x_1, x_2, x_3, x_4 \rangle_{\mathbb{C}},$$

and put

$$W := \mathrm{Sym}^5 V, \quad G := \mathrm{SL}(V) \simeq \mathrm{SL}(5), \quad \mathbb{P}(W) = \mathbb{P}(\mathrm{Sym}^5 V).$$

Thus an element $0 \neq f \in W$ is a quintic form in the variables $x_0, \dots, x_4$, and its projective class defines the quintic threefold

$$X_f := V(f) \subset \mathbb{P}(V^\vee) \simeq \mathbb{P}^4.$$

Let $T \subset G$ be the diagonal maximal torus with respect to the ordered basis $x_0, \dots, x_4$.

Action convention. Throughout the paper, G acts on $W = \text{Sym}^5 V$ through the fifth symmetric power of its standard representation on V . More explicitly, if $g = (g_{ij}) \in G$, then

$$g \cdot x_j := \sum_{i=0}^4 g_{ij} x_i, \quad j = 0, \dots, 4, \quad (1)$$

and this action is extended multiplicatively and linearly to $\text{Sym}^5 V$. Via the contragredient action on $\mathbb{P}(V^\vee)$, this is the usual projective change-of-coordinates action on quintic hypersurfaces.

If one instead writes the action on equations in the pullback form

$$(g \star f)(x) := f(g^{-1}x),$$

then, under the above basis identification,

$$g \star f = (g^{-T}) \cdot f.$$

Since $g \mapsto g^{-T}$ is an automorphism of G , the two conventions have the same orbits, invariant ring, stable and semistable loci, and projective GIT quotient. We use only the action $\cdot$ in (1) below.

For a vector $u = (u_0, \dots, u_4) \in \mathbb{Q}^5$, set

$$\text{wt}(u) := u_0 + \dots + u_4.$$

We use the notation

$$\mathbb{Z}^5(d) := \{u \in \mathbb{Z}^5 \mid \text{wt}(u) = d\}, \quad \mathbb{Z}_{\geq 0}^5 := \{u \in \mathbb{Z}^5 \mid u_i \geq 0 \text{ for all } i\},$$

and

$$I := \mathbb{Z}^5(5) \cap \mathbb{Z}_{\geq 0}^5.$$

Thus I is the set of exponent vectors of degree-five monomials. For $u \in I$, write

$$x^u := x_0^{u_0} x_1^{u_1} x_2^{u_2} x_3^{u_3} x_4^{u_4}.$$

The barycenter of the degree-five simplex is

$$\eta := (1, 1, 1, 1, 1) \in I.$$

For $r \in \mathbb{Q}^5$, define

$$I(r)_{\geq 0} := \{u \in I \mid r \cdot u \geq 0\}, \quad I(r)_{> 0} := \{u \in I \mid r \cdot u > 0\},$$

and

$$I(r)_{=0} := \{u \in I \mid r \cdot u = 0\}.$$

A nonzero vector $r \in \mathbb{Z}^5$ is called *reduced* if

$$\gcd(r_0, r_1, r_2, r_3, r_4) = 1.$$

Let

$$f = \sum_{u \in I} a_u x^u \in W.$$

Its support is

$$\text{Supp}(f) := \{u \in I \mid a_u \neq 0\}.$$

A one-parameter subgroup of T is written as

$$\lambda_r(t) = \text{diag}(t^{r_0}, t^{r_1}, t^{r_2}, t^{r_3}, t^{r_4}), \quad r = (r_0, \dots, r_4) \in \mathbb{Z}^5(0).$$

By the action convention above,

$$\lambda_r(t) \cdot x^u = t^{r \cdot u} x^u. \quad (2)$$

Accordingly, if

$$f = \sum_{m \in \mathbb{Z}} f_m^{(r)}, \quad f_m^{(r)} \in \text{Span}_{\mathbb{C}}\{x^u \mid u \in I, r \cdot u = m\},$$

then

$$\lambda_r(t) \cdot f = \sum_{m \in \mathbb{Z}} t^m f_m^{(r)}.$$

In particular, if $\text{Supp}(f) \subset I(r)_{\geq 0}$, then the affine limit exists and equals the weight-zero part:

$$\lim_{t \rightarrow 0} \lambda_r(t) \cdot f = f_0^{(r)} = \sum_{u \in I(r)=0} a_u x^u. \quad (3)$$

Thus, with our convention, the zero-weight projection is obtained by $\lambda_r(t)$ itself, rather than by $\lambda_r(t)^{-1}$.

Definition 2.1. Let $s \subset I$. We say that s is *not stable with respect to T* if

$$s \subset I(r)_{\geq 0}$$

for some nonzero $r \in \mathbb{Z}^5(0)$. We say that s is *unstable with respect to T* if

$$s \subset I(r)_{> 0}$$

for some nonzero $r \in \mathbb{Z}^5(0)$.

For $0 \neq f \in W$, we say that f is not stable, respectively unstable, with respect to T if $\text{Supp}(f)$ has the corresponding property.

We now record the equivalent convex-geometric formulation. Let

$$\mathcal{H}_5 := \{u \in \mathbb{R}^5 \mid \text{wt}(u) = 5\}.$$

Translation by $-\eta$ identifies $\mathcal{H}_5$ with the real character space of T , represented by the hyperplane

$$\mathbb{R}^5(0) := \{v \in \mathbb{R}^5 \mid v_0 + \cdots + v_4 = 0\}.$$

Indeed, the character of the monomial x^u is represented by $u - \eta$, and

$$r \cdot (u - \eta) = r \cdot u$$

for every $r \in \mathbb{Z}^5(0)$. The Hilbert–Mumford numerical criterion gives

$$f \text{ is } T\text{-semistable} \iff \eta \in \text{Conv}(\text{Supp}(f)),$$

and

$$f \text{ is } T\text{-stable} \iff \eta \in \text{Int}_{\mathcal{H}_5} \text{Conv}(\text{Supp}(f)),$$

where $\text{Int}_{\mathcal{H}_5}$ denotes the interior in the full affine hyperplane $\mathcal{H}_5$. By contrast, polystability is an orbit-closure condition. The affine torus orbit-closure criterion (see, for example, [Dol03, PV94]) gives

$$[f] \text{ is } T\text{-polystable} \iff T \cdot f \text{ is closed in } W \iff \eta \in \text{relint} \text{Conv}(\text{Supp}(f)),$$

where the relative interior is taken in the affine hull of $\text{Conv}(\text{Supp}(f))$. Consequently, f is strictly T -semistable precisely when

$$\eta \in \text{Conv}(\text{Supp}(f)) \quad \text{but} \quad \eta \notin \text{Int}_{\mathcal{H}_5} \text{Conv}(\text{Supp}(f)).$$

The distinction between relative interior and ambient interior will be important later: a polystable form may have a positive-dimensional torus stabilizer and hence fail to be stable.

Theorem 2.2 (Hilbert–Mumford numerical criterion). *Let $0 \neq f \in W = \text{Sym}^5 V$. The point $[f] \in \mathbb{P}(W)$, or equivalently the quintic threefold $X_f \subset \mathbb{P}^4$, is not stable for the G -action if and only if there exist $\sigma \in G$ and a nonzero vector $r \in \mathbb{Z}^5(0)$ such that*

$$\text{Supp}(\sigma \cdot f) \subset I(r)_{\geq 0}.$$

It is unstable for the G -action if and only if there exist $\sigma \in G$ and a nonzero $r \in \mathbb{Z}^5(0)$ such that

$$\text{Supp}(\sigma \cdot f) \subset I(r)_{> 0}.$$

In particular, $[f]$ is strictly semistable for G if and only if

- 1. there exists $\sigma \in G$ such that $\sigma \cdot f$ is not stable with respect to the fixed torus T ; and*
- 2. for every $\sigma \in G$, the form $\sigma \cdot f$ is semistable with respect to T .*

Proof. Every one-parameter subgroup of G is conjugate to one contained in the fixed maximal torus T . After the corresponding change of basis, it is therefore enough to test the subgroups λ_r with $0 \neq r \in \mathbb{Z}^5(0)$. By (2), the λ_r -weights of the nonzero monomials of $\sigma \cdot f$ are exactly the integers $r \cdot u$. The Hilbert–Mumford inequalities are consequently equivalent to the two displayed support containments. The final statement is the conjunction of nonstability and semistability. $\square$

2.2 Maximal strictly semistable supports for the fixed torus

We now enumerate the inclusion-maximal supports that can occur for strictly semistable quintics with respect to the fixed maximal torus. Let

$$\mathcal{S} := \{I(r)_{\geq 0} \mid 0 \neq r \in \mathbb{Z}^5(0)\},$$

ordered by inclusion. Although the set of vectors r is infinite, $\mathcal{S}$ is finite because I is finite.

For every $r \in \mathbb{Z}^5(0)$, one has

$$r \cdot \eta = 0.$$

Hence

$$\eta \in I(r)_{=0} \subset I(r)_{\geq 0}.$$

It follows that a form whose support is exactly $I(r)_{\geq 0}$ is T -semistable, while the defining half-space shows that it is not T -stable. Thus the maximal elements of $\mathcal{S}$ are precisely the maximal full supports of strictly T -semistable quintics.

Different vectors may define the same subset of I . The following lemma reduces the classification to finitely many supporting hyperplanes through the barycenter.

Lemma 2.3. *Let $I(r)_{\geq 0}$ be a maximal element of $\mathcal{S}$. Then there exist three exponent vectors*

$$u_1, u_2, u_3 \in I$$

and a reduced vector $r' \in \mathbb{Z}^5(0)$ such that

- 1. the $\mathbb{Q}$ -vector subspace of $\mathbb{Q}^5$ spanned by u_1, u_2, u_3, η has dimension 4;*
- 2. r' is orthogonal to that subspace; and*
- 3.*

$$I(r)_{\geq 0} = I(r')_{\geq 0}.$$

Proof. Put

$$S := I(r)_{\geq 0}$$

and work in the four-dimensional rational vector space

$$E := \mathbb{Q}^5(0) = \{q \in \mathbb{Q}^5 \mid q \cdot \eta = 0\}.$$

Consider the rational polyhedral cone

$$K_S := \left\{ q \in E \mid \begin{array}{ll} q \cdot u \geq 0 & (u \in S), \\ q \cdot v \leq 0 & (v \in I \setminus S) \end{array} \right\}.$$

The original vector r belongs to K_S , so this cone is nonzero. Moreover, if $0 \neq q \in K_S$ and $q \cdot v = 0$ for some $v \in I \setminus S$, then

$$S \subsetneq I(q)_{\geq 0},$$

contradicting the maximality of S . Hence every nonzero $q \in K_S$ satisfies

$$q \cdot v < 0 \quad (v \in I \setminus S),$$

and therefore defines exactly the same support:

$$I(q)_{\geq 0} = S.$$

The cone K_S is pointed. Indeed, if both q and $-q$ belonged to K_S , then q would vanish on every point of I ; since I affinely spans $\mathcal{H}_5$, this would force $q = 0$. Choose an extremal ray of K_S , and let $r' \in \mathbb{Z}^5(0)$ be its primitive integral generator. Since E has dimension 4, this ray is cut out by at least three linearly independent active inequalities. No inequality indexed by $I \setminus S$ can be active, by the strict inequality above. Consequently, there exist $u_1, u_2, u_3 \in S$ such that

$$r' \cdot u_1 = r' \cdot u_2 = r' \cdot u_3 = 0$$

and the three vectors

$$u_1 - \eta, \quad u_2 - \eta, \quad u_3 - \eta$$

are linearly independent in $\mathbb{Q}^5(0)$. Equivalently, u_1, u_2, u_3, η span a four-dimensional subspace of $\mathbb{Q}^5$. Finally, the preceding argument gives

$$I(r')_{\geq 0} = S = I(r)_{\geq 0},$$

as required. □

The lemma leads to the following finite enumeration.

Enumeration algorithm. Let $\mathcal{F}$ be the finite set of three-element subsets

$$\mathbf{u} = \{u_1, u_2, u_3\} \subset I.$$

Initialize a provisional collection of supports by $\mathcal{M}' := \emptyset$, and perform the following steps.

Step 1. For each $\mathbf{u} \in \mathcal{F}$, let $U_{\mathbf{u}} \subset \mathbb{Q}^5$ be the subspace spanned by

$$u_1, u_2, u_3, \eta.$$

If $\dim_{\mathbb{Q}} U_{\mathbf{u}} \neq 4$, discard the triple. If $\dim_{\mathbb{Q}} U_{\mathbf{u}} = 4$, compute a primitive integral normal vector

$$0 \neq r \in \mathbb{Z}^5(0)$$

to $U_{\mathbf{u}}$, determined up to sign.

Step 2. For each normal vector obtained in Step 1, add both supports

$$I(r)_{\geq 0} \quad \text{and} \quad I(-r)_{\geq 0}$$

to $\mathcal{M}'$.

Step 3. Replace $\mathcal{M}'$ by its closure under the Weyl group S_5 , acting by permutation of the coordinates, and delete repeated supports. Thus all possible containments

$$S \subseteq \pi(S'), \quad S, S' \in \mathcal{M}', \quad \pi \in S_5,$$

are represented as ordinary containments inside the resulting collection.

Step 4. Delete every support in $\mathcal{M}'$ that is properly contained in another support in $\mathcal{M}'$. Decompose the remaining maximal supports into S_5 -orbits, choose one representative from each orbit, and choose its reduced defining normal vector in the dominant Weyl chamber

$$r_0 \geq r_1 \geq r_2 \geq r_3 \geq r_4.$$

Lemma 2.3 proves that every maximal element of $\mathcal{S}$ occurs in this enumeration. No additional barycenter test is needed in the quintic case, because $\eta \in I(r)_{=0}$ for every $r \in \mathbb{Z}^5(0)$.

Remark 2.4. The computation may be accelerated by pruning the three-element subsets under the Weyl group, or by deleting duplicate normal hyperplanes before constructing their supports. These modifications change only the running time. The exhaustive calculation itself uses exact integer linear algebra, the two choices of sign of every normal vector, all coordinate permutations, and exact inclusion tests among subsets of the finite set I .

Proposition 2.5. *For $(\dim V, d) = (5, 5)$, the maximal T -strictly semistable supports are, up to permutation of the coordinates, the thirty-eight supports*

$$S_k := I(r_k)_{\geq 0} \quad (k = 1, \dots, 38)$$

listed in Table 5. Each r_k is reduced and is written in the dominant Weyl chamber

$$r_{k,0} \geq r_{k,1} \geq r_{k,2} \geq r_{k,3} \geq r_{k,4}.$$

Proof. There are

$$\binom{126}{3} = 325,500$$

three-element subsets of I . Exact integer row reduction shows that 309,150 of them satisfy

$$\dim_{\mathbb{Q}} \text{Span}_{\mathbb{Q}}\{u_1, u_2, u_3, \eta\} = 4.$$

Taking primitive normal vectors, both choices of sign, and all coordinate permutations produces 29,760 distinct supports of the form $I(r)_{\geq 0}$. Exact inclusion comparison in this S_5 -closed collection leaves 3,350 maximal supports, which decompose into exactly 38 orbits under S_5 .

The vectors displayed in Table 5 are the reduced dominant representatives of these 38 orbits. For every row, direct computation verifies

$$\sum_{i=0}^4 r_{k,i} = 0, \quad \gcd(r_{k,0}, \dots, r_{k,4}) = 1,$$

the stated dominant ordering, and the listed value of $|I(r_k)_{\geq 0}|$. Maximality is checked against every coordinate translate of every candidate support; hence no displayed support is properly contained in another element of $\mathcal{S}$. Since

$$\eta \in I(r_k)_{=0} \subset S_k,$$

a form with support exactly S_k is T -semistable, while the nonzero one-parameter subgroup λ_{r_k} shows that it is not T -stable. Thus every S_k is a maximal strictly semistable support. Conversely, Lemma 2.3 shows that every maximal element of $\mathcal{S}$ occurs in the exhaustive enumeration, up to coordinate permutation. $\square$

Table 5: The thirty-eight maximal strictly T -semistable monomial supports, represented by their reduced dominant normal vectors.

k	r_k	$ S_k $
1	(36, 1, -4, -9, -24)	58
2	(27, 2, -3, -8, -18)	59
3	(3, 0, 0, -1, -2)	62
4	(24, 9, -6, -11, -16)	62
5	(21, 6, -4, -9, -14)	62
6	(18, 3, -2, -7, -12)	62
7	(5, 1, 0, -2, -4)	63
8	(4, 2, -1, -2, -3)	63
9	(19, 4, -1, -6, -16)	64
10	(12, 2, -3, -3, -8)	64
11	(14, 4, -1, -6, -11)	65
12	(13, 8, -2, -7, -12)	66
13	(3, 2, 0, -2, -3)	67
14	(4, 1, 0, -1, -4)	67
15	(9, 4, -1, -6, -6)	68
16	(3, 2, 1, -2, -4)	69
17	(12, 7, 2, -8, -13)	69
18	(4, 2, 0, -1, -5)	69
19	(2, 1, 0, -1, -2)	69
20	(8, 3, -2, -2, -7)	69
21	(16, 11, 6, -9, -24)	70
22	(16, 6, 1, -4, -19)	70
23	(4, -1, -1, -1, -1)	70
24	(14, 9, 4, -6, -21)	71
25	(11, 6, 1, -4, -14)	71
26	(24, 9, 4, -1, -36)	73
27	(18, 8, 3, -2, -27)	73
28	(12, 7, 2, -3, -18)	73
29	(6, 1, 1, -4, -4)	73
30	(6, 6, 1, -4, -9)	73
31	(7, 2, 2, -3, -8)	74
32	(3, 3, -2, -2, -2)	75
33	(2, 1, 0, 0, -3)	76
34	(8, 3, 3, -2, -12)	76
35	(4, 4, -1, -1, -6)	76
36	(1, 0, 0, 0, -1)	80
37	(2, 2, 2, -3, -3)	81
38	(1, 1, 1, 1, -4)	91

3 Boundary families for the full $\mathrm{SL}(5)$ -action

In this section we pass from the maximal T -strictly semistable supports

$$S_k = I(r_k)_{\geq 0}, \quad k = 1, \dots, 38,$$

enumerated in Proposition 2.5 to boundary families for the full $\mathrm{SL}(5)$ -action. We keep the distinction between prequotient loci and quotient-side boundary families throughout the section.

Let

$$G = \mathrm{SL}(5), \quad W = \mathrm{Sym}^5 \mathbb{C}^5, \quad \mathcal{M}^{\mathrm{GIT}} = \mathbb{P}(W)^{ss} // G,$$

and let

$$\pi : \mathbb{P}(W)^{ss} \longrightarrow \mathcal{M}^{\mathrm{GIT}}$$

be the good quotient map. We write

$$\Sigma^{ss} := \mathbb{P}(W)^{ss} \setminus \mathbb{P}(W)^s$$

for the strictly semistable locus before taking the quotient, and

$$\partial \mathcal{M}^{\mathrm{GIT}} := \pi(\Sigma^{ss}) \subset \mathcal{M}^{\mathrm{GIT}}$$

for the GIT boundary in the quotient.

3.1 The prequotient and quotient-side families

For each maximal support S_k , set

$$V_k := \mathrm{Span}_{\mathbb{C}}\{x^u \mid u \in S_k\} \subset W.$$

We also use the notation

$$\mathbb{P}(V_k)^\circ := \{[f] \in \mathbb{P}(V_k) \mid \mathrm{Supp}(f) = S_k\}.$$

Thus $\mathbb{P}(V_k)^\circ$ is the open locus in the support family on which all monomials in S_k occur with nonzero coefficient. When we speak of a general quintic supported on S_k , we mean a point of a nonempty Zariski-open subset of $\mathbb{P}(V_k)^\circ$.

Define

$$Y_k := \overline{G \cdot \mathbb{P}(V_k)} \subset \mathbb{P}(W),$$

where the closure is taken in $\mathbb{P}(W)$. Since

$$S_k = I(r_k)_{\geq 0},$$

every point of $\mathbb{P}(V_k)$ is not stable with respect to the one-parameter subgroup λ_{r_k} . Hence every point of $G \cdot \mathbb{P}(V_k)$ is not stable for the G -action. As the stable locus is open, the closure Y_k is still contained in the non-stable locus:

$$Y_k \subset \mathbb{P}(W) \setminus \mathbb{P}(W)^s.$$

Consequently,

$$Y_k \cap \mathbb{P}(W)^{ss} = Y_k \cap \Sigma^{ss}.$$

We define the k -th prequotient boundary family by

$$\tilde{\Phi}_k := Y_k \cap \Sigma^{ss} = \overline{G \cdot \mathbb{P}(V_k)} \cap \Sigma^{ss} \subset \mathbb{P}(W)^{ss}.$$

The corresponding quotient-side boundary family is

$$\Phi_k := \pi(\tilde{\Phi}_k) \subset \partial \mathcal{M}^{\text{GIT}}.$$

Thus $\tilde{\Phi}_k$ is the G -saturated prequotient locus associated with the support S_k , whereas Φ_k is its image in the GIT quotient.

All inclusions among the symbols Φ_k are henceforth inclusions inside the quotient $\mathcal{M}^{\text{GIT}}$. In particular, a statement such as

$$\Phi_k \subset \Phi_\ell$$

is a quotient-side statement. Similarly, the notation

$$\dim \Phi_k$$

always denotes the dimension of the quotient-side family, not the dimension of the prequotient locus $\tilde{\Phi}_k$ inside $\mathbb{P}(W)^{ss}$.

3.2 Covering of the strictly semistable locus

The maximal supports from Proposition 2.5 cover all strictly semistable quintics after allowing an $\text{SL}(5)$ -change of coordinates. This gives the prequotient covering statement.

Proposition 3.1 (Covering by the maximal-support families). *For the natural action of $G = \text{SL}(5)$ on $\mathbb{P}(W)$, one has*

$$\Sigma^{ss} = \tilde{\Phi}_1 \cup \dots \cup \tilde{\Phi}_{38}.$$

Consequently,

$$\partial \mathcal{M}^{\text{GIT}} = \Phi_1 \cup \dots \cup \Phi_{38}.$$

Proof. The inclusion

$$\tilde{\Phi}_1 \cup \dots \cup \tilde{\Phi}_{38} \subset \Sigma^{ss}$$

is immediate from the definition of the prequotient families.

Conversely, let $[f] \in \Sigma^{ss}$. Since $[f]$ is not stable, the Hilbert–Mumford numerical criterion gives an element $\sigma \in G$ and a nonzero one-parameter subgroup

$$\lambda_r(t) = \text{diag}(t^{r_0}, \dots, t^{r_4}), \quad r = (r_0, \dots, r_4) \in \mathbb{Z}^5(0),$$

such that

$$\text{Supp}(\sigma \cdot f) \subset I(r)_{\geq 0}.$$

Because $[f]$ is semistable for G , the point $\sigma \cdot [f]$ is semistable for the fixed maximal torus T . Equivalently,

$$\eta = (1, 1, 1, 1, 1) \in \text{Conv}(\text{Supp}(\sigma \cdot f)).$$

Choose a maximal element of the finite collection

$$\{I(s)_{\geq 0} \mid 0 \neq s \in \mathbb{Z}^5(0)\}$$

which contains $\text{Supp}(\sigma \cdot f)$. Since this maximal support contains the support of a T -semistable form, its convex hull contains η . Hence, by Proposition 2.5, it is a coordinate permutation of one of the thirty-eight supports

$$S_1, \dots, S_{38}.$$

After composing σ with the corresponding coordinate change, we obtain some $g \in G$ and some k such that

$$g \cdot [f] \in \mathbb{P}(V_k).$$

Therefore

$$[f] \in G \cdot \mathbb{P}(V_k) \cap \Sigma^{ss} \subset \tilde{\Phi}_k.$$

This proves

$$\Sigma^{ss} \subset \tilde{\Phi}_1 \cup \dots \cup \tilde{\Phi}_{38}.$$

Applying the quotient map π gives

$$\partial \mathcal{M}^{\text{GIT}} = \pi(\Sigma^{ss}) = \Phi_1 \cup \dots \cup \Phi_{38}.$$

□

4 Quotient-side identifications from supporting hyperplanes

The thirty-eight labels

$$S_k = I(r_k)_{\geq 0}, \quad k = 1, \dots, 38,$$

record oriented half-space supports for the fixed diagonal torus. The orientation is relevant before taking the quotient: replacing r_k by $-r_k$ exchanges the two sides of the supporting hyperplane. The good quotient, however, retains only the weight-zero degeneration on the supporting hyperplane. We make this precise below and then apply it to the thirty-eight normal vectors of Proposition 2.5.

Throughout this section, put

$$X := \mathbb{P}(W), \quad X^{ss} := \mathbb{P}(W)^{ss},$$

and let

$$\pi : X^{ss} \longrightarrow \mathcal{M}^{\text{GIT}}$$

be the good quotient map. We use the standard facts that a good quotient maps closed G -invariant subsets to closed subsets and is constant on G -orbits; see, for example, [MFK94, Dol03].

4.1 The quotient image is determined by the weight-zero subspace

Let

$$0 \neq r = (r_0, \dots, r_4) \in \mathbb{Z}^5(0),$$

and write the corresponding weight decomposition as

$$W = \bigoplus_{m \in \mathbb{Z}} W_m(r), \quad W_m(r) := \text{Span}_{\mathbb{C}}\{x^u \mid r \cdot u = m\}.$$

Set

$$V(r) := \bigoplus_{m \geq 0} W_m(r) = \text{Span}_{\mathbb{C}}\{x^u \mid r \cdot u \geq 0\}$$

and

$$Z(r) := W_0(r) = \text{Span}_{\mathbb{C}}\{x^u \mid r \cdot u = 0\}.$$

If

$$H_r := \lambda_r(\mathbb{G}_m),$$

then

$$Z(r) = W^{H_r}.$$

Finally, define

$$Y(r) := \overline{G \cdot \mathbb{P}(V(r))} \subset X$$

and

$$\Phi(r) := \pi(Y(r) \cap X^{ss}) \subset \mathcal{M}^{\text{GIT}}.$$

Every point of $\mathbb{P}(V(r))$ is non-stable with respect to λ_r . Hence $Y(r)$ is contained in the non-stable locus, and

$$Y(r) \cap X^{ss} = Y(r) \cap \Sigma^{ss}.$$

For the normals in Proposition 2.5, one has

$$V(r_k) = V_k, \quad Y(r_k) = Y_k, \quad \Phi(r_k) = \Phi_k.$$

Lemma 4.1 (Zero-weight reduction in the good quotient). *For every nonzero $r \in \mathbb{Z}^5(0)$, one has*

$$\Phi(r) = \overline{\pi(\mathbb{P}(V(r)) \cap X^{ss})} = \overline{\pi(\mathbb{P}(Z(r)) \cap X^{ss})}, \quad (4)$$

where the closures are taken in $\mathcal{M}^{\text{GIT}}$.

Proof. Set

$$A(r) := \mathbb{P}(V(r)) \cap X^{ss}.$$

We first prove

$$\Phi(r) = \overline{\pi(A(r))}. \quad (5)$$

If $A(r) = \emptyset$, then $\mathbb{P}(V(r))$ is contained in the unstable locus, which is closed and G -invariant. Therefore $Y(r) \cap X^{ss} = \emptyset$, and (5) is immediate.

Assume now that $A(r) \neq \emptyset$. Since X^{ss} is open in X , the subset $A(r)$ is a nonempty open, hence dense, subset of the projective space $\mathbb{P}(V(r))$. Consequently,

$$Y(r) \cap X^{ss} = \overline{G \cdot A(r)}^{X^{ss}}.$$

Indeed, $G \times A(r)$ is dense in $G \times \mathbb{P}(V(r))$, so the action morphism shows that $G \cdot \mathbb{P}(V(r))$ is contained in the closure of $G \cdot A(r)$.

The subset $Y(r) \cap X^{ss}$ is closed and G -invariant in X^{ss} . Hence its image under the good quotient is closed. Since it contains $A(r)$, we obtain

$$\overline{\pi(A(r))} \subseteq \pi(Y(r) \cap X^{ss}).$$

Conversely, continuity of π , together with its G -invariance, gives

$$\begin{aligned} \pi(Y(r) \cap X^{ss}) &\subseteq \overline{\pi(G \cdot A(r))} \\ &= \overline{\pi(A(r))}. \end{aligned}$$

This proves (5).

It remains to replace $V(r)$ by its weight-zero subspace. Let $[f] \in A(r)$, and write

$$f = f_0 + f_{>0}, \quad f_0 \in Z(r), \quad f_{>0} \in \bigoplus_{m>0} W_m(r).$$

By (3),

$$\lim_{t \rightarrow 0} \lambda_r(t) \cdot f = f_0$$

in the affine space W . Since $[f]$ is semistable, there exists a homogeneous invariant

$$F \in \mathbb{C}[W]^G$$

of positive degree such that $F(f) \neq 0$. The invariance of F gives

$$F(f_0) = \lim_{t \rightarrow 0} F(\lambda_r(t) \cdot f) = F(f) \neq 0.$$

Thus $f_0 \neq 0$ and $[f_0] \in X^{ss}$.

The morphism

$$\mathbb{G}_m \longrightarrow X^{ss}, \quad t \longmapsto \lambda_r(t) \cdot [f],$$

extends to $\mathbb{A}^1$ with value $[f_0]$ at 0. Its composition with π is constant on $\mathbb{G}_m$. Since $\mathcal{M}^{\text{GIT}}$ is separated, the composition is constant on $\mathbb{A}^1$, and therefore

$$\pi([f]) = \pi([f_0]).$$

It follows that

$$\pi(\mathbb{P}(V(r)) \cap X^{ss}) \subseteq \pi(\mathbb{P}(Z(r)) \cap X^{ss}).$$

The reverse inclusion follows from $Z(r) \subseteq V(r)$. Hence the two images are equal. Combining this equality with (5) proves (4). $\square$

Corollary 4.2 (Supporting-hyperplane invariance). *Let $0 \neq r, s \in \mathbb{Z}^5(0)$. If there exists $g \in G$ such that*

$$Z(s) = gZ(r),$$

then

$$\Phi(s) = \Phi(r).$$

In particular:

1. *one has*

$$\Phi(r) = \Phi(-r);$$

2. *if*

$$H_s = gH_rg^{-1}$$

for some $g \in G$, then

$$\Phi(s) = \Phi(r).$$

Thus the quotient-side family is determined by the zero-weight monomial subspace, up to the G -action.

Proof. The element g preserves the semistable locus, and the quotient map satisfies $\pi(g \cdot x) = \pi(x)$. Hence

$$\pi(\mathbb{P}(Z(s)) \cap X^{ss}) = \pi(\mathbb{P}(Z(r)) \cap X^{ss}).$$

Taking closures and applying Lemma 4.1 gives $\Phi(s) = \Phi(r)$.

For the first special case, one has

$$Z(-r) = Z(r).$$

For the second, conjugacy of the one-dimensional subtori gives

$$Z(s) = W^{H_s} = gW^{H_r} = gZ(r).$$

Both assertions therefore follow from the general statement. $\square$

Remark 4.3. Lemma 4.1 applies to every semistable point of $\mathbb{P}(V(r))$, not only to a general full-support form or to a chosen normal-form locus. Thus the quotient-side identifications below do not require a case-by-case comparison of the closed-orbit normal forms.

4.2 The involution on the thirty-eight normal vectors

Let

$$\rho : \quad x_0 \mapsto x_4, \quad x_1 \mapsto x_3, \quad x_2 \mapsto x_2, \quad x_3 \mapsto x_1, \quad x_4 \mapsto x_0.$$

The underlying permutation is $(0\ 4)(1\ 3)$, so

$$\det(\rho) = 1$$

and hence $\rho \in G = \mathrm{SL}(5)$.

For

$$r = (r_0, r_1, r_2, r_3, r_4),$$

define

$$\iota(r) := (-r_4, -r_3, -r_2, -r_1, -r_0).$$

Conjugation by ρ reverses the entries of the weight vector:

$$\rho \lambda_r(t) \rho^{-1} = \lambda_{(r_4, r_3, r_2, r_1, r_0)}(t).$$

Since a one-parameter subgroup and its inverse have the same image,

$$H_{(r_4, r_3, r_2, r_1, r_0)} = H_{\iota(r)}.$$

Consequently,

$$\Phi(r) = \Phi(\iota(r)) \tag{6}$$

by Corollary 4.2.

For the dominant normal vectors in Proposition 2.5, the involution ι has the seventeen two-cycles displayed in Table 6.

Table 6: The seventeen opposite-half-space pairs.

pair (k, k')	r_k	$\iota(r_k) = r_{k'}$
(1, 26)	(36, 1, -4, -9, -24)	(24, 9, 4, -1, -36)
(2, 27)	(27, 2, -3, -8, -18)	(18, 8, 3, -2, -27)
(3, 33)	(3, 0, 0, -1, -2)	(2, 1, 0, 0, -3)
(4, 21)	(24, 9, -6, -11, -16)	(16, 11, 6, -9, -24)
(5, 24)	(21, 6, -4, -9, -14)	(14, 9, 4, -6, -21)
(6, 28)	(18, 3, -2, -7, -12)	(12, 7, 2, -3, -18)
(7, 18)	(5, 1, 0, -2, -4)	(4, 2, 0, -1, -5)
(8, 16)	(4, 2, -1, -2, -3)	(3, 2, 1, -2, -4)
(9, 22)	(19, 4, -1, -6, -16)	(16, 6, 1, -4, -19)
(10, 34)	(12, 2, -3, -3, -8)	(8, 3, 3, -2, -12)
(11, 25)	(14, 4, -1, -6, -11)	(11, 6, 1, -4, -14)
(12, 17)	(13, 8, -2, -7, -12)	(12, 7, 2, -8, -13)
(15, 30)	(9, 4, -1, -6, -6)	(6, 6, 1, -4, -9)
(20, 31)	(8, 3, -2, -2, -7)	(7, 2, 2, -3, -8)
(23, 38)	(4, -1, -1, -1, -1)	(1, 1, 1, 1, -4)
(29, 35)	(6, 1, 1, -4, -4)	(4, 4, -1, -1, -6)
(32, 37)	(3, 3, -2, -2, -2)	(2, 2, 2, -3, -3)

For every row (k, k') of Table 6, equation (6) gives

$$\Phi_k = \Phi(r_k) = \Phi(\iota(r_k)) = \Phi(r_{k'}) = \Phi_{k'}. \tag{7}$$

We record all seventeen equalities in one proposition.

Proposition 4.4 (Opposite-half-space identifications). *One has*

$$\begin{aligned}\Phi_1 &= \Phi_{26}, & \Phi_2 &= \Phi_{27}, & \Phi_3 &= \Phi_{33}, & \Phi_4 &= \Phi_{21}, & \Phi_5 &= \Phi_{24}, & \Phi_6 &= \Phi_{28}, \\ \Phi_7 &= \Phi_{18}, & \Phi_8 &= \Phi_{16}, & \Phi_9 &= \Phi_{22}, & \Phi_{10} &= \Phi_{34}, & \Phi_{11} &= \Phi_{25}, & \Phi_{12} &= \Phi_{17}, \\ \Phi_{15} &= \Phi_{30}, & \Phi_{20} &= \Phi_{31}, & \Phi_{23} &= \Phi_{38}, & \Phi_{29} &= \Phi_{35}, & \Phi_{32} &= \Phi_{37}.\end{aligned}$$

Proof. Apply (7) to the seventeen rows of Table 6. $\square$

The four remaining dominant normals are fixed by ι :

$$\begin{aligned}r_{13} &= (3, 2, 0, -2, -3), & r_{14} &= (4, 1, 0, -1, -4), \\ r_{19} &= (2, 1, 0, -1, -2), & r_{36} &= (1, 0, 0, 0, -1).\end{aligned}$$

Thus the involution on the thirty-eight dominant half-space labels has seventeen two-cycles and four fixed points. Equivalently, the thirty-eight oriented half-space supports arise from twenty-one supporting hyperplanes up to coordinate permutation and reversal of the normal.

Corollary 4.5 (Reduction to twenty-one candidate quotient images). *The GIT boundary is covered by the following twenty-one quotient images:*

$$\Phi_1, \Phi_2, \dots, \Phi_{15}, \Phi_{19}, \Phi_{20}, \Phi_{23}, \Phi_{29}, \Phi_{32}, \Phi_{36}.$$

Equivalently,

$$\begin{aligned}\partial\mathcal{M}^{\text{GIT}} &= \Phi_1 \cup \Phi_2 \cup \dots \cup \Phi_{15} \cup \Phi_{19} \cup \Phi_{20} \\ &\quad \cup \Phi_{23} \cup \Phi_{29} \cup \Phi_{32} \cup \Phi_{36}.\end{aligned}\tag{8}$$

Proof. Proposition 3.1 gives the cover by all thirty-eight support-indexed images. Replacing the second member of each two-cycle in Table 6 by the first member and applying Proposition 4.4 gives (8). $\square$

Remark 4.6. At this stage the twenty-one entries are quotient images selected from the support-indexed cover. Section 5 constructs a nonempty polystable locus in each entry and proves that these images are closed and irreducible. Section 6 then excludes all further equalities and containments.

5 The twenty-one closed-orbit families

The preceding section reduces the boundary cover from thirty-eight oriented support labels to twenty-one quotient images. From this point through the end of the main body we use the component index $a = 1, \dots, 21$. We choose

$$k(a) = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 19, 20, 23, 29, 32, 36$$

in that order, and put

$$\Theta_a := \Phi_{k(a)}, \quad F_a := \phi_{k(a)}^{\text{nf}}, \quad X_a := V(F_a) \subset \mathbb{P}^4.$$

Let k' be the partner of $k(a)$ in a two-cycle, and let

$$\rho(x_0, x_1, x_2, x_3, x_4) = (x_4, x_3, x_2, x_1, x_0).$$

Then ρ identifies the zero-weight spaces,

$$Z(r_{k'}) = \rho Z(r_{k(a)}),$$

and Corollary 4.2 gives $\Phi_{k'} = \Phi_{k(a)} = \Theta_a$. Thus one selected support label suffices at the level of quotient images. This statement does not assert that ρ carries the two displayed normal-form formulas coefficientwise into one another: residual centralizer normalization and a reparametrization may still be required after applying ρ . The relation between the paired normal-form loci and the general closed orbit of Θ_a is recorded precisely after Corollary 5.2.

5.1 Centralizer reduction and closed-orbit criteria

For an original support case k , let f_k be a general quintic supported on $S_k = I(r_k)_{\geq 0}$, and set

$$\phi_k := \lim_{t \rightarrow 0} \lambda_{r_k}(t) \cdot f_k, \quad H_k := \lambda_{r_k}(\mathbb{G}_m), \quad C_k := C_G(H_k).$$

The limit ϕ_k is the weight-zero part of f_k and belongs to the fixed space W^{H_k} . Since H_k is a torus, C_k is reductive; since H_k acts trivially on W^{H_k} , the action factors through C_k/H_k . Affine Luna reduction gives

$$G \cdot \phi_k \text{ is closed in } W \iff C_k \cdot \phi_k \text{ is closed in } W^{H_k}. \quad (9)$$

When the effective centralizer C_k/H_k is a torus, we use the standard relative-interior criterion: for a torus acting linearly on a vector space, the orbit of a vector is closed if and only if the origin lies in the relative interior of the convex hull of the characters occurring with nonzero coefficient. We abbreviate this test by C-H.

For the non-toric centralizers we use the following criterion. Let R be a complex reductive algebraic group acting on an affine variety X , and write

$$Y(R) := \text{Hom}_{\text{alg. grp}}(\mathbb{G}_m, R).$$

For $\eta \in Y(R)$, put

$$P_R(\eta) := \left\{ g \in R \mid \lim_{t \rightarrow 0} \eta(t) g \eta(t)^{-1} \text{ exists in } R \right\}.$$

Two one-parameter subgroups $\eta, \eta' \in Y(R)$ are called *opposite* if $P_R(\eta) \cap P_R(\eta')$ is a Levi subgroup of both parabolic subgroups. For $x \in X$, define

$$\Lambda_x^R := \left\{ \eta \in Y(R) \mid \lim_{t \rightarrow 0} \eta(t) \cdot x \text{ exists in } X \right\}.$$

A subset of $Y(R)$ is *symmetric* if every member has an opposite member in the subset.

Theorem 5.1 (Casimiro–Florentino criterion). *Let R be a complex reductive algebraic group acting on an affine variety X , and let $x \in X$. Then*

$$R \cdot x \text{ is closed in } X \iff \Lambda_x^R \text{ is symmetric}.$$

This is [CF12, Theorem 4.9]. The inverse one-parameter subgroup $\eta^{-1}(t) = \eta(t)^{-1}$ is opposite to η , because

$$P_R(\eta) \cap P_R(\eta^{-1}) = C_R(\eta(\mathbb{G}_m))$$

is a Levi subgroup of both parabolics. Hence invariance of Λ_x^R under $\eta \mapsto \eta^{-1}$ is sufficient for closedness. We abbreviate this test by C-F. In the support calculations of Appendix A, generic nonvanishing, discriminant, resultant, and residual-stability conditions combine with linear balancing identities to prove the required inversion symmetry.

More precisely, in a non-toric support case we start with an arbitrary

$$\eta \in \Lambda_{\phi_k}^{C_k},$$

diagonalize η inside the repeated-weight blocks of C_k , and translate existence of the affine limit into nonnegativity conditions on the weights of the monomials occurring in ϕ_k . The genericity conditions, the determinant-one relation, and the resulting balancing identities force η to lie in the distinguished torus H_k , which acts trivially on W^{H_k} . In each non-toric case the conclusion has the form

$$\Lambda_{\phi_k}^{C_k} = \{ \lambda_{r_k}(t^m) \mid m \in \mathbb{Z} \}.$$

This set is invariant under $m \mapsto -m$, hence symmetric. It follows that $C_k \cdot \phi_k$ is closed in W^{H_k} , and (9) then gives that $G \cdot \phi_k$ is closed in W .

After closedness has been established, the residual action of C_k , together with projective rescaling, is used to eliminate redundant coefficients and obtain an explicit normal form ϕ_k^{nf} . Thus every support calculation in Appendix A has three steps: extraction of the weight-zero limit, proof of polystability, and reduction to normal form. The support-by-support organization is retained in that appendix because these operations depend on the original thirty-eight maximal supports; the main body records only one representative for each quotient component.

5.2 Component representatives and dimensions

Table 7 records one general polystable weight-zero representative for each quotient image, the closed-orbit test, and its quotient dimension. The full support-indexed list and every case-by-case verification are given in Appendix A.

Let us put

$$B_\lambda(u, v) := uv(u - v)(u - \lambda v), \quad B_{\lambda, \mu}(u, v) := v(u - v)(u - \lambda v)(u - \mu v).$$

Here B_d, C_d, Q_d, L_d, M_d denote general forms of the indicated degree, and $Q_{2,2}$ denotes a bihomogeneous form of bidegree $(2, 2)$. All parameters are understood to lie in the relevant nonempty Zariski-open generic locus.

Table 7: Closed-orbit representatives for the twenty-one quotient components. The second column records the orbit of the oriented support label under reversal of the supporting half-space; a singleton is a fixed case. The displayed normal form is attached to the selected representative label, and no coefficientwise identification of paired displayed formulas by coordinate reversal is asserted. The tests C-H and C-F are the relative-interior and Casimiro–Florentino criteria described in this section.

a	support orbit	Test	Representative $F_a = \phi_{k(a)}^{\text{nf}}$	Parameters	d_a
1	$1 \leftrightarrow 26$	C-H	$F_1 = x_1^4 x_2 + x_0 x_3^4 + x_0 x_2^3 x_4$ $+ x_0 x_1 x_2 x_3 x_4 + \alpha x_0^2 x_4^3$	$\alpha \in \mathbb{C}^\times$	1
2	$2 \leftrightarrow 27$	C-H	$F_2 = x_1^3 x_2^2 + x_1^4 x_3 + x_0 x_2 x_3^3 + x_0 x_2^3 x_4$ $+ \alpha x_0 x_1 x_2 x_3 x_4 + \beta x_0^2 x_4^3$	$\alpha, \beta \in \mathbb{C}^\times$	2
3	$3 \leftrightarrow 33$	C-F	$F_3 = B_5(x_1, x_2) + x_0 x_2 x_3^3$ $+ x_0(x_1^2 + x_2^2)x_3 x_4 + x_0^2 x_4^3$	B_5 : general binary quintic	6
4	$4 \leftrightarrow 21$	C-H	$F_4 = x_1^2 x_2^3 + x_1^3 x_3 x_4 + x_0 x_2^4 + x_0 x_1 x_3^3$ $+ \alpha x_0 x_1 x_2 x_3 x_4 + \beta x_0^2 x_4^3$	$\alpha, \beta \in \mathbb{C}^\times$	2
5	$5 \leftrightarrow 24$	C-H	$F_5 = x_1^2 x_2^3 + x_1^3 x_3^2 + x_1^3 x_2 x_4 + x_0 x_2^3 x_3$ $+ \alpha x_0 x_1 x_3^3 + \beta x_0 x_1 x_2 x_3 x_4 + \gamma x_0^2 x_4^3$	$\alpha, \beta, \gamma \in \mathbb{C}^\times$	3
6	$6 \leftrightarrow 28$	C-H	$F_6 = x_1^2 x_2^3 + x_1^3 x_2 x_3 + x_1^4 x_4 + x_0 x_2^2 x_3^2$ $+ \alpha x_0 x_2^3 x_4 + \beta x_0 x_1 x_3^3 + \gamma x_0 x_1 x_2 x_3 x_4$ $+ \delta x_0 x_1^2 x_4^2 + \epsilon x_0^2 x_4^3$	$\alpha, \beta, \gamma, \delta,$ $\epsilon \in \mathbb{C}^\times$	5
7	$7 \leftrightarrow 18$	C-H	$F_7 = x_2^5 + x_1^2 x_2^2 x_3 + x_1^4 x_4 + x_0 x_1 x_3^3$ $+ \alpha x_0 x_1 x_2 x_3 x_4 + \beta x_0^2 x_3 x_4^2$	$\alpha, \beta \in \mathbb{C}^\times$	2
8	$8 \leftrightarrow 16$	C-H	$F_8 = x_1^2 x_2^2 x_3 + x_1^3 x_4^2 + x_0 x_4^4 + x_0 x_1 x_3^3$ $+ \alpha x_0 x_1 x_2 x_3 x_4 + \beta x_0^2 x_3 x_4^2$	$\alpha, \beta \in \mathbb{C}^\times$	2

continued on the next page

a	support orbit	Test	Representative $F_a = \phi_{k(a)}^{\text{nf}}$	Parameters	d_a
9	$9 \leftrightarrow 22$	C-H	$F_9 = x_1x_2^4 + x_1^2x_2^2x_3 + \alpha x_1^3x_3^2 + x_1^4x_4$ $+ x_0x_2x_3^3 + \beta x_0x_2^3x_4 + \gamma x_0x_1x_2x_3x_4$ $+ \delta x_0^2x_3x_4^2$	$\alpha, \beta, \gamma, \delta \in \mathbb{C}^\times$	4
10	$10 \leftrightarrow 34$	C-F	$F_{10} = x_1^3Q_{\mathbf{q}}(x_2, x_3) + x_1^4x_4 + x_0B_{\lambda}(x_2, x_3)$ $+ x_0x_1x_4R_{\mathbf{r}}(x_2, x_3) + \beta x_0x_1^2x_4^2 + x_0^2x_4^3$	$\lambda \in \mathbb{C} \setminus \{0, 1\};$ $\mathbf{q}, \mathbf{r} \in \mathbb{C}^3, \beta \in \mathbb{C}$	8
11	$11 \leftrightarrow 25$	C-H	$F_{11} = x_1x_2^4 + \alpha x_1^2x_2^2x_3 + \beta x_1^3x_3^2 + x_1^3x_2x_4$ $+ \gamma x_0x_2^2x_3^2 + x_0x_2^3x_4 + \delta x_0x_1x_3^3$ $+ \epsilon x_0x_1x_2x_3x_4 + \zeta x_0x_1^2x_4^2 + x_0^2x_3x_4^2$	$\alpha, \beta, \gamma, \delta,$ $\epsilon, \zeta \in \mathbb{C}^\times$	6
12	$12 \leftrightarrow 17$	C-H	$F_{12} = x_1x_2^4 + \alpha x_1^2x_2^2x_3 + \beta x_1^2x_2^2x_4 + x_1^3x_4^2$ $+ x_0x_2^3x_3 + \gamma x_0x_1x_3^3 + \delta x_0x_1x_2x_3x_4$ $+ \epsilon x_0^2x_3^2x_4 + x_0^2x_2x_4^2$	$\alpha, \beta, \gamma, \delta,$ $\epsilon \in \mathbb{C}^\times$	5
13	13	C-H	$F_{13} = x_2^5 + \alpha x_1x_2^3x_3 + \beta x_1^2x_2^2x_3^2 + x_1^3x_4^2$ $+ x_0x_2^3x_4 + \gamma x_0x_1x_2x_3x_4 + x_0^2x_3^3$ $+ \delta x_0^2x_2x_4^2$	$\alpha, \beta, \gamma, \delta \in \mathbb{C}^\times$	4
14	14	C-H	$F_{14} = x_2^5 + \alpha x_1x_2^3x_3 + \beta x_1^2x_2^2x_3^2 + x_1^4x_4 + x_0x_3^4$ $+ \gamma x_0x_2^3x_4 + \delta x_0x_1x_2x_3x_4 + x_0^2x_2x_4^2$	$\alpha, \beta, \gamma, \delta \in \mathbb{C}^\times$	4
15	$15 \leftrightarrow 30$	C-F	$F_{15} = x_1x_2^4 + x_1^2x_2^2x_3$ $+ x_1^3(\gamma x_3^2 + \beta x_3x_4 + x_4^2) + x_0x_2^3x_4$ $+ x_0x_1x_2R(x_3, x_4) + x_0^2B(x_3, x_4)$	$\gamma, \beta \in \mathbb{C};$ $R \in \text{Sym}^2\langle x_3, x_4 \rangle;$ $B \in \text{Sym}^3\langle x_3, x_4 \rangle$	9
16	19	C-H	$F_{16} = x_2^5 + \alpha x_1x_2^3x_3 + \beta x_1^2x_2^2x_3^2 + \gamma x_1^2x_2^2x_4$ $+ x_1^3x_3x_4 + \delta x_0x_2^2x_3^2 + \epsilon x_0x_2^3x_4 + x_0x_1x_3^3$ $+ \zeta x_0x_1x_2x_3x_4 + \xi x_0x_1^2x_4^2 + x_0^2x_3^2x_4$ $+ \theta x_0^2x_2x_4^2$	$\alpha, \beta, \gamma, \delta,$ $\epsilon, \zeta, \xi, \theta \in \mathbb{C}^\times$	8
17	$20 \leftrightarrow 31$	C-F	$F_{17} = x_0B_{\lambda, \mu}(x_2, x_3) + x_1^2C_{\mathbf{c}}(x_2, x_3) + x_1^3x_2x_4$ $+ x_0x_1x_4Q_{\mathbf{q}}(x_2, x_3) + x_0x_1^2x_4^2$ $+ x_0^2x_4^2M_{\mathbf{m}}(x_2, x_3)$	λ, μ distinct in $\mathbb{C} \setminus \{0, 1\};$ $\mathbf{c} \in \mathbb{C}^4, \mathbf{q} \in \mathbb{C}^3,$ $\mathbf{m} \in \mathbb{C}^2$	11
18	$23 \leftrightarrow 38$	C-F	$F_{18} = x_0B_4(x_1, x_2, x_3, x_4)$	B_4 : smooth quartic surface in $\mathbb{P}^3$	19
19	$29 \leftrightarrow 35$	C-F	$F_{19} = x_3A_4(x_1, x_2) + x_4C_4(x_1, x_2)$ $+ x_0Q_{2,2}(x_1, x_2; x_3, x_4) + x_0^2x_3x_4(x_3 - x_4)$	A_4, C_4 : binary quartics; $Q_{2,2}$ of bidegree $(2, 2)$	15
20	$32 \leftrightarrow 37$	C-F	$F_{20} = x_0^2C_0(x_2, x_3, x_4) + x_0x_1C_1(x_2, x_3, x_4)$ $+ x_1^2C_2(x_2, x_3, x_4)$	C_0, C_1, C_2 : ternary cubics	18
21	36	C-F	$F_{21} = B_5(x_1, x_2, x_3) + x_0x_4C_3(x_1, x_2, x_3)$ $+ x_0^2x_3x_4^2$	B_5, C_3 : ternary forms of degrees 5, 3	24

Corollary 5.2 (Twenty-one closed irreducible quotient-side families). *The twenty-one quotient images listed in Table 7 are nonempty closed irreducible subvarieties and cover the GIT boundary:*

$$\partial \mathcal{M}^{\text{GIT}} = \Theta_1 \cup \dots \cup \Theta_{21},$$

The subvarieties are indexed by $\Theta_a = \Phi_{k(a)}$, as fixed above, and their dimensions are the numbers d_a in the last column of the table.

Proof. Appendix A constructs, for every oriented support case, a nonempty open family of weight-zero limits with closed G -orbit and reduces it to the displayed normal form. Thus every

selected $\Phi_{k(a)}$ is nonempty. The variety $Y_{k(a)} \cap X^{ss}$ is a nonempty irreducible closed G -invariant subset of X^{ss} ; its image under the good quotient is therefore closed and irreducible. The covering statement is Corollary 4.5, and the dimension calculations are those in Appendix A. $\square$

Remark 5.3 (Paired normal-form loci). Let k' be paired with $k(a)$. The constructions in Appendix A start from nonempty open full-support loci for both labels and reduce their weight-zero limits to the respective displayed normal forms. The quotient images of these normal-form loci are therefore dense constructible subsets of $\Phi_{k'} = \Phi_{k(a)} = \Theta_a$. Since every fiber of a good quotient contains a unique closed orbit, a general point of Θ_a has that closed orbit represented, up to projective equivalence, by a general member of the selected family F_a . No coefficientwise identification of the two displayed formulas by ρ is asserted or needed.

Remark 5.4. This corollary is still a covering statement. Pairwise non-inclusion, and therefore the assertion that the Θ_a are exactly the irreducible components, is proved in the next section.

6 Generic stabilizers, non-inclusion, and the component theorem

Table 7 fixes the representative label $k(a)$ and records the dimension $d_a = \dim \Theta_a$ for every component. The dimensions are retained as part of the classification, but they are not needed for pairwise non-inclusion: the stabilizer argument below treats all ordered pairs simultaneously.

By Corollary 5.2,

$$\partial \mathcal{M}^{\text{GIT}} = \Theta_1 \cup \dots \cup \Theta_{21}. \quad (10)$$

We prove that this cover is already the irreducible-component decomposition.

Theorem 6.1 (Pairwise non-inclusion and boundary components). *For any two distinct indices $a, b \in \{1, \dots, 21\}$, one has*

$$\Theta_a \not\subseteq \Theta_b$$

inside $\mathcal{M}^{\text{GIT}}$. Consequently, $\Theta_1, \dots, \Theta_{21}$ are precisely the irreducible components of $\partial \mathcal{M}^{\text{GIT}}$; in particular, the strictly semistable GIT boundary has exactly twenty-one irreducible components.

The proof has three ingredients. First, the generic connected stabilizer along Θ_a is the one-dimensional torus defined by its supporting hyperplane. Second, these twenty-one tori are pairwise nonconjugate in G . Third, Luna's étale slice theorem implies that an inclusion $\Theta_a \subseteq \Theta_b$ would force a conjugate of the generic connected stabilizer of Θ_b to be contained in that of Θ_a . Since both are connected one-dimensional tori, such an inclusion is an equality, yielding the required contradiction.

6.1 Generic connected stabilizers

For the representative case $k(a)$ fixed in Table 7, set

$$r_a := r_{k(a)}, \quad H_a := \lambda_{r_a}(\mathbb{G}_m) \subset G. \quad (11)$$

Every monomial occurring in the Case $k(a)$ normal form has r_a -weight zero. Hence H_a fixes every member of that normal-form family as a vector, not merely projectively.

Let $\mathfrak{g} := \mathfrak{sl}_5$. For a nonzero quintic $F \in W = \text{Sym}^5 \mathbb{C}^5$, consider the linear map

$$\mathcal{D}_F : \mathfrak{g} \oplus \mathbb{C} \longrightarrow W, \quad (A, c) \longmapsto A \cdot F - cF, \quad (12)$$

where $A \cdot F$ is the infinitesimal action associated with the action convention in (1). We use the following basis of $\mathfrak{g}$:

$$x_i \frac{\partial}{\partial x_j} \quad (i \neq j), \quad x_i \frac{\partial}{\partial x_i} - x_4 \frac{\partial}{\partial x_4} \quad (0 \leq i \leq 3),$$

and append the projective-scaling column $-F$. With respect to the 126 degree-five monomials, $\mathcal{D}_F$ is represented by a 126×25 matrix.

Lemma 6.2. *For every nonzero quintic F , projection to the first factor induces an isomorphism*

$$\ker(\mathcal{D}_F) \xrightarrow{\sim} \text{Lie}(\text{Stab}_G([F])).$$

Consequently,

$$\dim \text{Stab}_G([F]) = 25 - \text{rank}(\mathcal{D}_F). \quad (13)$$

Proof. An element $A \in \mathfrak{g}$ is tangent to the projective stabilizer of $[F]$ precisely when $A \cdot F \in \mathbb{C}F$. This is equivalent to the existence of $c \in \mathbb{C}$ such that $\mathcal{D}_F(A, c) = 0$. Since $F \neq 0$, the scalar c is uniquely determined by A , so projection to $\mathfrak{g}$ is an isomorphism onto the projective stabilizer Lie algebra. Over $\mathbb{C}$, the stabilizer is a smooth algebraic group, and its dimension equals the dimension of its Lie algebra, giving (13). $\square$

For a primitive vector $r \in \mathbb{Z}^5(0)$, define its unsigned permutation signature by

$$\nu(r) := \min_{\text{lex}} \{\text{sort}(r), \text{sort}(-r)\}, \quad (14)$$

where sort denotes increasing rearrangement.

Proposition 6.3 (Generic connected stabilizers). *For a general closed-orbit normal form representing Θ_a , one has*

$$\text{Stab}_G([F])^\circ = H_a \quad (1 \leq a \leq 21).$$

Equivalently, the generic connected stabilizer along Θ_a is the conjugacy class of H_a .

Proof. Let P_a be the irreducible coefficient space of the displayed Case $k(a)$ normal form before deleting the genericity discriminants, and let $F_a(\mathbf{u})$ denote the corresponding universal form. Because all monomials of $F_a(\mathbf{u})$ have r_a -weight zero, the infinitesimal generator

$$R_a := \text{diag}(r_{a,0}, \dots, r_{a,4}) \in \mathfrak{sl}_5$$

satisfies $R_a \cdot F_a(\mathbf{u}) = 0$ identically on P_a . Thus

$$\text{rank } \mathcal{D}_{F_a(\mathbf{u})} \leq 24 \quad \text{for every } \mathbf{u} \in P_a. \quad (15)$$

We now certify that equality occurs. Fix

$$p = 1,000,003.$$

For each a , the ancillary file `filter2_certificates.json` records the following exact data:

1. an integral specialization F_a^* of the Case $k(a)$ normal form;
2. twenty-four row indices and twenty-four column indices in the integer matrix of $\mathcal{D}_{F_a^*}$; and
3. the determinant δ_a of the selected 24×24 submatrix, reduced modulo p .

The residues δ_a are listed in Table 8, and all are nonzero. Since the selected submatrices have integer entries,

$$\delta_a \not\equiv 0 \pmod{p} \implies \det(C_a) \neq 0 \text{ in } \mathbb{Z}.$$

Therefore the corresponding 24×24 minor of the universal matrix $\mathcal{D}_{F_a(\mathbf{u})}$ is a nonzero polynomial on P_a . Hence $\text{rank } \mathcal{D}_{F_a(\mathbf{u})} \geq 24$ on a nonempty Zariski-open subset of P_a . Together with (15), this gives

$$\text{rank } \mathcal{D}_{F_a(\mathbf{u})} = 24$$

for general $\mathbf{u} \in P_a$.

The integral test point F_a^* need not itself satisfy every genericity condition imposed in the normal-form table. Indeed, the rank-24 locus is a nonempty open subset of the irreducible space P_a , while the general closed-orbit locus constructed case by case in Appendix A and summarized in Section 5 is another nonempty open subset of P_a ; the two open subsets therefore intersect.

By Lemma 6.2, the general projective stabilizer has dimension one. It contains the connected one-dimensional torus H_a , so its identity component is exactly H_a . Finally, the quotient image of the general normal-form locus is a dense constructible subset of Θ_a , hence contains a dense open subset; this proves the assertion about the generic connected stabilizer along Θ_a . $\square$

Table 8: Exact modular certificates for the generic projective stabilizer computation. Here C_a is the selected 24×24 integer submatrix of $\mathcal{D}_{F_a^*}$, and $p = 1,000,003$.

a	$k(a)$	$\nu(r_a)$	$\det(C_a) \bmod p$
1	1	(-36, -1, 4, 9, 24)	28050
2	2	(-27, -2, 3, 8, 18)	818759
3	3	(-3, 0, 0, 1, 2)	341126
4	4	(-24, -9, 6, 11, 16)	749212
5	5	(-21, -6, 4, 9, 14)	871295
6	6	(-18, -3, 2, 7, 12)	515005
7	7	(-5, -1, 0, 2, 4)	419273
8	8	(-4, -2, 1, 2, 3)	226555
9	9	(-19, -4, 1, 6, 16)	925602
10	10	(-12, -2, 3, 3, 8)	658896
11	11	(-14, -4, 1, 6, 11)	266577
12	12	(-13, -8, 2, 7, 12)	910121
13	13	(-3, -2, 0, 2, 3)	157960
14	14	(-4, -1, 0, 1, 4)	506192
15	15	(-9, -4, 1, 6, 6)	882759
16	19	(-2, -1, 0, 1, 2)	843397
17	20	(-8, -3, 2, 2, 7)	538703
18	23	(-4, 1, 1, 1, 1)	247964
19	29	(-6, -1, -1, 4, 4)	133397
20	32	(-3, -3, 2, 2, 2)	392541
21	36	(-1, 0, 0, 0, 1)	206888

Remark 6.4 (Verification of the certificates). The script `filter2_stabilizer.py` reconstructs every specialized normal form and every 126×25 matrix from the data in `filter2_certificates.json`. It verifies the weight-zero condition, the infinitesimal kernel vector, each displayed nonzero modular determinant, and the pairwise distinct signatures. The independent script `independent_sympy_check.py` reconstructs the matrices by symbolic differentiation and verifies $\text{rank}_{\mathbb{Q}} \mathcal{D}_{F_a^*} = 24$ for all twenty-one cases. No floating-point rank computation is used.

6.2 Conjugacy classes of the generic tori

Lemma 6.5. *Let $r, s \in \mathbb{Z}^5(0)$ be primitive. The one-dimensional subtori $\lambda_r(\mathbb{G}_m)$ and $\lambda_s(\mathbb{G}_m)$ of $\text{SL}(5)$ are conjugate if and only if*

$$\nu(r) = \nu(s).$$

Proof. Because r and s are primitive, the cocharacters $\lambda_r : \mathbb{G}_m \rightarrow \lambda_r(\mathbb{G}_m)$ and $\lambda_s : \mathbb{G}_m \rightarrow \lambda_s(\mathbb{G}_m)$ are isomorphisms onto their images. A conjugacy between the two image tori therefore induces an algebraic-group automorphism of $\mathbb{G}_m$, necessarily $t \mapsto t$ or $t \mapsto t^{-1}$. The restrictions of the standard five-dimensional representation to the two tori must consequently have the same multiset of weights, up to a common sign. This is precisely the equality $\nu(r) = \nu(s)$.

Conversely, equality of the signatures gives a permutation of the coordinate weight spaces carrying the weights of r to those of s , possibly after replacing s by $-s$. A permutation matrix realizes this conjugacy in $\text{GL}(5)$; if its determinant is not one, multiply it by a diagonal matrix

centralizing the torus and having the inverse determinant. The resulting conjugating matrix lies in $\mathrm{SL}(5)$. $\square$

Corollary 6.6. *The tori $H_1, \dots, H_{21}$ are pairwise nonconjugate in G .*

Proof. The twenty-one signatures in Table 8 are pairwise distinct, so the claim follows from Lemma 6.5. $\square$

6.3 The Luna closure obstruction

We now isolate the specialization statement needed for the component proof. The saturation step below ensures that the stabilizer-subgroup property holds over an actual open neighborhood in the quotient, rather than only on a chosen slice chart.

Lemma 6.7 (Luna closure obstruction). *Let $Z_1, Z_2 \subset \mathcal{M}^{\mathrm{GIT}}$ be irreducible closed subvarieties. Suppose that there are dense open subsets $U_i \subseteq Z_i$ and connected reductive subgroups $K_i \subset G$ such that, for every $z \in U_i$, the identity component of the stabilizer of the unique closed orbit over z is conjugate to K_i . If $Z_1 \subseteq Z_2$, then a conjugate of K_2 is contained in K_1 .*

Proof. Choose $z \in U_1$, let $G \cdot [F]$ be the unique closed orbit over z , and put

$$S := \mathrm{Stab}_G([F]).$$

After conjugating K_1 , we may assume that

$$S^\circ = K_1.$$

Choose a homogeneous invariant $I \in \mathbb{C}[W]^G$ of positive degree with $I(F) \neq 0$, and set

$$X_I := \{[f] \in \mathbb{P}(W)^{ss} \mid I(f) \neq 0\}.$$

This is a G -invariant affine neighborhood of $[F]$. Its good quotient

$$\pi_I : X_I \longrightarrow Q_I := X_I // G$$

is an open neighborhood of z in $\mathcal{M}^{\mathrm{GIT}}$.

Apply Luna's étale slice theorem at the closed orbit $G \cdot [F]$ [Lun73]; compare also [Lun75]. After shrinking the slice, the strongly étale local model $G \times^S V$ gives a G -invariant open neighborhood $O \subseteq X_I$ of $G \cdot [F]$ such that, for every $y \in O$, the stabilizer G_y is conjugate to a subgroup of S . In the slice tube, this is the identity

$$G_{[g,v]} = gS_v g^{-1} \subseteq gSg^{-1}.$$

We pass from this equivariant neighborhood to a saturated neighborhood over the quotient. Put

$$C := X_I \setminus O.$$

Then C is closed and G -invariant, so $\pi_I(C)$ is closed in Q_I . Moreover,

$$z \notin \pi_I(C).$$

Indeed, if $y \in C$ satisfied $\pi_I(y) = z$, then the closure of $G \cdot y$ would contain the unique closed orbit $G \cdot [F]$ in that fibre. Since C is closed and G -invariant, this would imply $[F] \in C$, contrary to $[F] \in O$. Therefore

$$Q := Q_I \setminus \pi_I(C)$$

is an open neighborhood of z , and

$$\pi_I^{-1}(Q) \subseteq O. \tag{16}$$

Because $Z_1 \subseteq Z_2$, the point z lies in Z_2 . Hence $Q \cap Z_2$ is a nonempty open subset of the irreducible variety Z_2 , and it meets the dense open subset U_2 . Choose

$$q \in Q \cap U_2$$

and let $G \cdot [E]$ be the unique closed orbit over q . By (16), one has $[E] \in O$, so a conjugate of $\text{Stab}_G([E])$ is contained in S . Since $q \in U_2$, the identity component $\text{Stab}_G([E])^\circ$ is conjugate to K_2 . Consequently, a conjugate of K_2 is contained in S . Since K_2 is connected, that conjugate is contained in

$$S^\circ = K_1.$$

This proves the claim. □

Proof of Theorem 6.1. Suppose that $\Theta_a \subseteq \Theta_b$ for distinct indices a and b . By Proposition 6.3 and Lemma 6.7, there exists $g \in G$ such that

$$gH_b g^{-1} \subseteq H_a.$$

Both sides are connected one-dimensional tori. A proper closed subgroup of an irreducible one-dimensional algebraic group has dimension zero, so this inclusion must be an equality. Thus H_a and H_b are conjugate, contradicting Corollary 6.6. Hence

$$\Theta_a \not\subseteq \Theta_b \quad (a \neq b).$$

It remains to identify the irreducible components. Let C be an irreducible component of $\partial\mathcal{M}^{\text{GIT}}$. Since the boundary is the finite union in (10), the irreducible set C is contained in some Θ_j . Since Θ_j is itself an irreducible closed subset of the boundary, the maximality of C gives $C = \Theta_j$. Conversely, every Θ_i is contained in an irreducible component C_i , and the preceding argument gives $C_i = \Theta_j$ for some j . Thus $\Theta_i \subseteq \Theta_j$, so pairwise non-inclusion forces $i = j$ and hence $C_i = \Theta_i$. Therefore $\Theta_1, \dots, \Theta_{21}$ are exactly the twenty-one irreducible components of $\partial\mathcal{M}^{\text{GIT}}$. □

7 Comparison with Lakhani's classification

Having proved the complete component theorem, we now compare it with the earlier study of the GIT compactification of quintic threefolds by Lakhani [Lak10]. Since the two papers organize the boundary in rather different ways, we record the precise dictionary between their families and separate the results already obtained in [Lak10] from the additional results proved here. Lakhani uses *non-stable* for the union of the unstable and strictly semistable loci. He first isolates seven families $SS1, \dots, SS7$, then takes their one-parameter-subgroup limits to four families $MO-A, \dots, MO-D$, which he calls the first level of minimal orbits, and finally studies ten further families $MO2-I, \dots, MO2-X$ arising from special degenerations of those four families. The present paper instead takes as its primary object the quotient-side boundary

$$\partial \mathcal{M}^{\text{GIT}} = \pi(\mathbb{P}(W)^{ss} \setminus \mathbb{P}(W)^s)$$

and distinguishes throughout between a maximal monomial-support family before taking the quotient and its image in $\mathcal{M}^{\text{GIT}}$. This distinction is essential for comparing the two classifications.

The dictionary between the two classifications. There is first a sign convention to take into account. If $r = (r_0, \dots, r_4)$ is one of Lakhani's normalized one-parameter subgroups, then his associated non-stable monomial family is $M_{\leq 0}(r)$. With the action convention used here,

$$M_{\leq 0}(r) = I(-r)_{\geq 0}.$$

After returning $-r$ to the dominant Weyl chamber by reversing the coordinates, its dominant representative is

$$\iota(r) := (-r_4, -r_3, -r_2, -r_1, -r_0).$$

We retain Lakhani's notation $q_{a,b}(U \mid V)$ for a general bihomogeneous form of bidegree (a, b) in the indicated blocks. With this convention, the correspondence is the following. The equality in the fourth column is an equality of the closures of the quotient images of the general closed-orbit loci; it is not a literal equality of the displayed parameter spaces before coordinate changes and residual centralizer normalization.

Lakhani family	First-level form	Lakhani type / present support	Present quotient component	dim
MO-A	$q_{2,3}(x_0, x_1 \mid x_2, x_3, x_4)$	$SS1 \leftrightarrow S_{32}$ $SS3 \leftrightarrow S_{37}$	$\Theta_{20} = \Phi_{32} = \Phi_{37}$ normal form F_{20}	18
MO-B	$q_5 + x_0 x_4 q_3 + x_0^2 x_4^2 q_1$ (the q_i are in x_1, x_2, x_3)	$SS5 \leftrightarrow S_{36}$	$\Theta_{21} = \Phi_{36}$ normal form F_{21}	24
MO-C	$q_{1,4} + x_4 q_{2,2} + x_4^2 q_3$ (blocks $(x_0, x_1) \mid (x_2, x_3)$)	$SS6 \leftrightarrow S_{29}$ $SS7 \leftrightarrow S_{35}$	$\Theta_{19} = \Phi_{29} = \Phi_{35}$ normal form F_{19}	15
MO-D	$x_0 q_4(x_1, x_2, x_3, x_4)$	$SS2 \leftrightarrow S_{23}$ $SS4 \leftrightarrow S_{38}$	$\Theta_{18} = \Phi_{23} = \Phi_{38}$ normal form F_{18}	19

The normal-form match is immediate for MO-A and MO-D. For MO-B, a general nonzero linear form q_1 is sent to a coordinate by the residual linear action. For MO-C, after the coordinate permutation dictated by ι , a general square-free binary cubic is normalized to $uv(u - v)$. These are the normalizations appearing in F_{21} and F_{19} , respectively. Consequently, the seven support types occurring in [Lak10] correspond to

$$S_{23}, S_{29}, S_{32}, S_{35}, S_{36}, S_{37}, S_{38}.$$

Under reversal of the supporting half-space they form the three pairs

$$(23, 38), \quad (29, 35), \quad (32, 37),$$

together with the fixed case 36. They therefore give four quotient-side families. In the complete component catalog these are exactly the four components of largest dimension:

$$\dim \Theta_{21} = 24, \quad \dim \Theta_{18} = 19, \quad \dim \Theta_{20} = 18, \quad \dim \Theta_{19} = 15,$$

whereas $\dim \Theta_a \leq 11$ for $1 \leq a \leq 17$.

Results already obtained in [Lak10]. The following parts of the boundary analysis precede the present paper.

1. Lakhani developed a monomial-poset and linear-programming analysis of the Hilbert–Mumford criterion, exhibited the seven families $\text{SS1}, \dots, \text{SS7}$, and computed their destabilizing one-parameter subgroups and associated bad flags. He also characterized these families geometrically by, respectively: a double plane; a hyperplane component; a triple line; a quadruple point; a special triple point whose tangent cone is the union of a double plane and another hyperplane, together with a contact-order-five condition; a double line with constant double-plane tangent cone, together with a contact-order-four condition; and a triple point on a plane with the incidence and quartic-intersection conditions stated in his Theorem 2.
2. By taking the corresponding one-parameter-subgroup limits, Lakhani obtained the four explicit invariant families $\text{MO-A}, \dots, \text{MO-D}$ displayed above. Their centralizer representations have block types $2 + 3$, $1 + 3 + 1$, $2 + 2 + 1$, and $1 + 4$, respectively. Lakhani applied Luna’s criterion to these representations, proved that a general member of each family has closed $\text{SL}(5)$ -orbit, and separated the unstable members from those that admit a further degeneration. Thus the equations of these four large families and their generic closed-orbit property are prior to the present work.
3. Lakhani introduced the ten second-level families $\text{MO2-I}, \dots, \text{MO2-X}$, repeated the closed-orbit analysis on those families, and described a degeneration diagram inside the union of the four first-level families. In particular, he showed that the nonclosed orbits occurring in this recursive analysis eventually degenerate to

$$x_0x_1x_2x_3x_4,$$

the terminal normal-crossing orbit in his diagram. He also gave several partial criteria for the stable locus, which are not used in the present paper.

Comparison of the polystability arguments. Both papers use the same basic reduction: a non-stable form is first replaced by its weight-zero one-parameter-subgroup limit, and Luna’s criterion then reduces closedness of the $\text{SL}(5)$ -orbit to closedness for a centralizer action on the fixed subspace. Since the original one-parameter subgroup H acts trivially on that fixed subspace, closedness depends on the effective quotient $C_G(H)/H$. Accordingly, Lakhani’s phrase “stable with respect to the centralizer” is naturally read as stability for this effective centralizer action. At the level of G , the resulting limit is not stable, because its stabilizer contains H ; when its orbit is closed, it is the polystable representative of the corresponding S -equivalence class.

The proof presented in [Lak10] is a recursive Hilbert–Mumford calculation. For MO-A , Lakhani writes a normalized one-parameter subgroup of the $\text{SL}(2) \times \text{SL}(3)$ -type centralizer in block-diagonal form, orders the coefficient monomials by their centralizer weights, and uses a poset calculation to list five maximal semistable nonclosed subfamilies and eighteen unstable subfamilies. His Proposition 24 states that the former degenerate to second-level families, that the latter are unstable, and that every remaining member has closed orbit. For MO-B , MO-C , and MO-D , the same procedure yields, respectively, four semistable nonclosed and four unstable

subfamilies, one of each, and four semistable nonclosed and three unstable subfamilies. Lakhani also applies analogous calculations to the non-toric second-level centralizers. Thus his proof of generic closedness proceeds by classifying the exceptional centralizer-nonstable support loci and iterating the construction on their limits. An important additional output of this method is the explicit further-degeneration stratification inside the four large families.

The present paper organizes the closed-orbit step differently. For each of the thirty-eight maximal supports it first extracts the full weight-zero limit ϕ_k and then proves directly that a specified general limit is polystable. If the effective centralizer is a torus, closedness is checked by the relative-interior criterion for the character polytope. If the effective centralizer has a nontrivial linear block, the paper uses the Casimiro–Florentino criterion [CF12]. Writing $C_k = C_G(H_k)$, the relevant set is

$$\Lambda_{\phi_k}^{C_k} := \left\{ \lambda : \mathbb{G}_m \longrightarrow C_k \mid \lim_{t \rightarrow 0} \lambda(t) \cdot \phi_k \text{ exists} \right\}.$$

On an explicit nonempty open locus, nonvanishing coefficient, discriminant, and resultant conditions are combined with linear balancing identities for the one-parameter-subgroup weights. The case-by-case analysis proves that $\Lambda_{\phi_k}^{C_k}$ is invariant under $\lambda \mapsto \lambda^{-1}$, which gives closedness by the Casimiro–Florentino criterion. The residual centralizer action is then used to obtain an explicit normal form and compute the quotient dimension. In particular, the four representatives in the table are the non-toric cases $k = 23, 29, 32, 36$, so this gives an alternative direct proof of their generic polystability, together with explicit genericity conditions.

These two approaches should be viewed as complementary rather than as competing claims of priority. Lakhani’s centralizer Hilbert–Mumford analysis already establishes the generic closedness of the four first-level families and, because it classifies their exceptional loci, records many special degenerations. The present argument instead gives a uniform direct polystability certificate for a general weight-zero limit in every one of the thirty-eight maximal-support cases and retains the parameters needed for the quotient-side normal forms and dimension calculations. Conversely, the present paper does not attempt to reconstruct the full internal centralizer-stratification diagram for all thirty-eight support families.

The non-exhaustiveness of the four-family picture. The difference between the two classifications is therefore not the validity of Lakhani’s four first-level families, but their exhaustiveness. The exhaustive calculation in Proposition 2.5 produces thirty-eight maximal T -strictly semistable supports, and Proposition 3.1 proves that their G -saturation cover the entire strictly semistable locus. The seven Lakhani support types account for three opposite-half-space pairs and one fixed point. The remaining thirty-one supports account for fourteen further pairs and three further fixed points. Equivalently,

$$38 = (3 \cdot 2 + 1) + (14 \cdot 2 + 3), \quad 21 = 4 + 17.$$

Thus the exhaustive computation here shows that the seven-family list does not exhaust the maximal strictly semistable supports and that the four first-level quotient families do not exhaust $\partial \mathcal{M}^{\text{GIT}}$.

The second-level families MO2–I, $\dots$, MO2–X should in particular not be confused with the additional components $\Theta_1, \dots, \Theta_{17}$. The former are special further-degeneration loci arising inside the union of the four first-level families. The latter are additional irreducible components which, by Theorem 6.1, are not contained in any of $\Theta_{18}, \dots, \Theta_{21}$.

Additional results of the present paper. Beyond the four-family picture and its internal degeneration analysis, the present paper proves the following.

1. The Hilbert–Mumford support calculation is made exhaustive: exact integer linear algebra, both orientations of each candidate supporting hyperplane, all coordinate permutations, and exact inclusion tests give precisely the thirty-eight maximal T -strictly semistable supports. Their G -saturation covers the whole strictly semistable locus. This yields thirty-one additional oriented maximal supports to the seven support types occurring in [Lak10].
2. The S -equivalences among support-indexed families are explained by the uniform zero-weight principle

$$\Phi(r) = \overline{\pi(\mathbb{P}(V(r)) \cap \mathbb{P}(W)^{ss})} = \overline{\pi(\mathbb{P}(Z(r)) \cap \mathbb{P}(W)^{ss})} = \Phi(-r),$$

proved in Lemma 4.1 and Corollary 4.2. Hence the quotient remembers the supporting hyperplane but not the orientation of its normal. The thirty-eight oriented supports decompose into seventeen two-cycles and four fixed points, giving the twenty-one closed irreducible quotient-side families of Corollary 5.2. The three pairs and one fixed point visible in Lakhani’s list are exactly the restriction of this general involution to his seven support types.

3. For all thirty-eight support cases, the present paper constructs explicit polystable weight-zero normal forms and computes the corresponding quotient-side dimensions. For the resulting twenty-one quotient families, it determines the generic connected projective stabilizers and proves that they are pairwise nonconjugate one-dimensional tori. The exact rank certificates of Proposition 6.3, together with Luna’s étale slice theorem, exclude every containment

$$\Theta_a \subseteq \Theta_b \quad (a \neq b).$$

Theorem 6.1 therefore proves that the boundary has exactly twenty-one irreducible components. In particular, it proves that the four Lakhani families are genuine irreducible components in the complete quotient-side decomposition and that there are seventeen further components, none of which is contained in one of those four components.

4. Finally, the present paper gives a component-by-component scheme-theoretic and local analytic study of the general closed-orbit representatives. It determines the saturated Jacobian scheme and the local analytic stratification in every component, identifies the generic transverse structures along the positive-dimensional singular loci, proves that precisely eleven isolated weighted-homogeneous local normal-form families occur, and identifies their normalized weight systems with eleven rows of Yonemura’s list. It also proves

$$\tilde{\alpha}(X_a) = 1 = \frac{4+1}{5} \quad (a = 1, \dots, 21).$$

These scheme-theoretic singularity calculations, the occurrence-and- exhaustion statement for the eleven weight systems, and the minimal-exponent calculation do not appear in [Lak10].

In summary, Lakhani first found the four families that become the four largest components in the present classification and carried out a substantial recursive closed-orbit and degeneration analysis within their union. The present paper places those families in the complete quotient-side picture: it gives a uniform polystability construction for all maximal supports, identifies seventeen additional irreducible components, proves that the resulting twenty-one families are exactly the boundary components, and determines the generic orbit-theoretic and singularity-theoretic structure of every component.

8 Singular schemes, local types, and minimal exponents

For each $a = 1, \dots, 21$, let $k(a)$ be the representative selected in Table 7, let

$$F_a := \phi_{k(a)}^{\text{nf}}, \quad X_a := V(F_a) \subset \mathbb{P}^4,$$

and restrict the parameters to the fixed nonempty Zariski-open generic locus specified for that row. We determine the saturated Jacobian scheme

$$\text{Sing}(X_a) = V(J(F_a)), \quad J(F_a) := \left(\frac{\partial F_a}{\partial x_0}, \dots, \frac{\partial F_a}{\partial x_4} \right) : (x_0, \dots, x_4)^\infty.$$

The analysis is now indexed only by quotient components. If k' is the partner of $k(a)$ in a two-cycle, then ρ identifies $Z(r_{k(a)})$ with $Z(r_{k'})$, while Corollary 4.2 identifies the quotient images $\Phi_{k(a)}$ and $\Phi_{k'}$. By Remark 5.3, the fiber over a general point of their common quotient family contains a unique closed orbit, represented up to projective equivalence by a general member of F_a . This does not assert that the two displayed normal-form formulas are carried into one another by ρ alone; residual normalization and reparametrization may be needed. Since saturated Jacobian schemes, local analytic types, and minimal exponents are invariant under projective coordinate changes, the single calculation for F_a covers both oriented support labels. The correspondence is recorded in the support-orbit column of Table 7.

The reduced positive-dimensional singular supports are explicit low-degree configurations: lines and line arrangements, smooth conics, cuspidal plane curves, planes, and smooth quartic surfaces. Some components also contain isolated singular points. Exactly eleven weighted-homogeneous local normal-form families occur at those isolated points. Their defining equations, integral weights, Milnor numbers, Yonemura numbers, and component occurrences are collected in Table 4. We first place the weight systems in Yonemura's classification and record the precise genericity conditions, then give the component-indexed singular-support catalog. The complete local calculations are deferred to Appendix B.

Relation with Yonemura's list of hypersurface simple $K3$ weights

Recall first the class of singularities to which Yonemura's result applies. Following [Yon90, Definition–Proposition 1.4], a three-dimensional singularity is called a *simple $K3$ singularity* if it is a Gorenstein purely elliptic singularity of $(0, 2)$ -type; equivalently, the exceptional divisor on any minimal resolution is a normal $K3$ surface. In the hypersurface setting, Yonemura considers isolated singularities defined by nondegenerate polynomials. For a nondegenerate hypersurface simple $K3$ singularity, the defining polynomial is quasi-homogeneous with a uniquely determined rational weight

$$a = (a_1, a_2, a_3, a_4) \in \mathbb{Q}_{>0}^4, \quad a_1 + a_2 + a_3 + a_4 = 1;$$

see [Yon90, Theorem 1.6 and Section 2].

For such a rational weight a , put

$$T(a) := \{\nu \in \mathbb{Z}_{\geq 0}^4 \mid a \cdot \nu = 1\}, \quad C(a) := \sum_{\nu \in T(a)} \mathbb{R}_{\geq 0} \nu \subset \mathbb{R}^4.$$

Yonemura considers the set

$$W_4 := \left\{ a \in \mathbb{Q}_{>0}^4 \mid \begin{array}{l} a_1 + \dots + a_4 = 1, \quad a_1 \geq a_2 \geq a_3 \geq a_4, \\ (1, 1, 1, 1) \in \text{Int } C(a) \end{array} \right\}.$$

Yonemura proves in [Yon90, Proposition 2.1] that W_4 has cardinality 95, and Table 2.2 of the same paper gives the complete list of these weights together with one quasi-homogeneous polynomial realizing each weight as a nondegenerate hypersurface simple $K3$ singularity.

For an integral weight system

$$(w_1, w_2, w_3, w_4; D), \quad w_1 + w_2 + w_3 + w_4 = D,$$

let

$$a(w; D) := \left(\frac{w_{\sigma(1)}}{D}, \frac{w_{\sigma(2)}}{D}, \frac{w_{\sigma(3)}}{D}, \frac{w_{\sigma(4)}}{D} \right), \quad w_{\sigma(1)} \geq \cdots \geq w_{\sigma(4)},$$

be its ordered normalized weight. Whenever $a(w; D)$ occurs in Yonemura's Table 2.2, we call the corresponding row index the *Yonemura number* of the weight system. We write $\mathbf{S}_\mu^{YN}$ for a weighted-homogeneous local normal-form family with Milnor number μ whose weight system has Yonemura number N . Thus the subscript records the Milnor number, while the superscript records the corresponding row of Yonemura's table. The eleven weight systems that occur at isolated points of the quintic GIT boundary have the following correspondence.

Local family	Integral weight system	Ordered normalized weight	Yonemura No.
$\mathbf{S}_{81}^{Y1}$	(1, 1, 1, 1; 4)	$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$	1
$\mathbf{S}_{87}^{Y52}$	(7, 8, 9, 12; 36)	$\left(\frac{1}{3}, \frac{1}{4}, \frac{2}{9}, \frac{7}{36}\right)$	52
$\mathbf{S}_{88}^{Y84}$	(5, 6, 7, 9; 27)	$\left(\frac{1}{3}, \frac{7}{27}, \frac{2}{9}, \frac{5}{27}\right)$	84
$\mathbf{S}_{88}^{Y15}$	(3, 3, 4, 5; 15)	$\left(\frac{1}{3}, \frac{4}{15}, \frac{1}{5}, \frac{1}{5}\right)$	15
$\mathbf{S}_{90}^{Y2}$	(2, 3, 3, 4; 12)	$\left(\frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{6}\right)$	2
$\mathbf{S}_{91}^{Y53}$	(3, 4, 5, 6; 18)	$\left(\frac{1}{3}, \frac{5}{18}, \frac{2}{9}, \frac{1}{6}\right)$	53
$\mathbf{S}_{96}^{Y86}$	(4, 5, 7, 9; 25)	$\left(\frac{9}{25}, \frac{7}{25}, \frac{1}{5}, \frac{4}{25}\right)$	86
$\mathbf{S}_{96}^{Y94}$	(3, 4, 5, 7; 19)	$\left(\frac{7}{19}, \frac{5}{19}, \frac{4}{19}, \frac{3}{19}\right)$	94
$\mathbf{S}_{96}^{Y21}$	(1, 1, 1, 2; 5)	$\left(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)$	21
$\mathbf{S}_{100}^{Y3}$	(1, 1, 2, 2; 6)	$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right)$	3
$\mathbf{S}_{102}^{Y62}$	(3, 4, 5, 8; 20)	$\left(\frac{2}{5}, \frac{1}{4}, \frac{1}{5}, \frac{3}{20}\right)$	62

Table 9: The eleven isolated local normal-form families and their Yonemura numbers. In the notation $\mathbf{S}_\mu^{YN}$, the subscript is the Milnor number and the superscript records row N of Table 2.2 of [Yon90].

This correspondence is deliberately stated at the level of weight systems. Yonemura's classification is a classification, in terms of weights, of nondegenerate hypersurface simple $K3$ singularities, and the polynomial in each row of his table is an example realizing that weight. The row number does not assert that all parameter values in one of the local families below are mutually analytically equivalent, or that they are analytically equivalent to Yonemura's displayed example. Accordingly, the result of the present paper is not a new classification of isolated hypersurface singularities. It is instead an occurrence-and-exhaustion statement for the quintic GIT boundary: precisely the following eleven weighted-homogeneous local normal-form families occur at the isolated singular points of general closed-orbit representatives.

Every defining equation below is weighted homogeneous of degree D for positive weights (w_1, w_2, w_3, w_4) satisfying

$$w_1 + w_2 + w_3 + w_4 = D.$$

Once the indicated genericity condition guarantees that the critical point is isolated, the standard weighted-homogeneous formulas give

$$\tilde{\alpha}_0 = \frac{w_1 + w_2 + w_3 + w_4}{D} = 1, \quad \mu = \prod_{i=1}^4 \left(\frac{D}{w_i} - 1 \right);$$

see [Sai71, Sai94]. This is the critical value

$$\frac{n+1}{d} = \frac{4+1}{5} = 1$$

for quintic hypersurfaces in $\mathbb{P}^4$ in the minimal-exponent approach to GIT stability; see [Par25]. We therefore use the paper-specific term *extremal quintic threefold singularities* for these eleven local normal-form families. This terminology refers to their occurrence on the quintic GIT boundary and to the equality $\tilde{\alpha}_0 = 1$, not to a claim that their weight systems lie outside the classical Yonemura list.

Definition 8.1. Let $B_4(X_1, X_2, X_3, X_4)$ be a homogeneous quartic whose projectivization $V(B_4) \subset \mathbb{P}^3$ is smooth. We denote by

$$\text{QH}(1, 1, 1, 1; 4)_{81}(B_4)$$

the weighted-homogeneous family of isolated hypersurface germs represented by

$$B_4(X_1, X_2, X_3, X_4) = 0.$$

It is quasi-homogeneous of weighted degree 4 with respect to the weights

$$(w_{X_1}, w_{X_2}, w_{X_3}, w_{X_4}) = (1, 1, 1, 1).$$

Its ordered normalized weight is

$$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right),$$

and hence its Yonemura number is 1. Its Milnor number is $\mu = 81$, and its local minimal exponent is

$$\tilde{\alpha} = 1.$$

We abbreviate this weighted-homogeneous local normal-form family by

$$\mathbf{S}_{81}^{Y1}(B_4).$$

Definition 8.2. We denote by

$$\text{QH}(7, 8, 9, 12; 36)_{87}(\tau)$$

the weighted-homogeneous family of isolated hypersurface germs represented by

$$X_1^4 X_2 + X_3^4 + X_2^3 X_4 + \tau X_1 X_2 X_3 X_4 + X_4^3 = 0,$$

where

$$\tau \in \mathbb{C}^\times, \quad \tau^4 \neq 256.$$

It is quasi-homogeneous of weighted degree 36 with respect to the weights

$$(w_{X_1}, w_{X_2}, w_{X_3}, w_{X_4}) = (7, 8, 9, 12).$$

Its ordered normalized weight is

$$\left(\frac{1}{3}, \frac{1}{4}, \frac{2}{9}, \frac{7}{36}\right),$$

and hence its Yonemura number is 52. Its Milnor number is $\mu = 87$, and its local minimal exponent is

$$\tilde{\alpha} = 1.$$

We abbreviate this weighted-homogeneous local normal-form family by

$$\mathbf{S}_{87}^{Y52}(\tau).$$

Definition 8.3. We denote by

$$\text{QH}(5, 6, 7, 9; 27)_{88}(\alpha, \beta)$$

the weighted-homogeneous family of isolated hypersurface germs represented by

$$X_1^3 X_2^2 + X_1^4 X_3 + X_2 X_3^3 + X_2^3 X_4 + \alpha X_1 X_2 X_3 X_4 + \beta X_4^3 = 0,$$

where

$$\alpha, \beta \in \mathbb{C}^\times,$$

and

$$\alpha^6 - 2\alpha^5 + \alpha^4 + 54\alpha^3\beta - 378\alpha^2\beta + 576\alpha\beta + 729\beta^2 - 256\beta \neq 0.$$

It is quasi-homogeneous of weighted degree 27 with respect to the weights

$$(w_{X_1}, w_{X_2}, w_{X_3}, w_{X_4}) = (5, 6, 7, 9).$$

Its ordered normalized weight is

$$\left(\frac{1}{3}, \frac{7}{27}, \frac{2}{9}, \frac{5}{27}\right),$$

and hence its Yonemura number is 84. Its Milnor number is $\mu = 88$, and its local minimal exponent is

$$\tilde{\alpha} = 1.$$

We abbreviate this weighted-homogeneous local normal-form family by

$$\mathbf{S}_{88}^{\text{Y84}}(\alpha, \beta).$$

Definition 8.4. Let $B_5(X_1, X_2)$ be a binary quintic for which the hypersurface germ below is isolated at the origin. We denote by

$$\text{QH}(3, 3, 4, 5; 15)_{88}(B_5)$$

the weighted-homogeneous family of isolated hypersurface germs represented by

$$B_5(X_1, X_2) + X_2 X_3^3 + (X_1^2 + X_2^2) X_3 X_4 + X_4^3 = 0.$$

It is quasi-homogeneous of weighted degree 15 with respect to the weights

$$(w_{X_1}, w_{X_2}, w_{X_3}, w_{X_4}) = (3, 3, 4, 5).$$

Its ordered normalized weight is

$$\left(\frac{1}{3}, \frac{4}{15}, \frac{1}{5}, \frac{1}{5}\right),$$

and hence its Yonemura number is 15. Its Milnor number is $\mu = 88$, and its local minimal exponent is

$$\tilde{\alpha} = 1.$$

We abbreviate this weighted-homogeneous local normal-form family by

$$\mathbf{S}_{88}^{\text{Y15}}(B_5).$$

Definition 8.5. Let B_4 be a square-free binary quartic and let Q_2, R_2 be binary quadratic forms. Let

$$\mathcal{R}_{90}^I(B_4, Q_2, R_2, \beta)$$

denote the weighted discriminant condition for the weighted-homogeneous germ below to be isolated at the origin. We denote by

$$\text{QH}(2, 3, 3, 4; 12)_{90}(B_4, Q_2, R_2, \beta)$$

the weighted-homogeneous family of isolated hypersurface germs represented by

$$B_4(X_2, X_3) + X_1^3 Q_2(X_2, X_3) + X_1^4 X_4 + X_1 X_4 R_2(X_2, X_3) + \beta X_1^2 X_4^2 + X_4^3 = 0,$$

where

$$\beta \in \mathbb{C}, \quad \mathcal{R}_{90}^I(B_4, Q_2, R_2, \beta) \neq 0.$$

It is quasi-homogeneous of weighted degree 12 with respect to the weights

$$(w_{X_1}, w_{X_2}, w_{X_3}, w_{X_4}) = (2, 3, 3, 4).$$

Its ordered normalized weight is

$$\left(\frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{6}\right),$$

and hence its Yonemura number is 2. Its Milnor number is $\mu = 90$, and its local minimal exponent is

$$\tilde{\alpha} = 1.$$

We abbreviate this weighted-homogeneous local normal-form family by

$$\mathbf{S}_{90}^{Y2}(B_4, Q_2, R_2, \beta).$$

Definition 8.6. Let

$$\mathcal{R}_{91}(\alpha, \beta, \gamma, \delta, \epsilon)$$

denote the bad-parameter equation for which the weighted-homogeneous germ below is not isolated at the origin. We denote by

$$\text{QH}(3, 4, 5, 6; 18)_{91}(\alpha, \beta, \gamma, \delta, \epsilon)$$

the weighted-homogeneous family of isolated hypersurface germs represented by

$$X_1^2 X_2^3 + X_1^3 X_2 X_3 + X_1^4 X_4 + X_2^2 X_3^2 + \alpha X_2^3 X_4 + \beta X_1 X_3^3 + \gamma X_1 X_2 X_3 X_4 + \delta X_1^2 X_4^2 + \epsilon X_4^3 = 0,$$

where

$$\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{C}^\times, \quad \mathcal{R}_{91}(\alpha, \beta, \gamma, \delta, \epsilon) \neq 0.$$

It is quasi-homogeneous of weighted degree 18 with respect to the weights

$$(w_{X_1}, w_{X_2}, w_{X_3}, w_{X_4}) = (3, 4, 5, 6).$$

Its ordered normalized weight is

$$\left(\frac{1}{3}, \frac{5}{18}, \frac{2}{9}, \frac{1}{6}\right),$$

and hence its Yonemura number is 53. Its Milnor number is $\mu = 91$, and its local minimal exponent is

$$\tilde{\alpha} = 1.$$

We abbreviate this weighted-homogeneous local normal-form family by

$$\mathbf{S}_{91}^{Y53}(\alpha, \beta, \gamma, \delta, \epsilon).$$

Definition 8.7. Let

$$\mathcal{R}_{96}(\alpha, \beta)$$

denote the bad-parameter equation for which the weighted-homogeneous germ below is not isolated at the origin. We denote by

$$\text{QH}(4, 5, 7, 9; 25)_{96}(\alpha, \beta)$$

the weighted-homogeneous family of isolated hypersurface germs represented by

$$X_2^5 + X_1^2 X_2^2 X_3 + X_1^4 X_4 + X_1 X_3^3 + \alpha X_1 X_2 X_3 X_4 + \beta X_3 X_4^2 = 0,$$

where

$$\alpha, \beta \in \mathbb{C}^\times, \quad \mathcal{R}_{96}(\alpha, \beta) \neq 0.$$

It is quasi-homogeneous of weighted degree 25 with respect to the weights

$$(w_{X_1}, w_{X_2}, w_{X_3}, w_{X_4}) = (4, 5, 7, 9).$$

Its ordered normalized weight is

$$\left(\frac{9}{25}, \frac{7}{25}, \frac{1}{5}, \frac{4}{25} \right),$$

and hence its Yonemura number is 86. Its Milnor number is $\mu = 96$, and its local minimal exponent is

$$\tilde{\alpha} = 1.$$

We abbreviate this weighted-homogeneous local normal-form family by

$$\mathbf{S}_{96}^{\text{Y86}}(\alpha, \beta).$$

Definition 8.8. Let

$$\mathcal{R}_{96}^{\text{II}}(\alpha, \beta, \gamma, \delta)$$

denote the bad-parameter equation for which the weighted-homogeneous germ below is not isolated at the origin. We denote by

$$\text{QH}(3, 4, 5, 7; 19)_{96}(\alpha, \beta, \gamma, \delta)$$

the weighted-homogeneous family of isolated hypersurface germs represented by

$$\begin{aligned} X_1 X_2^4 + X_1^2 X_2^2 X_3 + \alpha X_1^3 X_3^2 + X_1^4 X_4 + X_2 X_3^3 + \beta X_2^3 X_4 \\ + \gamma X_1 X_2 X_3 X_4 + \delta X_3 X_4^2 = 0, \end{aligned}$$

where

$$\alpha, \beta, \gamma, \delta \in \mathbb{C}^\times, \quad \mathcal{R}_{96}^{\text{II}}(\alpha, \beta, \gamma, \delta) \neq 0.$$

It is quasi-homogeneous of weighted degree 19 with respect to the weights

$$(w_{X_1}, w_{X_2}, w_{X_3}, w_{X_4}) = (3, 4, 5, 7).$$

Its ordered normalized weight is

$$\left(\frac{7}{19}, \frac{5}{19}, \frac{4}{19}, \frac{3}{19} \right),$$

and hence its Yonemura number is 94. Its Milnor number is $\mu = 96$, and its local minimal exponent is

$$\tilde{\alpha} = 1.$$

We abbreviate this weighted-homogeneous local normal-form family by

$$\mathbf{S}_{96}^{\text{Y94}}(\alpha, \beta, \gamma, \delta).$$

Definition 8.9. Let B_5 be a ternary quintic, let C_3 be a ternary cubic, and let L_1 be a nonzero linear form in (X_1, X_2, X_3) . Let

$$\mathcal{R}_{96}^{\text{III}}(B_5, C_3, L_1)$$

denote the bad-parameter equation for which the weighted-homogeneous germ below is not isolated at the origin. We denote by

$$\text{QH}(1, 1, 1, 2; 5)_{96}(B_5, C_3, L_1)$$

the weighted-homogeneous family of isolated hypersurface germs represented by

$$B_5(X_1, X_2, X_3) + X_4 C_3(X_1, X_2, X_3) + X_4^2 L_1(X_1, X_2, X_3) = 0,$$

where

$$\mathcal{R}_{96}^{\text{III}}(B_5, C_3, L_1) \neq 0.$$

It is quasi-homogeneous of weighted degree 5 with respect to the weights

$$(w_{X_1}, w_{X_2}, w_{X_3}, w_{X_4}) = (1, 1, 1, 2).$$

Its ordered normalized weight is

$$\left(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right),$$

and hence its Yonemura number is 21. Its Milnor number is $\mu = 96$, and its local minimal exponent is

$$\tilde{\alpha} = 1.$$

We abbreviate this weighted-homogeneous local normal-form family by

$$\mathbf{S}_{96}^{\text{Y21}}(B_5, C_3, L_1).$$

Definition 8.10. Let A_4, C_4 be binary quartics in (X_1, X_2) , let $Q_{2,2}$ be a bihomogeneous form of bidegree $(2, 2)$ in (X_1, X_2) and (X_3, X_4) , and let B_3 be a binary cubic in (X_3, X_4) . Let

$$\mathcal{R}_{100}^{\text{I}}(A_4, C_4, Q_{2,2}, B_3)$$

denote the bad-parameter equation for which the weighted-homogeneous germ below is not isolated at the origin. We denote by

$$\text{QH}(1, 1, 2, 2; 6)_{100}(A_4, C_4, Q_{2,2}, B_3)$$

the weighted-homogeneous family of isolated hypersurface germs represented by

$$X_3 A_4(X_1, X_2) + X_4 C_4(X_1, X_2) + Q_{2,2}(X_1, X_2; X_3, X_4) + B_3(X_3, X_4) = 0,$$

where

$$\mathcal{R}_{100}^{\text{I}}(A_4, C_4, Q_{2,2}, B_3) \neq 0.$$

It is quasi-homogeneous of weighted degree 6 with respect to the weights

$$(w_{X_1}, w_{X_2}, w_{X_3}, w_{X_4}) = (1, 1, 2, 2).$$

Its ordered normalized weight is

$$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right),$$

and hence its Yonemura number is 3. Its Milnor number is $\mu = 100$, and its local minimal exponent is

$$\tilde{\alpha} = 1.$$

We abbreviate this weighted-homogeneous local normal-form family by

$$\mathbf{S}_{100}^{\text{Y3}}(A_4, C_4, Q_{2,2}, B_3).$$

Definition 8.11. Let

$$\mathcal{R}_{102}^I(\alpha, \beta, \gamma, \delta)$$

denote the bad-parameter equation for which the weighted-homogeneous germ below is not isolated at the origin. We denote by

$$\text{QH}(3, 4, 5, 8; 20)_{102}(\alpha, \beta, \gamma, \delta)$$

the weighted-homogeneous family of isolated hypersurface germs represented by

$$\begin{aligned} X_2^5 + \alpha X_1 X_2^3 X_3 + \beta X_1^2 X_2 X_3^2 + X_1^4 X_4 + X_3^4 + \gamma X_2^3 X_4 \\ + \delta X_1 X_2 X_3 X_4 + X_2 X_4^2 = 0, \end{aligned}$$

where

$$\alpha, \beta, \gamma, \delta \in \mathbb{C}^\times, \quad \mathcal{R}_{102}^I(\alpha, \beta, \gamma, \delta) \neq 0.$$

It is quasi-homogeneous of weighted degree 20 with respect to the weights

$$(w_{X_1}, w_{X_2}, w_{X_3}, w_{X_4}) = (3, 4, 5, 8).$$

Its ordered normalized weight is

$$\left(\frac{2}{5}, \frac{1}{4}, \frac{1}{5}, \frac{3}{20} \right),$$

and hence its Yonemura number is 62. Its Milnor number is $\mu = 102$, and its local minimal exponent is

$$\tilde{\alpha} = 1.$$

We abbreviate this weighted-homogeneous local normal-form family by

$$\mathbf{S}_{102}^{\text{Y62}}(\alpha, \beta, \gamma, \delta).$$

Equation-level catalog. The defining equations, integral weights, Milnor numbers, Yonemura numbers, and component occurrences of the eleven families are summarized in Table 4. Definitions 8.1– 8.11 record the exact nonempty Zariski-open genericity conditions.

The component-indexed singular-support catalog

All rows are for general parameters. We use

$$P_0 = (1 : 0 : 0 : 0 : 0), \quad P_\infty = (0 : 0 : 0 : 0 : 1),$$

and

$$B_\lambda(u, v) := uv(u - v)(u - \lambda v), \quad B_{\lambda, \mu}(u, v) := v(u - v)(u - \lambda v)(u - \mu v).$$

The last column records only isolated singular points; distinguished points on positive-dimensional strata and their transverse analytic types are listed in Table 3 and proved in Appendix B. In the notation $\mathbf{S}_\mu^{\text{YN}}$, μ is the Milnor number and N the Yonemura number. In the row $a = 19$, $B_3^{(29)} = x_3 x_4 (x_3 - x_4)$.

Table 10: Reduced singular supports of the twenty-one quotient-component representatives. Here $k = k(a)$ is the selected oriented support case. For a paired label, coordinate reversal identifies the zero-weight spaces and the quotient images agree, but it need not identify the displayed normal forms coefficientwise. The general singularity data therefore agree up to projective equivalence.

a	$k(a)$	Singular locus	Geometry / incidences	Isolated local family
1	1	$\text{Sing}(X_1) = C \cup \{P_0\}$ $C = V(x_0, x_1, x_3^4 + x_2^3 x_4)$	rational plane quartic; cusp at P_∞	$P_0 : \mathbf{S}_{87}^{Y52}(\tau)$
2	2	$\text{Sing}(X_2) = L \cup C \cup \{P_0\}$ $L = V(x_0, x_1, x_2)$ $C = V(x_0, x_1, x_3^3 + x_2^2 x_4)$	L line; C cuspidal cubic; $L \cap C = \{P_\infty\}$	$P_0 : \mathbf{S}_{88}^{Y84}(\alpha, \beta)$
3	3	$\text{Sing}(X_3) = L \cup \{P_0\}$ $L = V(x_0, x_1, x_2)$	L line; special point P_∞	$P_0 : \mathbf{S}_{88}^{Y15}(B_5)$
4	4	$\text{Sing}(X_4) = M \sqcup L$ $M = V(x_2, x_3, x_4)$ $L = V(x_0, x_1, x_2)$	two disjoint lines	$\emptyset$
5	5	$\text{Sing}(X_5) = N \cup L \cup M$ $N = V(x_2, x_3, x_4)$ $L = V(x_0, x_1, x_2)$ $M = V(x_0, x_1, x_3)$	three lines; $L \cap M = \{P_\infty\}$; $N \cap L = N \cap M = \emptyset$	$\emptyset$
6	6	$\text{Sing}(X_6) = L \cup C \cup \{P_0\}$ $L = V(x_0, x_1, x_2)$ $C = V(x_0, x_1, x_3^2 + \alpha x_2 x_4)$	L line; C smooth conic; $L \cap C = \{P_\infty\}$	$P_0 : \mathbf{S}_{91}^{Y53}$ $(\alpha, \beta, \gamma, \delta, \epsilon)$
7	7	$\text{Sing}(X_7) = L \cup \{P_0\}$ $L = V(x_0, x_1, x_2)$	L line; special point P_∞	$P_0 : \mathbf{S}_{96}^{Y86}(\alpha, \beta)$
8	8	$\text{Sing}(X_8) = N \cup L$ $N = V(x_2, x_3, x_4)$ $L = V(x_0, x_1, x_2)$	two disjoint lines	$\emptyset$
9	9	$\text{Sing}(X_9) = L \cup \{P_0\}$ $L = V(x_0, x_1, x_2)$	L line; special point P_∞	$P_0 : \mathbf{S}_{96}^{Y94}(\alpha, \beta, \gamma, \delta)$
10	10	$\text{Sing}(X_{10}) = C_\lambda \cup \{P_0\}$ $C_\lambda = V(x_0, x_1, B_\lambda(x_2, x_3))$	$C_\lambda = L_0 \cup L_\infty \cup L_1 \cup L_\lambda$; four lines through P_∞	$P_0 : \mathbf{S}_{90}^{Y2}$ $(B_\lambda, Q_{\mathbf{q}}, R_{\mathbf{r}}, \beta)$
11	11	$\text{Sing}(X_{11}) = N \cup L$ $N = V(x_2, x_3, x_4)$ $L = V(x_0, x_1, x_2)$	two disjoint lines	$\emptyset$
12	12	$\text{Sing}(X_{12}) = N \cup L$ $N = V(x_2, x_3, x_4)$ $L = V(x_0, x_1, x_2)$	two disjoint lines	$\emptyset$
13	13	$\text{Sing}(X_{13}) = N \cup L$ $N = V(x_2, x_3, x_4)$ $L = V(x_0, x_1, x_2)$	two disjoint lines	$\emptyset$
14	14	$\text{Sing}(X_{14}) = \{P_0, P_\infty\}$	two isolated points	$P_0, P_\infty : \mathbf{S}_{102}^{Y62}(\alpha, \beta, \gamma, \delta)$
15	15	$\text{Sing}(X_{15}) = N \cup L$ $N = V(x_2, x_3, x_4)$ $L = V(x_0, x_1, x_2)$	two disjoint lines	$\emptyset$

continued on the next page

a	$k(a)$	Singular locus	Geometry / incidences	Isolated local family
16	19	$\text{Sing}(X_{16}) = N \cup L$ $N = V(x_2, x_3, x_4)$ $L = V(x_0, x_1, x_2)$	two disjoint lines	$\emptyset$
17	20	$\text{Sing}(X_{17}) = N \cup C_{\lambda, \mu}$ $N = V(x_2, x_3, x_4)$ $C_{\lambda, \mu} = V(x_0, x_1, B_{\lambda, \mu}(x_2, x_3))$	N line; $C_{\lambda, \mu}$ four lines through P_∞ ; disjoint from N	$\emptyset$
18	23	$\text{Sing}(X_{18}) = S \cup \{P_0\}$ $S = V(x_0, B_4)$	S smooth quartic surface	$P_0 : \mathbf{S}_{81}^{Y1}(B_4)$
19	29	$\text{Sing}(X_{19}) = L \cup \{P_0\}$ $L = V(x_0, x_1, x_2)$	L line	$P_0 : \mathbf{S}_{100}^{Y3}$ $(A_4, C_4, Q_{2,2}, B_3^{(29)})$
20	32	$\text{Sing}(X_{20}) = N \cup \Pi$ $N = V(x_2, x_3, x_4)$ $\Pi = V(x_0, x_1)$	N line; $\Pi \simeq \mathbb{P}^2$; $N \cap \Pi = \emptyset$	$\emptyset$
21	36	$\text{Sing}(X_{21}) = \{P_0, P_\infty\}$	two isolated points	$P_0, P_\infty : \mathbf{S}_{96}^{Y21}(B_5, C_3, X_3)$

Theorem 8.12 (Component-indexed singularity and minimal-exponent catalog). *For a general closed-orbit representative X_a of every component Θ_a , the reduced singular support is the one in Table 10, and the full local analytic stratification is the one in Table 3. The isolated points belong to precisely the eleven weighted-homogeneous families of Table 4; no other isolated local family occurs for a general boundary representative. Moreover,*

$$\tilde{\alpha}(X_a) = 1 = \frac{4+1}{5}, \quad a = 1, \dots, 21.$$

Proof. The saturated-Jacobian, local analytic, and minimal-exponent calculations are carried out component by component in Appendix B. For every paired support label k' , coordinate reversal identifies its zero-weight space with that of the chosen representative, and Corollary 4.2 identifies their quotient images. Remark 5.3 therefore shows that the paired normal-form loci represent the same general closed orbit up to projective equivalence. Hence the twenty-one calculations cover all thirty-eight original oriented support labels; no coefficientwise identification of the displayed normal forms by ρ is used. $\square$

A Support-by-support construction of the closed-orbit normal forms

This appendix supplies the thirty-eight support-indexed calculations underlying Section 5. For each maximal strictly T -semistable support

$$S_k = I(r_k)_{\geq 0}, \quad k = 1, \dots, 38,$$

we extract the weight-zero limit ϕ_k , apply the centralizer reduction and the C-H or C-F criterion developed in Section 5, and use the residual centralizer action to obtain the explicit normal form ϕ_k^{nf} . The calculations are necessarily indexed by all thirty-eight oriented supports; the quotient-side reduction to twenty-one representatives has already been carried out in Sections 4 and 5.

For reference, the full support-indexed catalog is reproduced first. Its paired rows determine the same quotient component, but are retained here so that every case calculation can be checked directly against its original support label.

Let us put

$$B_\lambda(u, v) := uv(u - v)(u - \lambda v), \quad B_{\lambda, \mu}(u, v) := v(u - v)(u - \lambda v)(u - \mu v).$$

Here B_d, C_d, Q_d, L_d, M_d denote general forms of the indicated degree, and $Q_{2,2}$ denotes a bihomogeneous form of bidegree $(2, 2)$. All parameters are understood to lie in the relevant nonempty Zariski-open generic locus.

Table 11: Closed-orbit normal forms for the 38 quintic-threefold boundary families.

k	Test	Normal form	Parameters	Dim
1	C-H	$\phi_1^{\text{nf}} = x_1^4 x_2 + x_0 x_3^4 + x_0 x_2^3 x_4$ $+ x_0 x_1 x_2 x_3 x_4 + \alpha x_0^2 x_4^3$	$\alpha \in \mathbb{C}^\times$	1
2	C-H	$\phi_2^{\text{nf}} = x_1^3 x_2^2 + x_1^4 x_3 + x_0 x_2 x_3^3 + x_0 x_2^3 x_4$ $+ \alpha x_0 x_1 x_2 x_3 x_4 + \beta x_0^2 x_4^3$	$\alpha, \beta \in \mathbb{C}^\times$	2
3	C-F	$\phi_3^{\text{nf}} = B_5(x_1, x_2) + x_0 x_2 x_3^3$ $+ x_0(x_1^2 + x_2^2)x_3 x_4 + x_0^2 x_4^3$	B_5 : general binary quintic	6
4	C-H	$\phi_4^{\text{nf}} = x_1^2 x_2^3 + x_1^3 x_3 x_4 + x_0 x_2^4 + x_0 x_1 x_3^3$ $+ \alpha x_0 x_1 x_2 x_3 x_4 + \beta x_0^2 x_4^3$	$\alpha, \beta \in \mathbb{C}^\times$	2
5	C-H	$\phi_5^{\text{nf}} = x_1^2 x_2^3 + x_1^3 x_3^2 + x_1^3 x_2 x_4 + x_0 x_2^3 x_3$ $+ \alpha x_0 x_1 x_3^3 + \beta x_0 x_1 x_2 x_3 x_4 + \gamma x_0^2 x_4^3$	$\alpha, \beta, \gamma \in \mathbb{C}^\times$	3
6	C-H	$\phi_6^{\text{nf}} = x_1^2 x_2^3 + x_1^3 x_2 x_3 + x_1^4 x_4 + x_0 x_2^2 x_3^2$ $+ \alpha x_0 x_2^3 x_4 + \beta x_0 x_1 x_3^3 + \gamma x_0 x_1 x_2 x_3 x_4$ $+ \delta x_0 x_1^2 x_4^2 + \epsilon x_0^2 x_4^3$	$\alpha, \beta, \gamma, \delta,$ $\epsilon \in \mathbb{C}^\times$	5
7	C-H	$\phi_7^{\text{nf}} = x_2^5 + x_1^2 x_2^2 x_3 + x_1^4 x_4 + x_0 x_1 x_3^3$ $+ \alpha x_0 x_1 x_2 x_3 x_4 + \beta x_0^2 x_3 x_4^2$	$\alpha, \beta \in \mathbb{C}^\times$	2
8	C-H	$\phi_8^{\text{nf}} = x_1^2 x_2^2 x_3 + x_1^3 x_4^2 + x_0 x_2^4 + x_0 x_1 x_3^3$ $+ \alpha x_0 x_1 x_2 x_3 x_4 + \beta x_0^2 x_3 x_4^2$	$\alpha, \beta \in \mathbb{C}^\times$	2
9	C-H	$\phi_9^{\text{nf}} = x_1 x_2^4 + x_1^2 x_2^2 x_3 + \alpha x_1^3 x_3^2 + x_1^4 x_4$ $+ x_0 x_2 x_3^3 + \beta x_0 x_2^3 x_4 + \gamma x_0 x_1 x_2 x_3 x_4$ $+ \delta x_0^2 x_3 x_4^2$	$\alpha, \beta, \gamma, \delta \in \mathbb{C}^\times$	4

continued on the next page

k	Test	Normal form	Parameters	Dim
10	C-F	$\phi_{10}^{\text{nf}} = x_1^3 Q_{\mathbf{q}}(x_2, x_3) + x_1^4 x_4 + x_0 B_{\lambda}(x_2, x_3)$ $+ x_0 x_1 x_4 R_{\mathbf{r}}(x_2, x_3) + \beta x_0 x_1^2 x_4^2 + x_0^2 x_4^3$	$\lambda \in \mathbb{C} \setminus \{0, 1\};$ $\mathbf{q}, \mathbf{r} \in \mathbb{C}^3, \beta \in \mathbb{C}$	8
11	C-H	$\phi_{11}^{\text{nf}} = x_1 x_2^4 + \alpha x_1^2 x_2^2 x_3 + \beta x_1^3 x_3^2 + x_1^3 x_2 x_4$ $+ \gamma x_0 x_2^2 x_3^2 + x_0 x_2^3 x_4 + \delta x_0 x_1 x_3^3$ $+ \epsilon x_0 x_1 x_2 x_3 x_4 + \zeta x_0 x_1^2 x_4^2 + x_0^2 x_3 x_4^2$	$\alpha, \beta, \gamma, \delta,$ $\epsilon, \zeta \in \mathbb{C}^{\times}$	6
12	C-H	$\phi_{12}^{\text{nf}} = x_1 x_2^4 + \alpha x_1^2 x_2^2 x_3 + \beta x_1^2 x_2^2 x_4 + x_1^3 x_4^2$ $+ x_0 x_2^3 x_3 + \gamma x_0 x_1 x_3^3 + \delta x_0 x_1 x_2 x_3 x_4$ $+ \epsilon x_0^2 x_3^2 x_4 + x_0^2 x_2 x_4^2$	$\alpha, \beta, \gamma, \delta,$ $\epsilon \in \mathbb{C}^{\times}$	5
13	C-H	$\phi_{13}^{\text{nf}} = x_2^5 + \alpha x_1 x_2^3 x_3 + \beta x_1^2 x_2^2 x_3 + x_1^3 x_4^2$ $+ x_0 x_2^3 x_4 + \gamma x_0 x_1 x_2 x_3 x_4 + x_0^2 x_3^3$ $+ \delta x_0^2 x_2 x_4^2$	$\alpha, \beta, \gamma, \delta \in \mathbb{C}^{\times}$	4
14	C-H	$\phi_{14}^{\text{nf}} = x_2^5 + \alpha x_1 x_2^3 x_3 + \beta x_1^2 x_2^2 x_3 + x_1^4 x_4 + x_0 x_3^4$ $+ \gamma x_0 x_2^3 x_4 + \delta x_0 x_1 x_2 x_3 x_4 + x_0^2 x_2 x_4^2$	$\alpha, \beta, \gamma, \delta \in \mathbb{C}^{\times}$	4
15	C-F	$\phi_{15}^{\text{nf}} = x_1 x_2^4 + x_1^2 x_2^2 x_3$ $+ x_1^3 (\gamma x_3^2 + \beta x_3 x_4 + x_4^2) + x_0 x_2^3 x_4$ $+ x_0 x_1 x_2 R(x_3, x_4) + x_0^2 B(x_3, x_4)$	$\gamma, \beta \in \mathbb{C};$ $R \in \text{Sym}^2 \langle x_3, x_4 \rangle;$ $B \in \text{Sym}^3 \langle x_3, x_4 \rangle$	9
16	C-H	$\phi_{16}^{\text{nf}} = x_2^4 x_4 + x_1 x_2^2 x_3^2 + x_1^3 x_3 x_4 + x_0 x_1 x_2 x_3 x_4$ $+ \alpha x_0^2 x_3^3 + \beta x_0^2 x_1 x_4^2$	$\alpha, \beta \in \mathbb{C}^{\times}$	2
17	C-H	$\phi_{17}^{\text{nf}} = x_2^4 x_3 + x_1 x_2^3 x_4 + \alpha x_1^2 x_2^2 x_3 + x_1^3 x_3 x_4$ $+ \beta x_0 x_2^2 x_3^2 + \gamma x_0 x_1 x_2 x_3 x_4 + \delta x_0 x_1^2 x_4^2$ $+ x_0^2 x_3^3 + \epsilon x_0^2 x_2 x_4^2$	$\alpha, \beta, \gamma, \delta,$ $\epsilon \in \mathbb{C}^{\times}$	5
18	C-H	$\phi_{18}^{\text{nf}} = x_2^5 + x_1 x_2^2 x_3^2 + x_1^3 x_3 x_4 + x_0 x_3^4$ $+ \alpha x_0 x_1 x_2 x_3 x_4 + \beta x_0^2 x_1 x_4^2$	$\alpha, \beta \in \mathbb{C}^{\times}$	2
19	C-H	$\phi_{19}^{\text{nf}} = x_2^5 + \alpha x_1 x_2^3 x_3 + \beta x_1^2 x_2^2 x_3 + \gamma x_1^2 x_2^2 x_4$ $+ x_1^3 x_3 x_4 + \delta x_0 x_2^2 x_3^2 + \epsilon x_0 x_2^3 x_4 + x_0 x_1 x_3^3$ $+ \zeta x_0 x_1 x_2 x_3 x_4 + \xi x_0 x_1^2 x_4^2 + x_0^2 x_3^2 x_4$ $+ \theta x_0^2 x_2 x_4^2$	$\alpha, \beta, \gamma, \delta,$ $\epsilon, \zeta, \xi, \theta \in \mathbb{C}^{\times}$	8
20	C-F	$\phi_{20}^{\text{nf}} = x_0 B_{\lambda, \mu}(x_2, x_3) + x_1^2 C_{\mathbf{c}}(x_2, x_3) + x_1^3 x_2 x_4$ $+ x_0 x_1 x_4 Q_{\mathbf{q}}(x_2, x_3) + x_0 x_1^2 x_4^2$ $+ x_0^2 x_4^2 M_{\mathbf{m}}(x_2, x_3)$	λ, μ distinct in $\mathbb{C} \setminus \{0, 1\};$ $\mathbf{c} \in \mathbb{C}^4, \mathbf{q} \in \mathbb{C}^3,$ $\mathbf{m} \in \mathbb{C}^2$	11
21	C-H	$\phi_{21}^{\text{nf}} = x_2^3 x_3^2 + x_2^4 x_4 + x_1^3 x_3 x_4 + x_0 x_1 x_3^3$ $+ \alpha x_0 x_1 x_2 x_3 x_4 + \beta x_0^3 x_4^2$	$\alpha, \beta \in \mathbb{C}^{\times}$	2
22	C-H	$\phi_{22}^{\text{nf}} = x_2^4 x_3 + x_1 x_2^2 x_3^2 + \alpha x_1^2 x_3^3 + x_1^3 x_2 x_4$ $+ x_0 x_3^4 + \beta x_0 x_2^3 x_4 + \gamma x_0 x_1 x_2 x_3 x_4$ $+ \delta x_0^2 x_1 x_4^2$	$\alpha, \beta, \gamma, \delta \in \mathbb{C}^{\times}$	4
23	C-F	$\phi_{23}^{\text{nf}} = x_0 B_4(x_1, x_2, x_3, x_4)$	B_4 : smooth quartic surface in $\mathbb{P}^3$	19
24	C-H	$\phi_{24}^{\text{nf}} = x_2^3 x_3^2 + x_1 x_2^3 x_4 + \alpha x_1^2 x_3^3 + x_1^3 x_3 x_4$ $+ x_0 x_2 x_3^3 + \beta x_0 x_1 x_2 x_3 x_4 + \gamma x_0^3 x_4^2$	$\alpha, \beta, \gamma \in \mathbb{C}^{\times}$	3
25	C-H	$\phi_{25}^{\text{nf}} = x_2^4 x_3 + \alpha x_1 x_2^2 x_3^2 + \beta x_1^2 x_3^3 + x_1^2 x_2^2 x_4$ $+ x_1^3 x_3 x_4 + x_0 x_2 x_3^3 + \gamma x_0 x_2^3 x_4$ $+ \delta x_0 x_1 x_2 x_3 x_4 + \epsilon x_0^2 x_3^2 x_4 + \zeta x_0^2 x_1 x_4^2$	$\alpha, \beta, \gamma, \delta,$ $\epsilon, \zeta \in \mathbb{C}^{\times}$	6

continued on the next page

k	Test	Normal form	Parameters	Dim
26	C-H	$\phi_{26}^{\text{nf}} = x_2x_3^4 + x_1^4x_4 + x_0x_2^3x_4$ $+ \alpha x_0x_1x_2x_3x_4 + x_0^3x_4^2$	$\alpha \in \mathbb{C}^\times$	1
27	C-H	$\phi_{27}^{\text{nf}} = x_2^2x_3^3 + x_1x_3^4 + x_1^3x_2x_4 + x_0x_2^3x_4$ $+ \alpha x_0x_1x_2x_3x_4 + \beta x_0^3x_4^2$	$\alpha, \beta \in \mathbb{C}^\times$	2
28	C-H	$\phi_{28}^{\text{nf}} = x_2^3x_3^2 + x_1x_2x_3^3 + x_1^2x_2^2x_4 + \beta x_1^3x_3x_4$ $+ x_0x_3^4 + \alpha x_0x_2^3x_4 + \gamma x_0x_1x_2x_3x_4$ $+ \delta x_0^2x_3^2x_4 + \epsilon x_0^3x_4^2$	$\alpha, \beta, \gamma, \delta,$ $\epsilon \in \mathbb{C}^\times$	5
29	C-F	$\phi_{29}^{\text{nf}} = x_3A_4(x_1, x_2) + x_4C_4(x_1, x_2)$ $+ x_0Q_{2,2}(x_1, x_2; x_3, x_4) + x_0^2x_3x_4(x_3 - x_4)$	A_4, C_4 : binary quartics; $Q_{2,2}$ of bidegree (2, 2)	15
30	C-F	$\phi_{30}^{\text{nf}} = x_2^4x_3 + x_2^2x_3^2L_1(x_0, x_1) + x_2^3x_4L_2(x_0, x_1)$ $+ x_3^3Q_1(x_0, x_1) + x_2x_3x_4Q_2(x_0, x_1)$ $+ x_0x_1(x_0 - x_1)x_4^2$	L_1, L_2 : binary linear; Q_1, Q_2 : binary quadratic	9
31	C-F	$\phi_{31}^{\text{nf}} = x_4B_\lambda(x_1, x_2) + x_3^2C_3(x_1, x_2) + x_0x_3^3L_1(x_1, x_2)$ $+ x_0x_3x_4Q_2(x_1, x_2) + x_0^2x_3^2x_4$ $+ x_0^2x_4^2M_1(x_1, x_2)$	$\lambda \in \mathbb{C} \setminus \{0, 1\};$ C_3, L_1, Q_2, M_1 of degrees 3, 1, 2, 1	11
32	C-F	$\phi_{32}^{\text{nf}} = x_0^2C_0(x_2, x_3, x_4) + x_0x_1C_1(x_2, x_3, x_4)$ $+ x_1^2C_2(x_2, x_3, x_4)$	C_0, C_1, C_2 : ternary cubics	18
33	C-F	$\phi_{33}^{\text{nf}} = B_5(x_2, x_3) + x_1^3x_2x_4$ $+ x_0x_1x_4(x_2^2 + x_3^2) + x_0^3x_4^2$	B_5 : general binary quintic	6
34	C-F	$\phi_{34}^{\text{nf}} = x_4B_\lambda(x_1, x_2) + x_3^3Q_2(x_1, x_2) + x_0x_3^4$ $+ x_0x_3x_4R_2(x_1, x_2) + \beta x_0^2x_3^2x_4 + x_0^3x_4^2$	$\lambda \in \mathbb{C} \setminus \{0, 1\};$ Q_2, R_2 : binary quadratic; $\beta \in \mathbb{C}$	8
35	C-F	$\phi_{35}^{\text{nf}} = x_0A_4(x_2, x_3) + x_1C_4(x_2, x_3)$ $+ x_4Q_{2,2}(x_0, x_1; x_2, x_3) + x_4^2x_0x_1(x_0 - x_1)$	A_4, C_4 : binary quartics; $Q_{2,2}$ of bidegree (2, 2)	15
36	C-F	$\phi_{36}^{\text{nf}} = B_5(x_1, x_2, x_3) + x_0x_4C_3(x_1, x_2, x_3)$ $+ x_0^2x_3x_4^2$	B_5, C_3 : ternary forms of degrees 5, 3	24
37	C-F	$\phi_{37}^{\text{nf}} = x_3^2D_0(x_0, x_1, x_2) + x_3x_4D_1(x_0, x_1, x_2)$ $+ x_4^2D_2(x_0, x_1, x_2)$	D_0, D_1, D_2 : ternary cubics	18
38	C-F	$\phi_{38}^{\text{nf}} = x_4B_4(x_0, x_1, x_2, x_3)$	B_4 : smooth quartic surface in $\mathbb{P}^3$	19

Unless stated otherwise, “general” means that the coefficients belong to the specified nonempty Zariski-open subset of the relevant parameter space. The nonvanishing, discriminant, resultant, and residual-stability conditions used in the closed-orbit and normalization arguments are recorded in the individual cases.

Case $k = 1$

1-PS limit

The witness vector and the corresponding one-parameter subgroup are

$$r_1 = (36, 1, -4, -9, -24), \quad \lambda_1(t) = \text{diag}(t^{36}, t, t^{-4}, t^{-9}, t^{-24}), \quad t \in \mathbb{G}_m.$$

Since the entries of r_1 sum to zero, λ_1 is a one-parameter subgroup of $G = \text{SL}(5)$. Let f_1 be a general quintic supported on

$$S_1 = I(r_1)_{\geq 0}.$$

With the action convention of (2), its affine one-parameter-subgroup limit is the weight-zero part

$$\phi_1 := \lim_{t \rightarrow 0} \lambda_1(t) \cdot f_1.$$

The degree-five weight-zero equations have precisely the exponent vectors

$$(0, 4, 1, 0, 0), \quad (1, 0, 0, 4, 0), \quad (1, 0, 3, 0, 1), \quad (1, 1, 1, 1, 1), \quad (2, 0, 0, 0, 3).$$

Consequently,

$$\phi_1 = a_1 x_1^4 x_2 + a_2 x_0 x_3^4 + a_3 x_0 x_2^3 x_4 + a_4 x_0 x_1 x_2 x_3 x_4 + a_5 x_0^2 x_4^3, \quad a_i \in \mathbb{C}^\times.$$

Set $H_1 = \lambda_1(\mathbb{G}_m)$. The five weights of H_1 on $\langle x_0, \dots, x_4 \rangle$ are pairwise distinct, and their multiset is not invariant under multiplication by -1 . Hence

$$C_G(H_1) = N_G(H_1) = T,$$

and

$$W^{H_1} = \langle x_1^4 x_2, x_0 x_3^4, x_0 x_2^3 x_4, x_0 x_1 x_2 x_3 x_4, x_0^2 x_4^3 \rangle.$$

Polystability

Let $v_1, \dots, v_5$ be the five exponent vectors above, with $v_4 = (1, 1, 1, 1, 1)$, and set

$$\delta_i := v_i - v_4 \quad (i = 1, 2, 3, 5).$$

Explicitly,

$$\begin{aligned} \delta_1 &= (-1, 3, 0, -1, -1), & \delta_2 &= (0, -1, -1, 3, -1), \\ \delta_3 &= (0, -1, 2, -1, 0), & \delta_5 &= (1, -1, -1, -1, 2). \end{aligned}$$

The vectors $\delta_1, \delta_2, \delta_3$ are linearly independent and

$$\delta_1 + \delta_2 + \delta_3 + \delta_5 = 0.$$

Therefore

$$0 \in \text{relint Conv}\{\delta_1, \delta_2, \delta_3, \delta_5\}.$$

Since every coefficient of ϕ_1 is nonzero, the convex-hull criterion shows that $(T/H_1) \cdot \phi_1$ is closed in W^{H_1} . By affine Luna reduction and $N_G(H_1) = T$, the orbit $G \cdot \phi_1$ is closed in W . Thus $[\phi_1]$ is polystable.

Normal form

On the full-support chart of $\mathbb{P}(W^{H_1})$, rescale projectively so that $a_4 = 1$, and put

$$b_i := \frac{a_i}{a_4} \quad (i = 1, 2, 3, 5).$$

The relative characters $\delta_1, \delta_2, \delta_3$ form an integral basis of $X^*(T/H_1)$. Hence the character map

$$T/H_1 \xrightarrow{\sim} (\mathbb{C}^\times)^3, \quad [t] \mapsto (\delta_1(t), \delta_2(t), \delta_3(t)),$$

is an isomorphism. Choose $t \in T$ such that

$$\delta_i(t) = b_i^{-1} \quad (i = 1, 2, 3).$$

This normalizes the first four coefficients to 1. Since

$$\delta_5 = -(\delta_1 + \delta_2 + \delta_3),$$

the remaining coefficient is

$$\alpha = b_1 b_2 b_3 b_5 = \frac{a_1 a_2 a_3 a_5}{a_4^4} \in \mathbb{C}^\times.$$

Thus every general limit in Case 1 is equivalent to the unique normal form

$$\phi_1^{\text{nf}}(\alpha) = x_1^4 x_2 + x_0 x_3^4 + x_0 x_2^3 x_4 + x_0 x_1 x_2 x_3 x_4 + \alpha x_0^2 x_4^3, \quad \alpha \in \mathbb{C}^\times.$$

The parameter α is the complete invariant of the full-support T -orbits, so the corresponding quotient-side family is one-dimensional. Thus

$$\dim \Phi_1 = 1.$$

No further restriction on α is imposed at the normal-form stage; every member of this family is polystable.

Case $k = 2$

1-PS limit

The witness vector and the corresponding one-parameter subgroup are

$$r_2 = (27, 2, -3, -8, -18), \quad \lambda_2(t) = \text{diag}(t^{27}, t^2, t^{-3}, t^{-8}, t^{-18}), \quad t \in \mathbb{G}_m.$$

Since the entries of r_2 sum to zero, λ_2 is a one-parameter subgroup of $G = \text{SL}(5)$. Let f_2 be a general quintic supported on

$$S_2 = I(r_2)_{\geq 0}.$$

With the action convention of (2), its affine one-parameter-subgroup limit is the weight-zero part

$$\phi_2 := \lim_{t \rightarrow 0} \lambda_2(t) \cdot f_2.$$

The degree-five weight-zero equations have precisely the exponent vectors

$$\begin{aligned} (0, 3, 2, 0, 0), \quad (0, 4, 0, 1, 0), \quad (1, 0, 1, 3, 0), \\ (1, 0, 3, 0, 1), \quad (1, 1, 1, 1, 1), \quad (2, 0, 0, 0, 3). \end{aligned}$$

Consequently,

$$\begin{aligned} \phi_2 = a_1 x_1^3 x_2^2 + a_2 x_1^4 x_3 + a_3 x_0 x_2 x_3^3 \\ + a_4 x_0 x_2^3 x_4 + a_5 x_0 x_1 x_2 x_3 x_4 + a_6 x_0^2 x_4^3, \quad a_i \in \mathbb{C}^\times. \end{aligned}$$

Set $H_2 = \lambda_2(\mathbb{G}_m)$. The five weights of H_2 on $\langle x_0, \dots, x_4 \rangle$ are pairwise distinct, and their multiset is not invariant under multiplication by -1 . Hence

$$C_G(H_2) = N_G(H_2) = T,$$

and

$$W^{H_2} = \langle x_1^3 x_2^2, x_1^4 x_3, x_0 x_2 x_3^3, x_0 x_2^3 x_4, x_0 x_1 x_2 x_3 x_4, x_0^2 x_4^3 \rangle.$$

Polystability

Let $v_1, \dots, v_6$ be the six exponent vectors above, with $v_5 = (1, 1, 1, 1, 1)$, and set

$$\delta_i := v_i - v_5 \quad (i = 1, 2, 3, 4, 6).$$

Explicitly,

$$\begin{aligned} \delta_1 = (-1, 2, 1, -1, -1), \quad \delta_2 = (-1, 3, -1, 0, -1), \\ \delta_3 = (0, -1, 0, 2, -1), \quad \delta_4 = (0, -1, 2, -1, 0), \\ \delta_6 = (1, -1, -1, -1, 2). \end{aligned}$$

These characters span the three-dimensional character space of T/H_2 and satisfy the positive relation

$$\delta_1 + \delta_2 + 2\delta_3 + \delta_4 + 2\delta_6 = 0.$$

Therefore

$$0 \in \text{relint Conv}\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_6\}.$$

Since every coefficient of ϕ_2 is nonzero, the convex-hull criterion shows that $(T/H_2) \cdot \phi_2$ is closed in W^{H_2} . By affine Luna reduction and $N_G(H_2) = T$, the orbit $G \cdot \phi_2$ is closed in W . Thus $[\phi_2]$ is polystable.

Normal form

On the full-support chart of $\mathbb{P}(W^{H_2})$, use the fourth monomial as the projective reference and set

$$\varepsilon_i := v_i - v_4 \quad (i = 1, 2, 3, 5, 6).$$

A direct lattice calculation gives

$$X^*(T/H_2) = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2 \oplus \mathbb{Z}\varepsilon_3.$$

Hence

$$T/H_2 \xrightarrow{\sim} (\mathbb{C}^\times)^3, \quad [t] \mapsto (\varepsilon_1(t), \varepsilon_2(t), \varepsilon_3(t)).$$

Projectively rescale so that $a_4 = 1$, put

$$b_i := \frac{a_i}{a_4} \quad (i = 1, 2, 3, 5, 6),$$

and choose $t \in T$ such that

$$\varepsilon_i(t) = b_i^{-1} \quad (i = 1, 2, 3).$$

This normalizes the first four coefficients to 1. Since

$$\varepsilon_5 = -\varepsilon_1 + \varepsilon_2, \quad \varepsilon_6 = -4\varepsilon_1 + 3\varepsilon_2 - \varepsilon_3,$$

the two remaining coefficients are

$$\alpha = \frac{a_1 a_5}{a_2 a_4}, \quad \beta = \frac{a_1^4 a_3 a_6}{a_2^3 a_4^3}.$$

Thus every general limit in Case 2 is equivalent to the unique normal form

$$\begin{aligned} \phi_2^{\text{nf}}(\alpha, \beta) &= x_1^3 x_2^2 + x_1^4 x_3 + x_0 x_2 x_3^3 \\ &\quad + x_0 x_2^3 x_4 + \alpha x_0 x_1 x_2 x_3 x_4 + \beta x_0^2 x_4^3, \end{aligned} \quad (\alpha, \beta) \in (\mathbb{C}^\times)^2.$$

The pair (α, β) is the complete invariant of the full-support T -orbits. Since

$$\dim \mathbb{P}(W^{H_2})^\circ = 5, \quad \dim(T/H_2) = 3,$$

the corresponding quotient-side family is two-dimensional. Equivalently,

$$\dim \Phi_2 = 5 - 3 = 2.$$

No additional condition on (α, β) is imposed at the normal-form stage; every member of this family is polystable.

Case $k = 3$

1-PS limit

The witness vector and the corresponding one-parameter subgroup are

$$r_3 = (3, 0, 0, -1, -2), \quad \lambda_3(t) = \text{diag}(t^3, 1, 1, t^{-1}, t^{-2}), \quad t \in \mathbb{G}_m.$$

Since the entries of r_3 sum to zero, λ_3 is a one-parameter subgroup of $G = \text{SL}(5)$. Let f_3 be a general quintic supported on

$$S_3 = I(r_3)_{\geq 0}.$$

With the action convention of (2), its affine one-parameter-subgroup limit is the weight-zero part

$$\phi_3 := \lim_{t \rightarrow 0} \lambda_3(t) \cdot f_3.$$

Put $U := \langle x_1, x_2 \rangle$. Solving the degree-five weight-zero equations gives

$$W^{H_3} = \text{Sym}^5 U \oplus x_0 x_3^3 U \oplus x_0 x_3 x_4 \text{Sym}^2 U \oplus \mathbb{C} x_0^2 x_4^3, \quad \dim W^{H_3} = 12,$$

where $H_3 = \lambda_3(\mathbb{G}_m)$. Hence a general limit has the form

$$\phi_3 = B_5(x_1, x_2) + x_0 x_3^3 L_1(x_1, x_2) + x_0 x_3 x_4 Q_2(x_1, x_2) + \kappa x_0^2 x_4^3,$$

where B_5 , L_1 , and Q_2 are binary forms of degrees 5, 1, and 2, respectively. We work on the nonempty Zariski-open locus

$$\kappa \neq 0, \quad \text{Disc}(B_5) \neq 0, \quad \text{Disc}(Q_2) \neq 0, \quad \text{Res}(L_1, Q_2) \neq 0.$$

The weights of H_3 on $\langle x_0, \dots, x_4 \rangle$ are $3, 0, 0, -1, -2$. Therefore

$$C_G(H_3) = \{ \text{diag}(\tau_0) \oplus A \oplus \text{diag}(\tau_3) \oplus \text{diag}(\tau_4) \mid A \in \text{GL}(2), \tau_0 \tau_3 \tau_4 \det A = 1 \}.$$

In particular,

$$\dim C_G(H_3) = 6,$$

and H_3 acts trivially on W^{H_3} .

Polystability

Set $R := C_G(H_3)$. By affine Luna reduction, it is enough to prove that $R \cdot \phi_3$ is closed in W^{H_3} . Let $\lambda \in \Lambda_{\phi_3}^R$. After conjugation in the $\text{GL}(U)$ -block, choose an eigenbasis X, Y of U and write the weights of λ as

$$(\alpha, s + n, s - n, \beta, \gamma), \quad \sigma := \alpha + 2s + \beta + \gamma = 0.$$

Put

$$\ell := \alpha + s + 3\beta, \quad \omega := 2\alpha + 3\gamma.$$

The nonvanishing of the relative invariants above, together with the nonnegativity of all weights occurring in ϕ_3 , gives

$$\text{Disc}(B_5) \neq 0 \implies s \geq 0, \quad \text{Res}(L_1, Q_2) \neq 0 \implies \ell \geq 0, \quad \kappa \neq 0 \implies \omega \geq 0.$$

Since

$$5s + \ell + \omega = 3\sigma = 0,$$

we obtain

$$s = \ell = \omega = 0.$$

The condition $\text{Disc}(B_5) \neq 0$ then forces $n = 0$: if $n \neq 0$, the nonnegative-weight condition makes B_5 divisible by the cube of one of the eigenvectors, contradicting square-freeness. Consequently every one-parameter subgroup in $\Lambda_{\phi_3}^R$ has weights

$$(3m, 0, 0, -m, -2m) \quad (m \in \mathbb{Z}),$$

and hence

$$\Lambda_{\phi_3}^R = \{\lambda_3(t^m) \mid m \in \mathbb{Z}\}.$$

This set is symmetric. The Casimiro–Florentino criterion therefore shows that ϕ_3 is polystable for the R -action. By affine Luna reduction, $G \cdot \phi_3$ is closed in W , and hence $[\phi_3]$ is polystable.

Normal form

Because Q_2 is square-free and L_1 is not a factor of Q_2 , the divisor of $L_1 Q_2$ on $\mathbb{P}(U) \cong \mathbb{P}^1$ consists of three distinct points. The $\text{PGL}(U)$ -part of $C_G(H_3)$ sends these points to the roots of

$$x_2(x_1^2 + x_2^2).$$

Thus we may take

$$L_1 = x_2, \quad Q_2 = x_1^2 + x_2^2.$$

The remaining diagonal action and projective rescaling normalize the coefficients of the other three blocks to 1. Every general limit in Case 3 is therefore equivalent to a member of the normal-form family

$$\phi_3^{\text{nf}}(B_5) = B_5(x_1, x_2) + x_0 x_2 x_3^3 + x_0(x_1^2 + x_2^2)x_3 x_4 + x_0^2 x_4^3,$$

where B_5 ranges over a nonempty Zariski-open subset of the space of binary quintics, in particular $\text{Disc}(B_5) \neq 0$. The residual stabilizer is finite for a general B_5 . Since

$$\dim W^{H_3} = 12, \quad \dim(C_G(H_3)/H_3) = 5,$$

the corresponding quotient-side family has dimension

$$\dim \Phi_3 = 12 - 1 - 5 = 6.$$

Case $k = 4$

1-PS limit

The witness vector and the corresponding one-parameter subgroup are

$$r_4 = (24, 9, -6, -11, -16), \quad \lambda_4(t) = \text{diag}(t^{24}, t^9, t^{-6}, t^{-11}, t^{-16}), \quad t \in \mathbb{G}_m.$$

Since the entries of r_4 sum to zero, λ_4 is a one-parameter subgroup of $G = \text{SL}(5)$. Let f_4 be a general quintic supported on

$$S_4 = I(r_4)_{\geq 0}.$$

With the action convention of (2), its affine one-parameter-subgroup limit is the weight-zero part

$$\phi_4 := \lim_{t \rightarrow 0} \lambda_4(t) \cdot f_4.$$

The degree-five weight-zero equations have precisely the exponent vectors

$$\begin{aligned} (0, 2, 3, 0, 0), & \quad (0, 3, 0, 1, 1), & (1, 0, 4, 0, 0), \\ (1, 1, 0, 3, 0), & (1, 1, 1, 1, 1), & (2, 0, 0, 0, 3). \end{aligned}$$

Consequently,

$$\begin{aligned}\phi_4 = & a_1x_1^2x_2^3 + a_2x_1^3x_3x_4 + a_3x_0x_2^4 + a_4x_0x_1x_3^3 \\ & + a_5x_0x_1x_2x_3x_4 + a_6x_0^2x_4^3, \quad a_i \in \mathbb{C}^\times.\end{aligned}$$

Set $H_4 = \lambda_4(\mathbb{G}_m)$. The five weights of H_4 on $\langle x_0, \dots, x_4 \rangle$ are pairwise distinct, and their multiset is not invariant under multiplication by -1 . Hence

$$C_G(H_4) = N_G(H_4) = T,$$

and

$$W^{H_4} = \langle x_1^2x_2^3, x_1^3x_3x_4, x_0x_2^4, x_0x_1x_3^3, x_0x_1x_2x_3x_4, x_0^2x_4^3 \rangle, \quad \dim W^{H_4} = 6.$$

Polystability

Let $v_1, \dots, v_6$ be the six exponent vectors above, with $v_5 = (1, 1, 1, 1, 1)$, and set

$$\delta_i := v_i - v_5 \quad (i = 1, 2, 3, 4, 6).$$

Explicitly,

$$\begin{aligned}\delta_1 &= (-1, 1, 2, -1, -1), & \delta_2 &= (-1, 2, -1, 0, 0), \\ \delta_3 &= (0, -1, 3, -1, -1), & \delta_4 &= (0, 0, -1, 2, -1), \\ \delta_6 &= (1, -1, -1, -1, 2).\end{aligned}$$

These characters span the three-dimensional character space of T/H_4 and satisfy the positive relation

$$\delta_1 + \delta_2 + \delta_3 + 2\delta_4 + 2\delta_6 = 0.$$

Therefore

$$0 \in \text{relint Conv}\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_6\}.$$

Since every coefficient of ϕ_4 is nonzero, the convex-hull criterion shows that $(T/H_4) \cdot \phi_4$ is closed in W^{H_4} . By affine Luna reduction and $N_G(H_4) = T$, the orbit $G \cdot \phi_4$ is closed in W . Thus $[\phi_4]$ is polystable.

Normal form

On the full-support chart of $\mathbb{P}(W^{H_4})$, use the fourth monomial as the projective reference and set

$$\varepsilon_i := v_i - v_4 \quad (i = 1, 2, 3, 5, 6).$$

A direct lattice calculation shows that

$$X^*(T/H_4) = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2 \oplus \mathbb{Z}\varepsilon_3,$$

while

$$\varepsilon_5 = -\varepsilon_1 + \varepsilon_2 + \varepsilon_3, \quad \varepsilon_6 = -4\varepsilon_1 + 3\varepsilon_2 + 3\varepsilon_3.$$

Thus

$$T/H_4 \xrightarrow{\sim} (\mathbb{C}^\times)^3, \quad [t] \mapsto (\varepsilon_1(t), \varepsilon_2(t), \varepsilon_3(t)).$$

Projectively rescale so that $a_4 = 1$, and use the effective torus action to normalize a_1, a_2, a_3 to 1. The two remaining coefficients are

$$\alpha = \frac{a_1a_5}{a_2a_3}, \quad \beta = \frac{a_1^4a_4a_6}{a_2^3a_3^3}.$$

Hence every general limit in Case 4 is equivalent to the unique normal form

$$\begin{aligned}\phi_4^{\text{nf}}(\alpha, \beta) = & x_1^2 x_2^3 + x_1^3 x_3 x_4 + x_0 x_2^4 + x_0 x_1 x_3^3 \\ & + \alpha x_0 x_1 x_2 x_3 x_4 + \beta x_0^2 x_4^3, \quad (\alpha, \beta) \in (\mathbb{C}^\times)^2.\end{aligned}$$

The pair (α, β) is the complete invariant of the full-support T -orbits. Since

$$\dim \mathbb{P}(W^{H_4})^\circ = 5, \quad \dim(T/H_4) = 3,$$

the corresponding quotient-side family has dimension

$$\dim \Phi_4 = 5 - 3 = 2.$$

No additional condition on (α, β) is imposed at the normal-form stage; every member of this family is polystable.

Case $k = 5$

1-PS limit

The witness vector and the corresponding one-parameter subgroup are

$$r_5 = (21, 6, -4, -9, -14), \quad \lambda_5(t) = \text{diag}(t^{21}, t^6, t^{-4}, t^{-9}, t^{-14}), \quad t \in \mathbb{G}_m.$$

Since the entries of r_5 sum to zero, λ_5 is a one-parameter subgroup of $G = \text{SL}(5)$. Let f_5 be a general quintic supported on

$$S_5 = I(r_5)_{\geq 0}.$$

With the action convention of (2), its affine one-parameter-subgroup limit is the weight-zero part

$$\phi_5 := \lim_{t \rightarrow 0} \lambda_5(t) \cdot f_5.$$

The degree-five weight-zero equations have precisely the exponent vectors

$$\begin{aligned}(0, 2, 3, 0, 0), \quad (0, 3, 0, 2, 0), \quad (0, 3, 1, 0, 1), \quad (1, 0, 3, 1, 0), \\ (1, 1, 0, 3, 0), \quad (1, 1, 1, 1, 1), \quad (2, 0, 0, 0, 3).\end{aligned}$$

Consequently,

$$\begin{aligned}\phi_5 = & a_1 x_1^2 x_2^3 + a_2 x_1^3 x_3^2 + a_3 x_1^3 x_2 x_4 + a_4 x_0 x_2^3 x_3 \\ & + a_5 x_0 x_1 x_3^3 + a_6 x_0 x_1 x_2 x_3 x_4 + a_7 x_0^2 x_4^3, \quad a_i \in \mathbb{C}^\times.\end{aligned}$$

Set $H_5 = \lambda_5(\mathbb{G}_m)$. The five weights of H_5 on $\langle x_0, \dots, x_4 \rangle$ are pairwise distinct, and their multiset is not invariant under multiplication by -1 . Hence

$$C_G(H_5) = N_G(H_5) = T,$$

and

$$\begin{aligned}W^{H_5} = \langle & x_1^2 x_2^3, x_1^3 x_3^2, x_1^3 x_2 x_4, x_0 x_2^3 x_3, \\ & x_0 x_1 x_3^3, x_0 x_1 x_2 x_3 x_4, x_0^2 x_4^3 \rangle, \quad \dim W^{H_5} = 7.\end{aligned}$$

Polystability

Let $v_1, \dots, v_7$ be the seven exponent vectors above, with $v_6 = (1, 1, 1, 1, 1)$, and set

$$\delta_i := v_i - v_6 \quad (i = 1, \dots, 7).$$

Thus $\delta_6 = 0$, and the six nonzero characters span the three-dimensional character space of T/H_5 . They satisfy the positive relation

$$\delta_1 + \delta_2 + \delta_3 + 2\delta_4 + 2\delta_5 + 3\delta_7 = 0.$$

Therefore

$$0 \in \text{relint Conv}\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_7\}.$$

Since every coefficient of ϕ_5 is nonzero, the convex-hull criterion shows that $(T/H_5) \cdot \phi_5$ is closed in W^{H_5} . By affine Luna reduction and $N_G(H_5) = T$, the orbit $G \cdot \phi_5$ is closed in W . Thus $[\phi_5]$ is polystable.

Normal form

On the full-support chart of $\mathbb{P}(W^{H_5})$, use the fourth monomial as the projective reference and put

$$\varepsilon_i := v_i - v_4 \quad (i = 1, 2, 3, 5, 6, 7).$$

A direct character-lattice calculation gives

$$X^*(T/H_5) = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2 \oplus \mathbb{Z}\varepsilon_3,$$

together with

$$\varepsilon_5 = -\varepsilon_1 + \varepsilon_2, \quad \varepsilon_6 = -\varepsilon_1 + \varepsilon_3, \quad \varepsilon_7 = -3\varepsilon_1 - \varepsilon_2 + 3\varepsilon_3.$$

Projectively rescale so that $a_4 = 1$, and use the effective torus action to normalize a_1, a_2, a_3 to 1. The three remaining coefficients are

$$\alpha = \frac{a_1 a_5}{a_2 a_4}, \quad \beta = \frac{a_1 a_6}{a_3 a_4}, \quad \gamma = \frac{a_1^3 a_2 a_7}{a_3^3 a_4^2}.$$

Hence every general limit in Case 5 is equivalent to the unique normal form

$$\begin{aligned} \phi_5^{\text{nf}}(\alpha, \beta, \gamma) = & x_1^2 x_2^3 + x_1^3 x_3^2 + x_1^3 x_2 x_4 + x_0 x_2^3 x_3 \\ & + \alpha x_0 x_1 x_3^3 + \beta x_0 x_1 x_2 x_3 x_4 + \gamma x_0^2 x_4^3, \end{aligned}$$

where

$$(\alpha, \beta, \gamma) \in (\mathbb{C}^\times)^3.$$

The triple (α, β, γ) is the complete invariant of the full-support T -orbits. Since

$$\dim \mathbb{P}(W^{H_5})^\circ = 6, \quad \dim(T/H_5) = 3,$$

the corresponding quotient-side family has dimension

$$\dim \Phi_5 = 6 - 3 = 3.$$

No additional condition on (α, β, γ) is imposed at the normal-form stage; every member of this family is polystable.

Case $k = 6$

1-PS limit

The witness vector and the corresponding one-parameter subgroup are

$$r_6 = (18, 3, -2, -7, -12), \quad \lambda_6(t) = \text{diag}(t^{18}, t^3, t^{-2}, t^{-7}, t^{-12}), \quad t \in \mathbb{G}_m.$$

Since the entries of r_6 sum to zero, λ_6 is a one-parameter subgroup of $G = \text{SL}(5)$. Let f_6 be a general quintic supported on

$$S_6 = I(r_6)_{\geq 0}.$$

With the action convention of (2), its affine one-parameter-subgroup limit is the weight-zero part

$$\phi_6 := \lim_{t \rightarrow 0} \lambda_6(t) \cdot f_6.$$

The degree-five weight-zero equations have precisely the exponent vectors

$$\begin{aligned} (0, 2, 3, 0, 0), \quad (0, 3, 1, 1, 0), \quad (0, 4, 0, 0, 1), \quad (1, 0, 2, 2, 0), \quad (1, 0, 3, 0, 1), \\ (1, 1, 0, 3, 0), \quad (1, 1, 1, 1, 1), \quad (1, 2, 0, 0, 2), \quad (2, 0, 0, 0, 3). \end{aligned}$$

Consequently,

$$\begin{aligned} \phi_6 = & a_1 x_1^2 x_2^3 + a_2 x_1^3 x_2 x_3 + a_3 x_1^4 x_4 + a_4 x_0 x_2^2 x_3^2 + a_5 x_0 x_2^3 x_4 \\ & + a_6 x_0 x_1 x_3^3 + a_7 x_0 x_1 x_2 x_3 x_4 + a_8 x_0 x_1^2 x_4^2 + a_9 x_0^2 x_4^3, \quad a_i \in \mathbb{C}^\times. \end{aligned}$$

Set $H_6 = \lambda_6(\mathbb{G}_m)$. The five weights of H_6 on $\langle x_0, \dots, x_4 \rangle$ are pairwise distinct, and their multiset is not invariant under multiplication by -1 . Hence

$$C_G(H_6) = N_G(H_6) = T.$$

Moreover,

$$\begin{aligned} W^{H_6} = \langle & x_1^2 x_2^3, x_1^3 x_2 x_3, x_1^4 x_4, x_0 x_2^2 x_3^2, x_0 x_2^3 x_4, \\ & x_0 x_1 x_3^3, x_0 x_1 x_2 x_3 x_4, x_0 x_1^2 x_4^2, x_0^2 x_4^3 \rangle, \quad \dim W^{H_6} = 9. \end{aligned}$$

Polystability

Let $v_1, \dots, v_9$ be the nine exponent vectors above, with $v_7 = (1, 1, 1, 1, 1)$, and set

$$\delta_i := v_i - v_7 \quad (i = 1, \dots, 9).$$

Thus $\delta_7 = 0$. The eight nonzero characters span the three-dimensional character space of T/H_6 and satisfy the positive relation

$$\delta_1 + \delta_2 + \delta_3 + 3\delta_4 + \delta_5 + 2\delta_6 + \delta_8 + 3\delta_9 = 0.$$

Therefore

$$0 \in \text{relint Conv}\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_8, \delta_9\}.$$

Since every coefficient of ϕ_6 is nonzero, the convex-hull criterion shows that $(T/H_6) \cdot \phi_6$ is closed in W^{H_6} . By affine Luna reduction and $C_G(H_6) = T$, the orbit $G \cdot \phi_6$ is closed in W . Thus $[\phi_6]$ is polystable.

Normal form

On the full-support chart

$$\mathbb{P}(W^{H_6})^\circ = \{[a_1 : \cdots : a_9] \mid a_i \neq 0\} \cong (\mathbb{C}^\times)^8,$$

the effective torus T/H_6 has dimension 3. Combining its action with overall projective rescaling, one may normalize the first four coefficients to 1. The affine relations

$$\begin{aligned} v_5 &= v_1 - 2v_2 + v_3 + v_4, & v_6 &= -v_1 + v_2 + v_4, \\ v_7 &= -v_2 + v_3 + v_4, & v_8 &= -2v_2 + 2v_3 + v_4, \\ v_9 &= -4v_2 + 3v_3 + 2v_4 \end{aligned}$$

give the five remaining quotient parameters

$$\begin{aligned} \alpha &= \frac{a_2^2 a_5}{a_1 a_3 a_4}, & \beta &= \frac{a_1 a_6}{a_2 a_4}, & \gamma &= \frac{a_2 a_7}{a_3 a_4}, \\ \delta &= \frac{a_2^2 a_8}{a_3^2 a_4}, & \epsilon &= \frac{a_2^4 a_9}{a_3^3 a_4^2}. \end{aligned}$$

Hence every general limit in Case 6 is equivalent to the unique normal form

$$\begin{aligned} \phi_6^{\text{nf}}(\alpha, \beta, \gamma, \delta, \epsilon) &= x_1^2 x_2^3 + x_1^3 x_2 x_3 + x_1^4 x_4 + x_0 x_2^2 x_3^2 + \alpha x_0 x_2^3 x_4 \\ &\quad + \beta x_0 x_1 x_3^3 + \gamma x_0 x_1 x_2 x_3 x_4 + \delta x_0 x_1^2 x_4^2 + \epsilon x_0^2 x_4^3, \end{aligned}$$

where

$$(\alpha, \beta, \gamma, \delta, \epsilon) \in (\mathbb{C}^\times)^5.$$

These five parameters form a complete invariant of the full-support T -orbits. Since

$$\dim \mathbb{P}(W^{H_6})^\circ = 8, \quad \dim(T/H_6) = 3,$$

and Luna reduction induces a finite morphism to the ambient GIT quotient, the corresponding quotient-side family has dimension

$$\dim \Phi_6 = 8 - 3 = 5.$$

No additional condition on $(\alpha, \beta, \gamma, \delta, \epsilon)$ is imposed at the normal-form stage; every member of this family is polystable.

Case $k = 7$

1-PS limit

The witness vector and the corresponding one-parameter subgroup are

$$r_7 = (5, 1, 0, -2, -4), \quad \lambda_7(t) = \text{diag}(t^5, t, 1, t^{-2}, t^{-4}), \quad t \in \mathbb{G}_m.$$

Since the entries of r_7 sum to zero, λ_7 is a one-parameter subgroup of $G = \text{SL}(5)$. Let f_7 be a general quintic supported on

$$S_7 = I(r_7)_{\geq 0}.$$

With the action convention of (2), its affine one-parameter-subgroup limit is the weight-zero part

$$\phi_7 := \lim_{t \rightarrow 0} \lambda_7(t) \cdot f_7.$$

The degree-five weight-zero equations have precisely the exponent vectors

$$(0, 0, 5, 0, 0), \quad (0, 2, 2, 1, 0), \quad (0, 4, 0, 0, 1), \quad (1, 1, 0, 3, 0), \quad (1, 1, 1, 1, 1), \quad (2, 0, 0, 1, 2).$$

Consequently,

$$\begin{aligned} \phi_7 = & a_1 x_2^5 + a_2 x_1^2 x_2^2 x_3 + a_3 x_1^4 x_4 + a_4 x_0 x_1 x_3^3 \\ & + a_5 x_0 x_1 x_2 x_3 x_4 + a_6 x_0^2 x_3 x_4^2, \quad a_i \in \mathbb{C}^\times. \end{aligned}$$

Set $H_7 = \lambda_7(\mathbb{G}_m)$. The five weights of H_7 on $\langle x_0, \dots, x_4 \rangle$ are pairwise distinct, and their multiset is not invariant under multiplication by -1 . Hence

$$C_G(H_7) = N_G(H_7) = T.$$

Moreover,

$$\begin{aligned} W^{H_7} = & \langle x_2^5, x_1^2 x_2^2 x_3, x_1^4 x_4, x_0 x_1 x_3^3, \\ & x_0 x_1 x_2 x_3 x_4, x_0^2 x_3 x_4^2 \rangle, \quad \dim W^{H_7} = 6. \end{aligned}$$

Polystability

Let $v_1, \dots, v_6$ be the six exponent vectors above, with $v_5 = (1, 1, 1, 1, 1)$, and set

$$\delta_i := v_i - v_5 \quad (i = 1, \dots, 6).$$

Thus $\delta_5 = 0$. The five nonzero characters span the three-dimensional character space of T/H_7 and satisfy the positive relation

$$\delta_1 + 2\delta_2 + \delta_3 + \delta_4 + 4\delta_6 = 0.$$

Therefore

$$0 \in \text{relint Conv}\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_6\}.$$

Since every coefficient of ϕ_7 is nonzero, the convex-hull criterion shows that $(T/H_7) \cdot \phi_7$ is closed in W^{H_7} . By affine Luna reduction and $C_G(H_7) = T$, the orbit $G \cdot \phi_7$ is closed in W . Thus $[\phi_7]$ is polystable.

Normal form

On the full-support chart

$$\mathbb{P}(W^{H_7})^\circ = \{[a_1 : \dots : a_6] \mid a_i \neq 0\} \cong (\mathbb{C}^\times)^5,$$

the effective torus T/H_7 has dimension 3. Combining its action with overall projective rescaling, one may normalize the first four coefficients to 1. The affine relations

$$v_5 = v_1 - 2v_2 + v_3 + v_4, \quad v_6 = 2v_1 - 5v_2 + 2v_3 + 2v_4$$

give the two quotient parameters

$$\alpha = \frac{a_2^2 a_5}{a_1 a_3 a_4}, \quad \beta = \frac{a_2^5 a_6}{a_1^2 a_3^2 a_4^2}.$$

Hence every general limit in Case 7 is equivalent to a normal form

$$\begin{aligned} \phi_7^{\text{nf}}(\alpha, \beta) = & x_2^5 + x_1^2 x_2^2 x_3 + x_1^4 x_4 + x_0 x_1 x_3^3 \\ & + \alpha x_0 x_1 x_2 x_3 x_4 + \beta x_0^2 x_3 x_4^2, \end{aligned}$$

where

$$(\alpha, \beta) \in (\mathbb{C}^\times)^2.$$

The parameters α and β form coordinates on the full-support torus quotient. Since

$$\dim \mathbb{P}(W^{H_7})^\circ = 5, \quad \dim(T/H_7) = 3,$$

the corresponding quotient-side family has dimension

$$\dim \Phi_7 = 5 - 3 = 2.$$

No additional condition on (α, β) is imposed at the normal-form stage; every member of this family is polystable.

Case $k = 8$

1-PS limit

The witness vector and the corresponding one-parameter subgroup are

$$r_8 = (4, 2, -1, -2, -3), \quad \lambda_8(t) = \text{diag}(t^4, t^2, t^{-1}, t^{-2}, t^{-3}), \quad t \in \mathbb{G}_m.$$

Since the entries of r_8 sum to zero, λ_8 is a one-parameter subgroup of $G = \text{SL}(5)$. Let f_8 be a general quintic supported on

$$S_8 = I(r_8)_{\geq 0}.$$

With the action convention of (2), its affine one-parameter-subgroup limit is the weight-zero part

$$\phi_8 := \lim_{t \rightarrow 0} \lambda_8(t) \cdot f_8.$$

The degree-five weight-zero equations have precisely the exponent vectors

$$(0, 2, 2, 1, 0), \quad (0, 3, 0, 0, 2), \quad (1, 0, 4, 0, 0), \quad (1, 1, 0, 3, 0), \quad (1, 1, 1, 1, 1), \quad (2, 0, 0, 1, 2).$$

Consequently,

$$\begin{aligned} \phi_8 = & a_1 x_1^2 x_2^2 x_3 + a_2 x_1^3 x_4^2 + a_3 x_0 x_2^4 + a_4 x_0 x_1 x_3^3 \\ & + a_5 x_0 x_1 x_2 x_3 x_4 + a_6 x_0^2 x_3 x_4^2, \quad a_i \in \mathbb{C}^\times. \end{aligned}$$

Set $H_8 = \lambda_8(\mathbb{G}_m)$. The five weights of H_8 on $\langle x_0, \dots, x_4 \rangle$ are pairwise distinct, and their multiset is not invariant under multiplication by -1 . Hence

$$C_G(H_8) = N_G(H_8) = T.$$

Moreover,

$$\begin{aligned} W^{H_8} = & \langle x_1^2 x_2^2 x_3, x_1^3 x_4^2, x_0 x_2^4, x_0 x_1 x_3^3, \\ & x_0 x_1 x_2 x_3 x_4, x_0^2 x_3 x_4^2 \rangle, \quad \dim W^{H_8} = 6. \end{aligned}$$

Polystability

Let $v_1, \dots, v_6$ be the six exponent vectors above, with $v_5 = (1, 1, 1, 1, 1)$, and set

$$\delta_i := v_i - v_5 \quad (i = 1, \dots, 6).$$

Thus $\delta_5 = 0$. The five nonzero characters span the three-dimensional character space of T/H_8 and satisfy the positive relation

$$\delta_1 + \delta_2 + \delta_3 + \delta_4 + 2\delta_6 = 0.$$

Therefore

$$0 \in \text{relint Conv}\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_6\}.$$

Since every coefficient of ϕ_8 is nonzero, the convex-hull criterion shows that $(T/H_8) \cdot \phi_8$ is closed in W^{H_8} . By affine Luna reduction and $C_G(H_8) = T$, the orbit $G \cdot \phi_8$ is closed in W . Thus $[\phi_8]$ is polystable.

Normal form

On the full-support chart

$$\mathbb{P}(W^{H_8})^\circ = \{[a_1 : \cdots : a_6] \mid a_i \neq 0\} \cong (\mathbb{C}^\times)^5,$$

the effective torus T/H_8 has dimension 3. Combining its action with overall projective rescaling, one may normalize the first four coefficients to 1. The affine relations

$$2v_5 = -v_1 + v_2 + v_3 + v_4, \quad v_6 = -2v_1 + v_2 + v_3 + v_4$$

show that the remaining coefficients may be parametrized by

$$\alpha^2 = \frac{a_1 a_5^2}{a_2 a_3 a_4}, \quad \beta = \frac{a_1^2 a_6}{a_2 a_3 a_4}.$$

After choosing α , every general limit in Case 8 is equivalent to a normal form

$$\begin{aligned} \phi_8^{\text{nf}}(\alpha, \beta) = & x_1^2 x_2^2 x_3 + x_1^3 x_4^2 + x_0 x_2^4 + x_0 x_1 x_3^3 \\ & + \alpha x_0 x_1 x_2 x_3 x_4 + \beta x_0^2 x_3 x_4^2, \end{aligned}$$

where

$$(\alpha, \beta) \in (\mathbb{C}^\times)^2.$$

Since

$$\dim \mathbb{P}(W^{H_8})^\circ = 5, \quad \dim(T/H_8) = 3,$$

the corresponding quotient-side family has dimension

$$\dim \Phi_8 = 5 - 3 = 2.$$

No additional condition on (α, β) is imposed at the normal-form stage; every member of this family is polystable.

Case $k = 9$

1-PS limit

The witness vector and the corresponding one-parameter subgroup are

$$r_9 = (19, 4, -1, -6, -16), \quad \lambda_9(t) = \text{diag}(t^{19}, t^4, t^{-1}, t^{-6}, t^{-16}), \quad t \in \mathbb{G}_m.$$

Since the entries of r_9 sum to zero, λ_9 is a one-parameter subgroup of $G = \text{SL}(5)$. Let f_9 be a general quintic supported on

$$S_9 = I(r_9)_{\geq 0}.$$

With the action convention of (2), its affine one-parameter-subgroup limit is the weight-zero part

$$\phi_9 := \lim_{t \rightarrow 0} \lambda_9(t) \cdot f_9.$$

The degree-five weight-zero equations have precisely the exponent vectors

$$\begin{aligned} (0, 1, 4, 0, 0), \quad (0, 2, 2, 1, 0), \quad (0, 3, 0, 2, 0), \quad (0, 4, 0, 0, 1), \\ (1, 0, 1, 3, 0), \quad (1, 0, 3, 0, 1), \quad (1, 1, 1, 1, 1), \quad (2, 0, 0, 1, 2). \end{aligned}$$

Consequently,

$$\begin{aligned} \phi_9 = & a_1 x_1 x_2^4 + a_2 x_1^2 x_2^2 x_3 + a_3 x_1^3 x_3^2 + a_4 x_1^4 x_4 \\ & + a_5 x_0 x_2 x_3^3 + a_6 x_0 x_2^3 x_4 + a_7 x_0 x_1 x_2 x_3 x_4 + a_8 x_0^2 x_3 x_4^2, \quad a_i \in \mathbb{C}^\times. \end{aligned}$$

Set $H_9 = \lambda_9(\mathbb{G}_m)$. The five weights of H_9 on $\langle x_0, \dots, x_4 \rangle$ are pairwise distinct, and their multiset is not invariant under multiplication by -1 . Hence

$$C_G(H_9) = N_G(H_9) = T.$$

Moreover,

$$W^{H_9} = \langle x_1 x_2^4, x_1^2 x_2^2 x_3, x_1^3 x_3^2, x_1^4 x_4, \\ x_0 x_2 x_3^3, x_0 x_2^3 x_4, x_0 x_1 x_2 x_3 x_4, x_0^2 x_3 x_4^2 \rangle, \quad \dim W^{H_9} = 8.$$

Polystability

Let $v_1, \dots, v_8$ be the eight exponent vectors above, with $v_7 = (1, 1, 1, 1, 1)$, and set

$$\delta_i := v_i - v_7 \quad (i = 1, \dots, 8).$$

Thus $\delta_7 = 0$. The seven nonzero characters span the three-dimensional character space of T/H_9 and satisfy the positive relation

$$\delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 + \delta_6 + 4\delta_8 = 0.$$

Therefore

$$0 \in \text{relint Conv}\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_8\}.$$

Since every coefficient of ϕ_9 is nonzero, the convex-hull criterion shows that $(T/H_9) \cdot \phi_9$ is closed in W^{H_9} . By affine Luna reduction and $C_G(H_9) = T$, the orbit $G \cdot \phi_9$ is closed in W . Thus $[\phi_9]$ is polystable.

Normal form

On the full-support chart

$$\mathbb{P}(W^{H_9})^\circ = \{[a_1 : \dots : a_8] \mid a_i \neq 0\} \cong (\mathbb{C}^\times)^7,$$

the effective torus T/H_9 has dimension 3. The character differences

$$v_2 - v_1 = (0, 1, -2, 1, 0),$$

$$v_4 - v_1 = (0, 3, -4, 0, 1),$$

$$v_5 - v_1 = (1, -1, -3, 3, 0)$$

have rank 3 in the character space of T/H_9 . Combining the torus action with overall projective rescaling, one may therefore normalize the coefficients of

$$x_1 x_2^4, \quad x_1^2 x_2^2 x_3, \quad x_1^4 x_4, \quad x_0 x_2 x_3^3$$

to 1. Every general limit in Case 9 is consequently equivalent to

$$\phi_9^{\text{nf}}(\alpha, \beta, \gamma, \delta) = x_1 x_2^4 + x_1^2 x_2^2 x_3 + \alpha x_1^3 x_3^2 + x_1^4 x_4 + x_0 x_2 x_3^3 \\ + \beta x_0 x_2^3 x_4 + \gamma x_0 x_1 x_2 x_3 x_4 + \delta x_0^2 x_3 x_4^2,$$

where

$$(\alpha, \beta, \gamma, \delta) \in (\mathbb{C}^\times)^4.$$

Since

$$\dim \mathbb{P}(W^{H_9})^\circ = 7, \quad \dim(T/H_9) = 3,$$

the corresponding quotient-side family has dimension

$$\dim \Phi_9 = 7 - 3 = 4.$$

No additional condition on $(\alpha, \beta, \gamma, \delta)$ is imposed at the normal-form stage; every member of this family is polystable.

Case $k = 10$

1-PS limit

The witness vector and the corresponding one-parameter subgroup are

$$r_{10} = (12, 2, -3, -3, -8), \quad \lambda_{10}(t) = \text{diag}(t^{12}, t^2, t^{-3}, t^{-3}, t^{-8}), \quad t \in \mathbb{G}_m.$$

Since the entries of r_{10} sum to zero, λ_{10} is a one-parameter subgroup of $G = \text{SL}(5)$. Let f_{10} be a general quintic supported on

$$S_{10} = I(r_{10})_{\geq 0}.$$

With the action convention of (2), its affine one-parameter-subgroup limit is the weight-zero part

$$\phi_{10} := \lim_{t \rightarrow 0} \lambda_{10}(t) \cdot f_{10}.$$

Put

$$U := \langle x_2, x_3 \rangle.$$

Solving the degree-five weight-zero equations gives

$$\begin{aligned} W^{H_{10}} &= x_1^3 \text{Sym}^2 U \oplus \mathbb{C}x_1^4 x_4 \oplus x_0 \text{Sym}^4 U \oplus x_0 x_1 x_4 \text{Sym}^2 U \\ &\quad \oplus \mathbb{C}x_0 x_1^2 x_4^2 \oplus \mathbb{C}x_0^2 x_4^3, \quad \dim W^{H_{10}} = 14, \end{aligned}$$

where $H_{10} = \lambda_{10}(\mathbb{G}_m)$. Hence a general limit has the form

$$\begin{aligned} \phi_{10} &= x_1^3 Q_2(x_2, x_3) + \kappa x_1^4 x_4 + x_0 B_4(x_2, x_3) \\ &\quad + x_0 x_1 x_4 R_2(x_2, x_3) + \beta x_0 x_1^2 x_4^2 + \epsilon x_0^2 x_4^3, \end{aligned}$$

where Q_2, R_2 are binary quadratics and B_4 is a binary quartic. For the closed-orbit construction we work on the nonempty Zariski-open locus

$$\kappa \epsilon \neq 0, \quad \text{Disc}(B_4) \neq 0, \quad \text{Res}(B_4, Q_2) \neq 0.$$

No nonvanishing condition on β is required.

The weight -3 of H_{10} has multiplicity two on U . Therefore

$$C_G(H_{10}) = \{ \text{diag}(\mu_0, \mu_1) \oplus A \oplus \text{diag}(\mu_4) \mid A \in \text{GL}(U), \mu_0 \mu_1 \mu_4 \det A = 1 \}.$$

In particular,

$$\dim C_G(H_{10}) = 6, \quad \dim(C_G(H_{10})/H_{10}) = 5.$$

Polystability

Set $R := C_G(H_{10})$. By affine Luna reduction, it is enough to prove that $R \cdot \phi_{10}$ is closed in $W^{H_{10}}$. Let $\rho \in \Lambda_{\phi_{10}}^R$. After conjugation in the $\text{GL}(U)$ -block, choose an eigenbasis X, Y of U and write the weights of ρ as

$$(a, b, s + n, s - n, c), \quad a + b + 2s + c = 0.$$

Define

$$p_B := a + 4s, \quad p_Q := 3b + 2s, \quad p_\kappa := 4b + c, \quad p_\epsilon := 2a + 3c.$$

The nonnegative-weight condition, together with the indicated relative invariants, gives

$$\begin{aligned} \text{Disc}(B_4) \neq 0 &\implies p_B \geq 0, \\ \text{Res}(B_4, Q_2) \neq 0 &\implies p_B + 2p_Q \geq 0, \\ \kappa \neq 0 &\implies p_\kappa \geq 0, \\ \epsilon \neq 0 &\implies p_\epsilon \geq 0. \end{aligned}$$

These quantities satisfy the balancing identity

$$5p_B + (p_B + 2p_Q) + 2p_\kappa + 4p_\epsilon = 14(a + b + 2s + c) = 0.$$

Hence

$$p_B = p_Q = p_\kappa = p_\epsilon = 0.$$

Since $p_B = 0$, the weights of the monomials in $B_4(X, Y)$ are $(4 - 2j)n$, $j = 0, \dots, 4$. If $n \neq 0$, the nonnegative-weight condition forces B_4 to have a multiple root, contradicting $\text{Disc}(B_4) \neq 0$. Thus $n = 0$.

Solving the remaining equalities gives

$$(a, b, s, s, c) = m(12, 2, -3, -3, -8) \quad (m \in \mathbb{Z}).$$

Consequently,

$$\Lambda_{\phi_{10}}^R = \{\lambda_{10}(t^m) \mid m \in \mathbb{Z}\}.$$

This rank-one set is symmetric. The Casimiro–Florentino criterion therefore shows that $R \cdot \phi_{10}$ is closed in $W^{H_{10}}$. By affine Luna reduction, $G \cdot \phi_{10}$ is closed in W , and hence $[\phi_{10}]$ is polystable.

Normal form

Because B_4 is square-free, the $\text{GL}(U)$ -action puts it in the form

$$B_\lambda(u, v) := uv(u - v)(u - \lambda v), \quad \lambda \in \mathbb{C} \setminus \{0, 1\}.$$

Write

$$Q_{\mathbf{q}}(u, v) = q_0 u^2 + q_1 uv + q_2 v^2, \quad R_{\mathbf{r}}(u, v) = r_0 u^2 + r_1 uv + r_2 v^2.$$

Using the remaining diagonal factors in $C_G(H_{10})$, together with overall projective rescaling, we normalize the coefficients of

$$x_0 B_\lambda(x_2, x_3), \quad x_1^4 x_4, \quad x_0^2 x_4^3$$

to 1. Every general limit in Case 10 is therefore equivalent to a member of the normal-form family

$$\begin{aligned} \phi_{10}^{\text{nf}}(\lambda, \mathbf{q}, \mathbf{r}, \beta) &= x_1^3 Q_{\mathbf{q}}(x_2, x_3) + x_1^4 x_4 + x_0 B_\lambda(x_2, x_3) \\ &\quad + x_0 x_1 x_4 R_{\mathbf{r}}(x_2, x_3) + \beta x_0 x_1^2 x_4^2 + x_0^2 x_4^3, \end{aligned}$$

where

$$\lambda \in \mathbb{C} \setminus \{0, 1\}, \quad (\mathbf{q}, \mathbf{r}, \beta) \in \mathbb{C}^3 \times \mathbb{C}^3 \times \mathbb{C}, \quad \text{Res}(B_\lambda, Q_{\mathbf{q}}) \neq 0.$$

The residual stabilizer is finite for a general parameter point. Therefore

$$\dim \Phi_{10} = \dim \mathbb{P}(W^{H_{10}}) - \dim(C_G(H_{10})/H_{10}) = 13 - 5 = 8.$$

Case $k = 11$

1-PS limit

The witness vector and the corresponding one-parameter subgroup are

$$r_{11} = (14, 4, -1, -6, -11), \quad \lambda_{11}(t) = \text{diag}(t^{14}, t^4, t^{-1}, t^{-6}, t^{-11}), \quad t \in \mathbb{G}_m.$$

Since the entries of r_{11} sum to zero, λ_{11} is a one-parameter subgroup of $G = \text{SL}(5)$. Let f_{11} be a general quintic supported on

$$S_{11} = I(r_{11})_{\geq 0}.$$

With the action convention of (2), its affine one-parameter-subgroup limit is the weight-zero part

$$\phi_{11} := \lim_{t \rightarrow 0} \lambda_{11}(t) \cdot f_{11}.$$

The degree-five weight-zero equations have precisely the exponent vectors

$$\begin{aligned} (0, 1, 4, 0, 0), & \quad (0, 2, 2, 1, 0), & (0, 3, 0, 2, 0), & \quad (0, 3, 1, 0, 1), & (1, 0, 2, 2, 0), \\ (1, 0, 3, 0, 1), & \quad (1, 1, 0, 3, 0), & (1, 1, 1, 1, 1), & \quad (1, 2, 0, 0, 2), & (2, 0, 0, 1, 2). \end{aligned}$$

Consequently,

$$\begin{aligned} \phi_{11} = & a_1 x_1 x_2^4 + a_2 x_1^2 x_2^2 x_3 + a_3 x_1^3 x_3^2 + a_4 x_1^3 x_2 x_4 + a_5 x_0 x_2^2 x_3^2 \\ & + a_6 x_0 x_2^3 x_4 + a_7 x_0 x_1 x_3^3 + a_8 x_0 x_1 x_2 x_3 x_4 + a_9 x_0 x_1^2 x_4^2 + a_{10} x_0^2 x_3 x_4^2, \end{aligned}$$

where $a_i \in \mathbb{C}^\times$ for all i .

Set $H_{11} = \lambda_{11}(\mathbb{G}_m)$. The five weights of H_{11} on $\langle x_0, \dots, x_4 \rangle$ are pairwise distinct, and their multiset is not invariant under multiplication by -1 . Hence

$$C_G(H_{11}) = N_G(H_{11}) = T.$$

Moreover,

$$\begin{aligned} W^{H_{11}} = & \langle x_1 x_2^4, x_1^2 x_2^2 x_3, x_1^3 x_3^2, x_1^3 x_2 x_4, x_0 x_2^2 x_3^2, \\ & x_0 x_2^3 x_4, x_0 x_1 x_3^3, x_0 x_1 x_2 x_3 x_4, x_0 x_1^2 x_4^2, x_0^2 x_3 x_4^2 \rangle, \quad \dim W^{H_{11}} = 10. \end{aligned}$$

Polystability

Let $v_1, \dots, v_{10}$ be the ten exponent vectors above, with $v_8 = (1, 1, 1, 1, 1)$, and set

$$\delta_i := v_i - v_8 \quad (i = 1, \dots, 10).$$

Thus $\delta_8 = 0$, and

$$\delta_2 = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_3.$$

The eight characters

$$\delta_1, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7, \delta_9, \delta_{10}$$

span the three-dimensional character space of T/H_{11} and satisfy the positive relation

$$\delta_1 + \delta_3 + \delta_4 + \delta_5 + \delta_6 + \delta_7 + \delta_9 + 3\delta_{10} = 0.$$

Therefore

$$0 \in \text{relint Conv}\{\delta_1, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7, \delta_9, \delta_{10}\}.$$

Adding δ_2 and δ_8 does not change this conclusion. Since all ten coefficients of ϕ_{11} are nonzero, the convex-hull criterion shows that $(T/H_{11}) \cdot \phi_{11}$ is closed in $W^{H_{11}}$. By affine Luna reduction and $C_G(H_{11}) = T$, the orbit $G \cdot \phi_{11}$ is closed in W . Thus $[\phi_{11}]$ is polystable.

Normal form

On the full-support chart

$$\mathbb{P}(W^{H_{11}})^\circ = \{[a_1 : \dots : a_{10}] \mid a_i \neq 0\} \cong (\mathbb{C}^\times)^9,$$

the effective torus T/H_{11} has dimension 3. The character differences

$$\begin{aligned} v_4 - v_1 &= (0, 2, -3, 0, 1), \\ v_6 - v_1 &= (1, -1, -1, 0, 1), \\ v_{10} - v_1 &= (2, -1, -4, 1, 2) \end{aligned}$$

have rank 3 in the character space of T/H_{11} . Combining the torus action with overall projective rescaling, one may therefore normalize the coefficients of

$$x_1 x_2^4, \quad x_1^3 x_2 x_4, \quad x_0 x_2^3 x_4, \quad x_0^2 x_3 x_4^2$$

to 1. Every general limit in Case 11 is consequently equivalent to a member of the normal-form family

$$\begin{aligned} \phi_{11}^{\text{nf}}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta) &= x_1 x_2^4 + \alpha x_1^2 x_2^2 x_3 + \beta x_1^3 x_3^2 + x_1^3 x_2 x_4 + \gamma x_0 x_2^2 x_3^2 \\ &\quad + x_0 x_2^3 x_4 + \delta x_0 x_1 x_3^3 + \varepsilon x_0 x_1 x_2 x_3 x_4 + \zeta x_0 x_1^2 x_4^2 + x_0^2 x_3 x_4^2, \end{aligned}$$

where

$$(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta) \in (\mathbb{C}^\times)^6.$$

Since

$$\dim \mathbb{P}(W^{H_{11}})^\circ = 9, \quad \dim(T/H_{11}) = 3,$$

the corresponding quotient-side family has dimension

$$\dim \Phi_{11} = 9 - 3 = 6.$$

No additional condition on the six parameters is imposed at the normal-form stage; every full-support member of this family is polystable.

Case $k = 12$

1-PS limit

The witness vector and the corresponding one-parameter subgroup are

$$r_{12} = (13, 8, -2, -7, -12), \quad \lambda_{12}(t) = \text{diag}(t^{13}, t^8, t^{-2}, t^{-7}, t^{-12}), \quad t \in \mathbb{G}_m.$$

Since the entries of r_{12} sum to zero, λ_{12} is a one-parameter subgroup of $G = \text{SL}(5)$. Let f_{12} be a general quintic supported on

$$S_{12} = I(r_{12})_{\geq 0}.$$

With the action convention of (2), its affine one-parameter-subgroup limit is the weight-zero part

$$\phi_{12} := \lim_{t \rightarrow 0} \lambda_{12}(t) \cdot f_{12}.$$

The degree-five weight-zero equations have precisely the exponent vectors

$$\begin{aligned} (0, 1, 4, 0, 0), \quad (0, 2, 1, 2, 0), \quad (0, 2, 2, 0, 1), \quad (0, 3, 0, 0, 2), \quad (1, 0, 3, 1, 0), \\ (1, 1, 0, 3, 0), \quad (1, 1, 1, 1, 1), \quad (2, 0, 0, 2, 1), \quad (2, 0, 1, 0, 2). \end{aligned}$$

Consequently,

$$\begin{aligned} \phi_{12} &= a_1 x_1 x_2^4 + a_2 x_1^2 x_2^2 x_3^2 + a_3 x_1^2 x_2^2 x_4 + a_4 x_1^3 x_4^2 + a_5 x_0 x_2^3 x_3 \\ &\quad + a_6 x_0 x_1 x_3^3 + a_7 x_0 x_1 x_2 x_3 x_4 + a_8 x_0^2 x_3^2 x_4 + a_9 x_0^2 x_2 x_4^2, \end{aligned}$$

where $a_i \in \mathbb{C}^\times$ for all i .

Set $H_{12} = \lambda_{12}(\mathbb{G}_m)$. The five weights of H_{12} on $\langle x_0, \dots, x_4 \rangle$ are pairwise distinct, and their multiset is not invariant under multiplication by -1 . Hence

$$C_G(H_{12}) = N_G(H_{12}) = T.$$

Moreover,

$$W^{H_{12}} = \langle x_1 x_2^4, x_1^2 x_2 x_3^2, x_1^2 x_2^2 x_4, x_1^3 x_4^2, x_0 x_2^3 x_3, \\ x_0 x_1 x_3^3, x_0 x_1 x_2 x_3 x_4, x_0^2 x_3^2 x_4, x_0^2 x_2 x_4^2 \rangle, \quad \dim W^{H_{12}} = 9.$$

Polystability

Let $v_1, \dots, v_9$ be the nine exponent vectors above, with $v_7 = (1, 1, 1, 1, 1)$, and set

$$\delta_i := v_i - v_7 \quad (i = 1, \dots, 9).$$

Thus $\delta_7 = 0$. The eight nonzero characters span the three-dimensional character space of T/H_{12} and satisfy the positive relation

$$\delta_1 + \delta_2 + 2\delta_3 + 2\delta_4 + \delta_5 + \delta_6 + 4\delta_8 + 2\delta_9 = 0.$$

Therefore

$$0 \in \text{relint Conv}\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_8, \delta_9\}.$$

Adding $\delta_7 = 0$ does not change this conclusion. Since all nine coefficients of ϕ_{12} are nonzero, the convex-hull criterion shows that $(T/H_{12}) \cdot \phi_{12}$ is closed in $W^{H_{12}}$. By affine Luna reduction and $C_G(H_{12}) = T$, the orbit $G \cdot \phi_{12}$ is closed in W . Thus $[\phi_{12}]$ is polystable.

Normal form

On the full-support chart

$$\mathbb{P}(W^{H_{12}})^\circ = \{[a_1 : \dots : a_9] \mid a_i \neq 0\} \cong (\mathbb{C}^\times)^8,$$

the effective torus T/H_{12} has dimension 3. The character differences

$$\begin{aligned} v_4 - v_1 &= (0, 2, -4, 0, 2), \\ v_5 - v_1 &= (1, -1, -1, 1, 0), \\ v_9 - v_1 &= (2, -1, -3, 0, 2) \end{aligned}$$

have rank 3 in the character space of T/H_{12} . Combining the torus action with overall projective rescaling, one may therefore normalize the coefficients of

$$x_1 x_2^4, \quad x_1^3 x_4^2, \quad x_0 x_2^3 x_3, \quad x_0^2 x_2 x_4^2$$

to 1. Every general limit in Case 12 is consequently equivalent to a member of the normal-form family

$$\begin{aligned} \phi_{12}^{\text{nf}}(\alpha, \beta, \gamma, \delta, \varepsilon) &= x_1 x_2^4 + \alpha x_1^2 x_2 x_3^2 + \beta x_1^2 x_2^2 x_4 + x_1^3 x_4^2 + x_0 x_2^3 x_3 \\ &\quad + \gamma x_0 x_1 x_3^3 + \delta x_0 x_1 x_2 x_3 x_4 + \varepsilon x_0^2 x_3^2 x_4 + x_0^2 x_2 x_4^2, \end{aligned}$$

where

$$(\alpha, \beta, \gamma, \delta, \varepsilon) \in (\mathbb{C}^\times)^5.$$

Since

$$\dim \mathbb{P}(W^{H_{12}})^\circ = 8, \quad \dim(T/H_{12}) = 3,$$

the corresponding quotient-side family has dimension

$$\dim \Phi_{12} = 8 - 3 = 5.$$

No additional condition on the five parameters is imposed at the normal-form stage; every full-support member of this family is polystable.

Case $k = 13$

1-PS limit

The witness vector and the corresponding one-parameter subgroup are

$$r_{13} = (3, 2, 0, -2, -3), \quad \lambda_{13}(t) = \text{diag}(t^3, t^2, 1, t^{-2}, t^{-3}), \quad t \in \mathbb{G}_m.$$

Since the entries of r_{13} sum to zero, λ_{13} is a one-parameter subgroup of $G = \text{SL}(5)$. Let f_{13} be a general quintic supported on

$$S_{13} = I(r_{13})_{\geq 0}.$$

With the action convention of (2), its affine one-parameter-subgroup limit is the weight-zero part

$$\phi_{13} := \lim_{t \rightarrow 0} \lambda_{13}(t) \cdot f_{13}.$$

The degree-five weight-zero equations have precisely the exponent vectors

$$\begin{aligned} (0, 0, 5, 0, 0), \quad (0, 1, 3, 1, 0), \quad (0, 2, 1, 2, 0), \quad (0, 3, 0, 0, 2), \\ (1, 0, 3, 0, 1), \quad (1, 1, 1, 1, 1), \quad (2, 0, 0, 3, 0), \quad (2, 0, 1, 0, 2). \end{aligned}$$

Consequently,

$$\begin{aligned} \phi_{13} = & a_1 x_2^5 + a_2 x_1 x_2^3 x_3 + a_3 x_1^2 x_2 x_3^2 + a_4 x_1^3 x_4^2 + a_5 x_0 x_2^3 x_4 \\ & + a_6 x_0 x_1 x_2 x_3 x_4 + a_7 x_0^2 x_3^3 + a_8 x_0^2 x_2 x_4^2, \end{aligned}$$

where $a_i \in \mathbb{C}^\times$ for all i .

Set $H_{13} = \lambda_{13}(\mathbb{G}_m)$. The five weights of H_{13} on $\langle x_0, \dots, x_4 \rangle$ are pairwise distinct, and hence

$$C_G(H_{13}) = T.$$

Moreover,

$$\begin{aligned} W^{H_{13}} = \langle & x_2^5, x_1 x_2^3 x_3, x_1^2 x_2 x_3^2, x_1^3 x_4^2, x_0 x_2^3 x_4, \\ & x_0 x_1 x_2 x_3 x_4, x_0^2 x_3^3, x_0^2 x_2 x_4^2 \rangle, \quad \dim W^{H_{13}} = 8. \end{aligned}$$

Polystability

Let $v_1, \dots, v_8$ be the eight exponent vectors above, with $v_6 = (1, 1, 1, 1, 1)$, and set

$$\delta_i := v_i - v_6 \quad (i = 1, \dots, 8).$$

Thus $\delta_6 = 0$. The seven nonzero characters span the three-dimensional character space of T/H_{13} and satisfy the positive relation

$$\delta_1 + \delta_2 + \delta_3 + 4\delta_4 + \delta_5 + 4\delta_7 + 3\delta_8 = 0.$$

Therefore

$$0 \in \text{relint Conv}\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_7, \delta_8\}.$$

Adding $\delta_6 = 0$ does not change this conclusion. Since all eight coefficients of ϕ_{13} are nonzero, the convex-hull criterion shows that $(T/H_{13}) \cdot \phi_{13}$ is closed in $W^{H_{13}}$. By affine Luna reduction and $C_G(H_{13}) = T$, the orbit $G \cdot \phi_{13}$ is closed in W . Thus $[\phi_{13}]$ is polystable.

Normal form

On the full-support chart

$$\mathbb{P}(W^{H_{13}})^\circ = \{[a_1 : \dots : a_8] \mid a_i \neq 0\} \cong (\mathbb{C}^\times)^7,$$

the effective torus T/H_{13} has dimension 3. After adjoining overall projective rescaling, the characters of the coefficients of

$$x_2^5, \quad x_1^3 x_4^2, \quad x_0 x_2^3 x_4, \quad x_0^2 x_3^3$$

are independent. These four coefficients may therefore be normalized to 1. Every general limit in Case 13 is consequently equivalent to a member of the normal-form family

$$\begin{aligned} \phi_{13}^{\text{nf}}(\alpha, \beta, \gamma, \delta) = & x_2^5 + \alpha x_1 x_2^3 x_3 + \beta x_1^2 x_2 x_3^2 + x_1^3 x_4^2 + x_0 x_2^3 x_4 \\ & + \gamma x_0 x_1 x_2 x_3 x_4 + x_0^2 x_3^3 + \delta x_0^2 x_2 x_4^2, \end{aligned}$$

where

$$(\alpha, \beta, \gamma, \delta) \in (\mathbb{C}^\times)^4.$$

Since

$$\dim \mathbb{P}(W^{H_{13}})^\circ = 7, \quad \dim(T/H_{13}) = 3,$$

the corresponding quotient-side family has dimension

$$\dim \Phi_{13} = 7 - 3 = 4.$$

No additional condition on the four parameters is imposed at the normal-form stage; every full-support member of this family is polystable.

Case $k = 14$

1-PS limit

The witness vector and the corresponding one-parameter subgroup are

$$r_{14} = (4, 1, 0, -1, -4), \quad \lambda_{14}(t) = \text{diag}(t^4, t, 1, t^{-1}, t^{-4}), \quad t \in \mathbb{G}_m.$$

Since the entries of r_{14} sum to zero, λ_{14} is a one-parameter subgroup of $G = \text{SL}(5)$. Let f_{14} be a general quintic supported on

$$S_{14} = I(r_{14})_{\geq 0}.$$

With the action convention of (2), its affine one-parameter-subgroup limit is the weight-zero part

$$\phi_{14} := \lim_{t \rightarrow 0} \lambda_{14}(t) \cdot f_{14}.$$

The degree-five weight-zero equations have precisely the exponent vectors

$$\begin{aligned} (0, 0, 5, 0, 0), \quad (0, 1, 3, 1, 0), \quad (0, 2, 1, 2, 0), \quad (0, 4, 0, 0, 1), \\ (1, 0, 0, 4, 0), \quad (1, 0, 3, 0, 1), \quad (1, 1, 1, 1, 1), \quad (2, 0, 1, 0, 2). \end{aligned}$$

Consequently,

$$\begin{aligned} \phi_{14} = & a_1 x_2^5 + a_2 x_1 x_2^3 x_3 + a_3 x_1^2 x_2 x_3^2 + a_4 x_1^4 x_4 + a_5 x_0 x_3^4 \\ & + a_6 x_0 x_2^3 x_4 + a_7 x_0 x_1 x_2 x_3 x_4 + a_8 x_0^2 x_2 x_4^2, \end{aligned}$$

where $a_i \in \mathbb{C}^\times$ for all i .

Set $H_{14} = \lambda_{14}(\mathbb{G}_m)$. The five weights of H_{14} on $\langle x_0, \dots, x_4 \rangle$ are pairwise distinct. Hence

$$C_G(H_{14}) = T.$$

Moreover,

$$\begin{aligned} W^{H_{14}} = & \langle x_2^5, x_1 x_2^3 x_3, x_1^2 x_2 x_3^2, x_1^4 x_4, x_0 x_3^4, \\ & x_0 x_2^3 x_4, x_0 x_1 x_2 x_3 x_4, x_0^2 x_2 x_4^2 \rangle, \quad \dim W^{H_{14}} = 8. \end{aligned}$$

Polystability

Let $v_1, \dots, v_8$ be the eight exponent vectors above, with $v_7 = (1, 1, 1, 1, 1)$, and set

$$\delta_i := v_i - v_7 \quad (i = 1, \dots, 8).$$

Thus $\delta_7 = 0$. The seven nonzero characters span the three-dimensional character space of T/H_{14} and satisfy the positive relation

$$\delta_1 + \delta_2 + \delta_3 + 4\delta_4 + 4\delta_5 + \delta_6 + 7\delta_8 = 0.$$

Therefore

$$0 \in \text{relint Conv}\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_8\}.$$

Adding $\delta_7 = 0$ does not change this conclusion. Since all eight coefficients of ϕ_{14} are nonzero, the convex-hull criterion shows that $(T/H_{14}) \cdot \phi_{14}$ is closed in $W^{H_{14}}$. By affine Luna reduction and $C_G(H_{14}) = T$, the orbit $G \cdot \phi_{14}$ is closed in W . Thus $[\phi_{14}]$ is polystable.

Normal form

On the full-support chart

$$\mathbb{P}(W^{H_{14}})^\circ = \{[a_1 : \dots : a_8] \mid a_i \neq 0\} \cong (\mathbb{C}^\times)^7,$$

the effective torus T/H_{14} has dimension 3. The character differences

$$\begin{aligned} v_4 - v_1 &= (0, 4, -5, 0, 1), \\ v_5 - v_1 &= (1, 0, -5, 4, 0), \\ v_8 - v_1 &= (2, 0, -4, 0, 2) \end{aligned}$$

have rank 3 in the character space of T/H_{14} . Combining the torus action with overall projective rescaling, one may therefore normalize the coefficients of

$$x_2^5, \quad x_1^4 x_4, \quad x_0 x_3^4, \quad x_0^2 x_2 x_4^2$$

to 1. Every general limit in Case 14 is consequently equivalent to a member of the normal-form family

$$\begin{aligned} \phi_{14}^{\text{nf}}(\alpha, \beta, \gamma, \delta) &= x_2^5 + \alpha x_1 x_2^3 x_3 + \beta x_1^2 x_2 x_3^2 + x_1^4 x_4 + x_0 x_3^4 \\ &\quad + \gamma x_0 x_2^3 x_4 + \delta x_0 x_1 x_2 x_3 x_4 + x_0^2 x_2 x_4^2, \end{aligned}$$

where

$$(\alpha, \beta, \gamma, \delta) \in (\mathbb{C}^\times)^4.$$

Since

$$\dim \mathbb{P}(W^{H_{14}})^\circ = 7, \quad \dim(T/H_{14}) = 3,$$

the corresponding quotient-side family has dimension

$$\dim \Phi_{14} = 7 - 3 = 4.$$

No additional condition on the four parameters is imposed at the normal-form stage; every full-support member of this family is polystable.

Case $k = 15$

1-PS limit

The witness vector and the corresponding one-parameter subgroup are

$$r_{15} = (9, 4, -1, -6, -6), \quad \lambda_{15}(t) = \text{diag}(t^9, t^4, t^{-1}, t^{-6}, t^{-6}), \quad t \in \mathbb{G}_m.$$

Since the entries of r_{15} sum to zero, λ_{15} is a one-parameter subgroup of $G = \text{SL}(5)$. Let f_{15} be a general quintic supported on

$$S_{15} = I(r_{15})_{\geq 0}.$$

With the action convention of (2), its affine one-parameter-subgroup limit is the weight-zero part

$$\phi_{15} := \lim_{t \rightarrow 0} \lambda_{15}(t) \cdot f_{15}.$$

Put

$$U := \langle x_3, x_4 \rangle.$$

Solving the degree-five weight-zero equations gives

$$\begin{aligned} W^{H_{15}} = & \langle x_1 x_2^4 \rangle \oplus x_1^2 x_2^2 \text{Sym}^1 U \oplus x_1^3 \text{Sym}^2 U \oplus x_0 x_2^3 \text{Sym}^1 U \\ & \oplus x_0 x_1 x_2 \text{Sym}^2 U \oplus x_0^2 \text{Sym}^3 U, \quad \dim W^{H_{15}} = 15, \end{aligned}$$

where $H_{15} = \lambda_{15}(\mathbb{G}_m)$. Thus a general limit can be written as

$$\begin{aligned} \phi_{15} = & \kappa x_1 x_2^4 + x_1^2 x_2^2 L_1(x_3, x_4) + x_1^3 Q_1(x_3, x_4) \\ & + x_0 x_2^3 L_2(x_3, x_4) + x_0 x_1 x_2 Q_2(x_3, x_4) + x_0^2 B_3(x_3, x_4), \end{aligned}$$

with L_1, L_2 linear, Q_1, Q_2 quadratic, and B_3 cubic in x_3, x_4 . We work on the nonempty open locus

$$\kappa \neq 0, \quad L_1 \wedge L_2 \neq 0, \quad \text{Disc}(Q_1) \neq 0, \quad \text{Disc}(B_3) \neq 0.$$

The weight -6 occurs twice on U , so the centralizer is non-toric:

$$C_G(H_{15}) = \{ \text{diag}(\mu_0, \mu_1, \mu_2) \oplus A \mid A \in \text{GL}(U), \mu_0 \mu_1 \mu_2 \det A = 1 \}.$$

Hence

$$\dim C_G(H_{15}) = 6, \quad \dim(C_G(H_{15})/H_{15}) = 5.$$

Polystability

Set $C := C_G(H_{15})$. By affine Luna reduction, it is enough to prove that $C \cdot \phi_{15}$ is closed in $W^{H_{15}}$. Let $\lambda \in \Lambda_{\phi_{15}}^C$. After conjugation in the $\text{GL}(U)$ -block, its weights may be written as

$$\text{wt}_\lambda(x_0, x_1, x_2, X, Y) = (a, b, c, s + n, s - n), \quad a + b + c + 2s = 0,$$

for a suitable eigenbasis X, Y of U .

Introduce the averaged weights

$$\nu := b + 4c, \quad q := 3b + 2s, \quad p := 2a + 3s.$$

The nonvanishing of κ , $\text{Disc}(Q_1)$, and $\text{Disc}(B_3)$, together with the nonnegativity of all occurring monomial weights, gives

$$\nu \geq 0, \quad q \geq 0, \quad p \geq 0.$$

On the other hand,

$$\nu + q + 2p = 4(a + b + c + 2s) = 0.$$

Therefore

$$\nu = q = p = 0.$$

Since $p = 0$, a nonzero relative weight n would force the binary cubic B_3 to be divisible by X^2 or by Y^2 , contradicting $\text{Disc}(B_3) \neq 0$. Hence $n = 0$. Solving the remaining weight equations gives

$$(a, b, c, s) = (9m, 4m, -m, -6m)$$

for some $m \in \mathbb{Z}$. Consequently,

$$\Lambda_{\phi_{15}}^C = \{\lambda_{15}(t^m) \mid m \in \mathbb{Z}\}.$$

This set is symmetric under $m \mapsto -m$. The Casimiro–Florentino criterion therefore shows that $C \cdot \phi_{15}$ is closed in $W^{H_{15}}$. Luna reduction then implies that $G \cdot \phi_{15}$ is closed in W , so $[\phi_{15}]$ is polystable.

Normal form

Assume that L_1, L_2 are linearly independent and write

$$Q_1 = eL_1^2 + fL_1L_2 + hL_2^2.$$

The coefficient

$$J(\phi_{15}) := \kappa e$$

is invariant under the centralizer action and projective rescaling. Thus this coefficient cannot in general be normalized away; equivalently, in monomial coordinates the ratio

$$\frac{a_1 a_4}{a_2^2}$$

is invariant.

Use $\text{GL}(U)$ to send L_1 and L_2 to x_3 and x_4 , and then use the diagonal factors of C , together with projective rescaling, to normalize the coefficients of

$$x_1 x_2^4, \quad x_1^2 x_2^2 x_3, \quad x_0 x_2^3 x_4$$

to 1. On the dense open locus $h \neq 0$, the residual one-parameter subgroup

$$\lambda_*(t) = \text{diag}(t, 1, 1, 1, t^{-1})$$

normalizes the coefficient of $x_1^3 x_4^2$ to 1. After renaming the remaining coefficients, every general limit is equivalent to a member of

$$\begin{aligned} \phi_{15}^{\text{nf}}(Q, R, B) = & x_1 x_2^4 + x_1^2 x_2^2 x_3 + x_1^3 Q(x_3, x_4) + x_0 x_2^3 x_4 \\ & + x_0 x_1 x_2 R(x_3, x_4) + x_0^2 B(x_3, x_4), \end{aligned}$$

where

$$\begin{aligned} Q &= \gamma x_3^2 + \beta x_3 x_4 + x_4^2, \\ R &= \rho_0 x_3^2 + \rho_1 x_3 x_4 + \rho_2 x_4^2, \end{aligned}$$

and

$$B = \sigma_0 x_3^3 + \sigma_1 x_3^2 x_4 + \sigma_2 x_3 x_4^2 + \sigma_3 x_4^3.$$

The nine parameters

$$(\gamma, \beta, \rho_0, \rho_1, \rho_2, \sigma_0, \sigma_1, \sigma_2, \sigma_3)$$

range over the nonempty open subset of $\mathbb{C}^9$ on which

$$\beta^2 - 4\gamma \neq 0, \quad \text{Disc}(B) \neq 0.$$

The root choices in the normalization are finite, and the remaining λ_* -ambiguity is finite. Hence this slice maps generically finitely onto a dense subset of Φ_{15} . Since

$$\dim \mathbb{P}(W^{H_{15}}) = 14, \quad \dim(C_G(H_{15})/H_{15}) = 5,$$

we obtain

$$\dim \Phi_{15} = 14 - 5 = 9.$$

No condition arising from the later singular-locus analysis is imposed at the normal-form stage.

Case $k = 16$

1-PS limit

The witness vector and the corresponding one-parameter subgroup are

$$r_{16} = (3, 2, 1, -2, -4), \quad \lambda_{16}(t) = \text{diag}(t^3, t^2, t, t^{-2}, t^{-4}), \quad t \in \mathbb{G}_m.$$

Since the entries of r_{16} sum to zero, λ_{16} is a one-parameter subgroup of $G = \text{SL}(5)$. Let f_{16} be a general quintic supported on

$$S_{16} = I(r_{16})_{\geq 0}.$$

With the action convention of (2), its affine one-parameter-subgroup limit is the weight-zero part

$$\phi_{16} := \lim_{t \rightarrow 0} \lambda_{16}(t) \cdot f_{16}.$$

The degree-five weight-zero equations have precisely the exponent vectors

$$\begin{aligned} (0, 0, 4, 0, 1), \quad (0, 1, 2, 2, 0), \quad (0, 3, 0, 1, 1), \quad (1, 1, 1, 1, 1), \\ (2, 0, 0, 3, 0), \quad (2, 1, 0, 0, 2). \end{aligned}$$

Consequently,

$$\begin{aligned} \phi_{16} = & a_1 x_2^4 x_4 + a_2 x_1 x_2^2 x_3^2 + a_3 x_1^3 x_3 x_4 + a_4 x_0 x_1 x_2 x_3 x_4 \\ & + a_5 x_0^2 x_3^3 + a_6 x_0^2 x_1 x_4^2, \end{aligned}$$

where $a_i \in \mathbb{C}^\times$ for all i .

Set $H_{16} = \lambda_{16}(\mathbb{G}_m)$. The five weights of H_{16} on $\langle x_0, \dots, x_4 \rangle$ are pairwise distinct. Hence

$$C_G(H_{16}) = T.$$

Moreover,

$$\begin{aligned} W^{H_{16}} = \langle & x_2^4 x_4, x_1 x_2^2 x_3^2, x_1^3 x_3 x_4, x_0 x_1 x_2 x_3 x_4, \\ & x_0^2 x_3^3, x_0^2 x_1 x_4^2 \rangle, \quad \dim W^{H_{16}} = 6. \end{aligned}$$

Polystability

Let $v_1, \dots, v_6$ be the six exponent vectors above, with $v_4 = (1, 1, 1, 1, 1)$, and set

$$\delta_i := v_i - v_4 \quad (i = 1, \dots, 6).$$

Thus $\delta_4 = 0$. The five nonzero characters span the three-dimensional character space of T/H_{16} and satisfy the positive relation

$$\delta_1 + \delta_2 + \delta_3 + \delta_5 + 2\delta_6 = 0.$$

Therefore

$$0 \in \text{relint Conv}\{\delta_1, \delta_2, \delta_3, \delta_5, \delta_6\}.$$

Adding $\delta_4 = 0$ does not change this conclusion. Since all six coefficients of ϕ_{16} are nonzero, the convex-hull criterion shows that $(T/H_{16}) \cdot \phi_{16}$ is closed in $W^{H_{16}}$. By affine Luna reduction and $C_G(H_{16}) = T$, the orbit $G \cdot \phi_{16}$ is closed in W . Thus $[\phi_{16}]$ is polystable.

Normal form

On the full-support chart

$$\mathbb{P}(W^{H_{16}})^\circ = \{[a_1 : \cdots : a_6] \mid a_i \neq 0\} \cong (\mathbb{C}^\times)^5,$$

the effective torus T/H_{16} has dimension 3. The character differences

$$v_2 - v_1 = (0, 1, -2, 2, -1),$$

$$v_3 - v_1 = (0, 3, -4, 1, 0),$$

$$v_4 - v_1 = (1, 1, -3, 1, 0)$$

have rank 3 in its character space. Combining the torus action with overall projective rescaling, one may therefore normalize the coefficients of

$$x_2^4 x_4, \quad x_1 x_2^2 x_3^2, \quad x_1^3 x_3 x_4, \quad x_0 x_1 x_2 x_3 x_4$$

to 1. Every general limit in Case 16 is consequently equivalent to a member of the normal-form family

$$\begin{aligned} \phi_{16}^{\text{nf}}(\alpha, \beta) = & x_2^4 x_4 + x_1 x_2^2 x_3^2 + x_1^3 x_3 x_4 + x_0 x_1 x_2 x_3 x_4 \\ & + \alpha x_0^2 x_3^3 + \beta x_0^2 x_1 x_4^2, \end{aligned}$$

where

$$(\alpha, \beta) \in (\mathbb{C}^\times)^2.$$

Since

$$\dim \mathbb{P}(W^{H_{16}})^\circ = 5, \quad \dim(T/H_{16}) = 3,$$

the corresponding quotient-side family has dimension

$$\dim \Phi_{16} = 5 - 3 = 2.$$

No additional condition on α or β is imposed at the normal-form stage; every full-support member of this family is polystable.

Case $k = 17$

1-PS limit

The witness vector and the corresponding one-parameter subgroup are

$$r_{17} = (12, 7, 2, -8, -13), \quad \lambda_{17}(t) = \text{diag}(t^{12}, t^7, t^2, t^{-8}, t^{-13}), \quad t \in \mathbb{G}_m.$$

Since the entries of r_{17} sum to zero, λ_{17} is a one-parameter subgroup of $G = \text{SL}(5)$. Let f_{17} be a general quintic supported on

$$S_{17} = I(r_{17})_{\geq 0}.$$

With the action convention of (2), its affine one-parameter-subgroup limit is the weight-zero part

$$\phi_{17} := \lim_{t \rightarrow 0} \lambda_{17}(t) \cdot f_{17}.$$

The degree-five weight-zero equations have precisely the exponent vectors

$$(0, 0, 4, 1, 0), \quad (0, 1, 3, 0, 1), \quad (0, 2, 1, 2, 0), \quad (0, 3, 0, 1, 1), \quad (1, 0, 2, 2, 0), \\ (1, 1, 1, 1, 1), \quad (1, 2, 0, 0, 2), \quad (2, 0, 0, 3, 0), \quad (2, 0, 1, 0, 2).$$

Consequently,

$$\phi_{17} = a_1 x_2^4 x_3 + a_2 x_1 x_2^3 x_4 + a_3 x_1^2 x_2 x_3^2 + a_4 x_1^3 x_3 x_4 + a_5 x_0 x_2^2 x_3^2 \\ + a_6 x_0 x_1 x_2 x_3 x_4 + a_7 x_0 x_1^2 x_4^2 + a_8 x_0^2 x_3^3 + a_9 x_0^2 x_2 x_4^2,$$

where $a_i \in \mathbb{C}^\times$ for all i .

Set $H_{17} = \lambda_{17}(\mathbb{G}_m)$. The five weights of H_{17} on $\langle x_0, \dots, x_4 \rangle$ are pairwise distinct. Hence

$$C_G(H_{17}) = T.$$

Moreover,

$$W^{H_{17}} = \langle x_2^4 x_3, x_1 x_2^3 x_4, x_1^2 x_2 x_3^2, x_1^3 x_3 x_4, x_0 x_2^2 x_3^2, \\ x_0 x_1 x_2 x_3 x_4, x_0 x_1^2 x_4^2, x_0^2 x_3^3, x_0^2 x_2 x_4^2 \rangle, \quad \dim W^{H_{17}} = 9.$$

Polystability

Let $v_1, \dots, v_9$ be the nine exponent vectors above, with $v_6 = (1, 1, 1, 1, 1)$, and set

$$\delta_i := v_i - v_6 \quad (i = 1, \dots, 9).$$

Thus $\delta_6 = 0$. The characters δ_i span the three-dimensional character space of T/H_{17} and satisfy the positive relation

$$\delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 + \delta_6 + 3\delta_7 + 2\delta_8 + 2\delta_9 = 0.$$

Therefore

$$0 \in \text{relint Conv}\{\delta_1, \dots, \delta_9\}.$$

Since all nine coefficients of ϕ_{17} are nonzero, the convex-hull criterion shows that $(T/H_{17}) \cdot \phi_{17}$ is closed in $W^{H_{17}}$. By affine Luna reduction and $C_G(H_{17}) = T$, the orbit $G \cdot \phi_{17}$ is closed in W . Thus $[\phi_{17}]$ is polystable.

Normal form

On the full-support chart

$$\mathbb{P}(W^{H_{17}})^\circ = \{[a_1 : \dots : a_9] \mid a_i \neq 0\} \cong (\mathbb{C}^\times)^8,$$

the effective torus T/H_{17} has dimension 3. The character differences

$$v_2 - v_1 = (0, 1, -1, -1, 1), \\ v_4 - v_1 = (0, 3, -4, 0, 1), \\ v_8 - v_1 = (2, 0, -4, 2, 0)$$

have rank 3 in its character space. Combining the torus action with overall projective rescaling, one may therefore normalize the coefficients of

$$x_2^4 x_3, \quad x_1 x_2^3 x_4, \quad x_1^3 x_3 x_4, \quad x_0^2 x_3^3$$

to 1. Every general limit in Case 17 is consequently equivalent to a member of the normal-form family

$$\begin{aligned}\phi_{17}^{\text{nf}}(\alpha, \beta, \gamma, \delta, \varepsilon) = & x_2^4 x_3 + x_1 x_2^3 x_4 + \alpha x_1^2 x_2 x_3^2 + x_1^3 x_3 x_4 + \beta x_0 x_2^2 x_3^2 \\ & + \gamma x_0 x_1 x_2 x_3 x_4 + \delta x_0 x_1^2 x_4^2 + x_0^2 x_3^3 + \varepsilon x_0^2 x_2 x_4^2,\end{aligned}$$

where

$$(\alpha, \beta, \gamma, \delta, \varepsilon) \in (\mathbb{C}^\times)^5.$$

Since

$$\dim \mathbb{P}(W^{H_{17}})^\circ = 8, \quad \dim(T/H_{17}) = 3,$$

the corresponding quotient-side family has dimension

$$\dim \Phi_{17} = 8 - 3 = 5.$$

No additional condition on the five parameters is imposed at the normal-form stage; every full-support member of this family is polystable.

Case $k = 18$

1-PS limit

The witness vector and the corresponding one-parameter subgroup are

$$r_{18} = (4, 2, 0, -1, -5), \quad \lambda_{18}(t) = \text{diag}(t^4, t^2, 1, t^{-1}, t^{-5}), \quad t \in \mathbb{G}_m.$$

Since the entries of r_{18} sum to zero, λ_{18} is a one-parameter subgroup of $G = \text{SL}(5)$. Let f_{18} be a general quintic supported on

$$S_{18} = I(r_{18})_{\geq 0}.$$

With the action convention of (2), its affine one-parameter-subgroup limit is the weight-zero part

$$\phi_{18} := \lim_{t \rightarrow 0} \lambda_{18}(t) \cdot f_{18}.$$

The degree-five weight-zero equations have precisely the exponent vectors

$$(0, 0, 5, 0, 0), \quad (0, 1, 2, 2, 0), \quad (0, 3, 0, 1, 1), \quad (1, 0, 0, 4, 0), \quad (1, 1, 1, 1, 1), \quad (2, 1, 0, 0, 2).$$

Consequently,

$$\phi_{18} = a_1 x_2^5 + a_2 x_1 x_2^2 x_3^2 + a_3 x_1^3 x_3 x_4 + a_4 x_0 x_3^4 + a_5 x_0 x_1 x_2 x_3 x_4 + a_6 x_0^2 x_1 x_4^2,$$

where $a_i \in \mathbb{C}^\times$ for all i .

Set $H_{18} = \lambda_{18}(\mathbb{G}_m)$. The five weights of H_{18} on $\langle x_0, \dots, x_4 \rangle$ are pairwise distinct. Moreover, their multiset is not invariant under multiplication by -1 . Hence

$$C_G(H_{18}) = N_G(H_{18}) = T.$$

The fixed subspace is

$$W^{H_{18}} = \left\langle x_2^5, x_1 x_2^2 x_3^2, x_1^3 x_3 x_4, x_0 x_3^4, x_0 x_1 x_2 x_3 x_4, x_0^2 x_1 x_4^2 \right\rangle, \quad \dim W^{H_{18}} = 6.$$

Polystability

Let $v_1, \dots, v_6$ be the six exponent vectors above, with $v_5 = (1, 1, 1, 1, 1)$, and put

$$\delta_i := v_i - v_5 \quad (i = 1, \dots, 6).$$

Thus $\delta_5 = 0$. These characters span the three-dimensional character space of T/H_{18} and satisfy the positive relation

$$\delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 + 3\delta_6 = 0.$$

Therefore

$$0 \in \text{relint Conv}\{\delta_1, \dots, \delta_6\}.$$

Since all six coefficients of ϕ_{18} are nonzero, the convex-hull criterion shows that $(T/H_{18}) \cdot \phi_{18}$ is closed in $W^{H_{18}}$. By affine Luna reduction and $C_G(H_{18}) = T$, the orbit $G \cdot \phi_{18}$ is closed in W . Thus $[\phi_{18}]$ is polystable.

Normal form

On the full-support chart

$$U_{18} := \mathbb{P}(W^{H_{18}})^\circ = \{[a_1 : \dots : a_6] \mid a_i \neq 0\} \cong (\mathbb{C}^\times)^5,$$

the effective torus T/H_{18} has dimension 3. Combining its action with overall projective rescaling, one may normalize

$$a_1 = a_2 = a_3 = a_4 = 1.$$

Indeed, if $w_i = (v_i, 1)$ denotes the coefficient character for the action of $T \times \mathbb{G}_m$, then $w_1, \dots, w_4$ are independent and

$$w_5 = w_1 - 2w_2 + w_3 + w_4, \quad w_6 = 2w_1 - 5w_2 + 2w_3 + 2w_4.$$

The two remaining invariant coefficients are therefore

$$\alpha = \frac{a_2^2 a_5}{a_1 a_3 a_4}, \quad \beta = \frac{a_2^5 a_6}{a_1^2 a_3^2 a_4^2}.$$

Thus every general limit in Case 18 is equivalent to a unique member of the normal-form family

$$\phi_{18}^{\text{nf}}(\alpha, \beta) = x_2^5 + x_1 x_2^2 x_3^2 + x_1^3 x_3 x_4 + x_0 x_3^4 + \alpha x_0 x_1 x_2 x_3 x_4 + \beta x_0^2 x_1 x_4^2,$$

where

$$(\alpha, \beta) \in (\mathbb{C}^\times)^2.$$

The two displayed character relations form a basis of the relation lattice, so

$$U_{18}/(T/H_{18}) \cong (\mathbb{C}^\times)^2$$

with coordinates (α, β) . Moreover, for a full-support point $x \in W^{H_{18}}$, the rank computation gives $G_x^\circ = H_{18}$. Consequently, any element of G carrying one full-support point of $W^{H_{18}}$ to another lies in $N_G(H_{18}) = T$. Hence the full G -quotient introduces no further identifications on this locus.

It follows that

$$\dim \Phi_{18} = \dim U_{18} - \dim(T/H_{18}) = 5 - 3 = 2.$$

No additional condition on (α, β) is imposed at the normal-form stage; every full-support member of this family is polystable.

Case $k = 19$

1-PS limit

The witness vector and the corresponding one-parameter subgroup are

$$r_{19} = (2, 1, 0, -1, -2), \quad \lambda_{19}(t) = \text{diag}(t^2, t, 1, t^{-1}, t^{-2}), \quad t \in \mathbb{G}_m.$$

Since the entries of r_{19} sum to zero, λ_{19} is a one-parameter subgroup of $G = \text{SL}(5)$. Let f_{19} be a general quintic supported on

$$S_{19} = I(r_{19})_{\geq 0}.$$

With the action convention of (2), its affine one-parameter-subgroup limit is the weight-zero part

$$\phi_{19} := \lim_{t \rightarrow 0} \lambda_{19}(t) \cdot f_{19}.$$

The degree-five weight-zero equations have precisely the exponent vectors

$$\begin{aligned} &(0, 0, 5, 0, 0), \quad (0, 1, 3, 1, 0), \quad (0, 2, 1, 2, 0), \quad (0, 2, 2, 0, 1), \\ &(0, 3, 0, 1, 1), \quad (1, 0, 2, 2, 0), \quad (1, 0, 3, 0, 1), \quad (1, 1, 0, 3, 0), \\ &(1, 1, 1, 1, 1), \quad (1, 2, 0, 0, 2), \quad (2, 0, 0, 2, 1), \quad (2, 0, 1, 0, 2). \end{aligned}$$

Consequently,

$$\begin{aligned} \phi_{19} = & a_1 x_2^5 + a_2 x_1 x_2^3 x_3 + a_3 x_1^2 x_2 x_3^2 + a_4 x_1^2 x_2^2 x_4 + a_5 x_1^3 x_3 x_4 \\ & + a_6 x_0 x_2^2 x_3^2 + a_7 x_0 x_2^3 x_4 + a_8 x_0 x_1 x_3^3 + a_9 x_0 x_1 x_2 x_3 x_4 \\ & + a_{10} x_0 x_1^2 x_4^2 + a_{11} x_0^2 x_3^2 x_4 + a_{12} x_0^2 x_2 x_4^2, \end{aligned}$$

where $a_i \in \mathbb{C}^\times$ for all i .

Set $H_{19} = \lambda_{19}(\mathbb{G}_m)$. Its five weights on $\langle x_0, \dots, x_4 \rangle$ are pairwise distinct, and hence

$$C_G(H_{19}) = T.$$

The fixed subspace is

$$\begin{aligned} W^{H_{19}} = & \langle x_2^5, x_1 x_2^3 x_3, x_1^2 x_2 x_3^2, x_1^2 x_2^2 x_4, x_1^3 x_3 x_4, \\ & x_0 x_2^2 x_3^2, x_0 x_2^3 x_4, x_0 x_1 x_3^3, x_0 x_1 x_2 x_3 x_4, \\ & x_0 x_1^2 x_4^2, x_0^2 x_3^2 x_4, x_0^2 x_2 x_4^2 \rangle, \quad \dim W^{H_{19}} = 12. \end{aligned}$$

Polystability

Let $v_1, \dots, v_{12}$ be the exponent vectors above, with $v_9 = (1, 1, 1, 1, 1)$, and put

$$\delta_i := v_i - v_9 \quad (i = 1, \dots, 12).$$

Thus $\delta_9 = 0$. The characters δ_i span the three-dimensional character space of T/H_{19} and satisfy the positive relation

$$\begin{aligned} &\delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 + \delta_6 + \delta_7 + \delta_8 + \delta_9 \\ &\quad + 4\delta_{10} + 4\delta_{11} + \delta_{12} = 0. \end{aligned}$$

Therefore

$$0 \in \text{relint Conv}\{\delta_1, \dots, \delta_{12}\}.$$

Since all twelve coefficients of ϕ_{19} are nonzero, the convex-hull criterion shows that $(T/H_{19}) \cdot \phi_{19}$ is closed in $W^{H_{19}}$. By affine Luna reduction and $C_G(H_{19}) = T$, the orbit $G \cdot \phi_{19}$ is closed in W . Hence $[\phi_{19}]$ is polystable.

Normal form

The full-support chart is

$$\mathbb{P}(W^{H_{19}})^\circ = \{[a_1 : \cdots : a_{12}] \mid a_i \neq 0\} \cong (\mathbb{C}^\times)^{11}.$$

The effective torus T/H_{19} has dimension 3. The character differences associated with the coefficients of

$$x_2^5, \quad x_1^3 x_3 x_4, \quad x_0 x_1 x_3^3, \quad x_0^2 x_3^2 x_4$$

have full rank after including overall projective rescaling. We may therefore normalize

$$a_1 = a_5 = a_8 = a_{11} = 1.$$

Renaming the remaining coefficients gives the normal form

$$\begin{aligned} \phi_{19}^{\text{nf}}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \xi, \theta) = & x_2^5 + \alpha x_1 x_2^3 x_3 + \beta x_1^2 x_2 x_3^2 + \gamma x_1^2 x_2^2 x_4 \\ & + x_1^3 x_3 x_4 + \delta x_0 x_2^2 x_3^2 + \varepsilon x_0 x_2^3 x_4 + x_0 x_1 x_3^3 \\ & + \zeta x_0 x_1 x_2 x_3 x_4 + \xi x_0 x_1^2 x_4^2 + x_0^2 x_3^2 x_4 + \theta x_0^2 x_2 x_4^2, \end{aligned}$$

where

$$(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \xi, \theta) \in (\mathbb{C}^\times)^8.$$

Thus eight parameters remain after diagonal torus scaling and projective rescaling. Accordingly,

$$\dim \Phi_{19} = \dim \mathbb{P}(W^{H_{19}})^\circ - \dim(T/H_{19}) = 11 - 3 = 8.$$

No additional parameter condition is imposed at the normal-form stage; every full-support member of this family is polystable.

Case $k = 20$

1-PS limit

The witness vector and the corresponding one-parameter subgroup are

$$r_{20} = (8, 3, -2, -2, -7), \quad \lambda_{20}(t) = \text{diag}(t^8, t^3, t^{-2}, t^{-2}, t^{-7}), \quad t \in \mathbb{G}_m.$$

Since the entries of r_{20} sum to zero, λ_{20} is a one-parameter subgroup of $G = \text{SL}(5)$. Let f_{20} be a general quintic supported on

$$S_{20} = I(r_{20})_{\geq 0}.$$

With the action convention of (2), its affine one-parameter-subgroup limit is the weight-zero part

$$\phi_{20} := \lim_{t \rightarrow 0} \lambda_{20}(t) \cdot f_{20}.$$

Put

$$U := \langle x_2, x_3 \rangle, \quad H_{20} := \lambda_{20}(\mathbb{G}_m).$$

Solving the degree-five weight-zero equations gives

$$\begin{aligned} W^{H_{20}} = & x_1^2 \text{Sym}^3 U \oplus x_1^3 x_4 \text{Sym}^1 U \oplus x_0 \text{Sym}^4 U \oplus x_0 x_1 x_4 \text{Sym}^2 U \\ & \oplus \langle x_0 x_1^2 x_4^2 \rangle \oplus x_0^2 x_4^2 \text{Sym}^1 U, \quad \dim W^{H_{20}} = 17. \end{aligned}$$

Thus a general limit can be written as

$$\begin{aligned} \phi_{20} = & x_1^2 C_3(x_2, x_3) + x_1^3 x_4 L_1(x_2, x_3) + x_0 B_4(x_2, x_3) \\ & + x_0 x_1 x_4 Q_2(x_2, x_3) + \kappa x_0 x_1^2 x_4^2 + x_0^2 x_4^2 M_1(x_2, x_3), \end{aligned}$$

where C_3 , B_4 , and Q_2 are binary forms of degrees 3, 4, and 2, respectively, and L_1, M_1 are binary linear forms. We work on the nonempty open locus

$$\kappa \neq 0, \quad \text{Disc}(B_4) \neq 0, \quad \text{Disc}(C_3) \neq 0,$$

$$\text{Res}(B_4, L_1) \neq 0, \quad \text{Res}(B_4, M_1) \neq 0.$$

The weight -2 occurs twice on U , so the centralizer is non-toric:

$$C_G(H_{20}) = \{\text{diag}(\mu_0, \mu_1) \oplus A \oplus \text{diag}(\mu_4) \mid A \in \text{GL}(U), \mu_0\mu_1\mu_4 \det A = 1\}.$$

Hence

$$\dim C_G(H_{20}) = 6, \quad \dim(C_G(H_{20})/H_{20}) = 5.$$

Polystability

Set $R := C_G(H_{20})$. By affine Luna reduction, it is enough to prove that $R \cdot \phi_{20}$ is closed in $W^{H_{20}}$. Let $\rho \in \Lambda_{\phi_{20}}^R$. After conjugation in the $\text{GL}(U)$ -block, its weights may be written as

$$\text{wt}_\rho(x_0, x_1, X, Y, x_4) = (a, b, s + n, s - n, c), \quad a + b + 2s + c = 0,$$

for a suitable eigenbasis X, Y of U .

Introduce the averaged block weights

$$p_B := a + 4s, \quad p_C := 2b + 3s, \quad p_L := 3b + s + c,$$

$$e := a + 2b + 2c, \quad p_M := 2a + s + 2c.$$

The nonvanishing of the discriminants, resultants, and κ , together with nonnegativity of all occurring monomial weights, gives

$$p_C \geq 0, \quad p_B \geq 0, \quad e \geq 0,$$

$$p_B + 4p_L \geq 0, \quad p_B + 4p_M \geq 0.$$

These quantities satisfy the balancing identity

$$4p_C + 2p_B + e + (p_B + 4p_L) + 2(p_B + 4p_M) = 22(a + b + 2s + c) = 0.$$

Consequently,

$$p_B = p_C = p_L = p_M = e = 0.$$

Since $p_B = 0$, a nonzero relative weight n would force the binary quartic B_4 to be divisible by X^2 or by Y^2 , contradicting $\text{Disc}(B_4) \neq 0$. Hence $n = 0$. Solving the remaining weight equations gives

$$(a, b, s, c) = m(8, 3, -2, -7)$$

for some $m \in \mathbb{Z}$. Therefore

$$\Lambda_{\phi_{20}}^R = \{\lambda_{20}(t^m) \mid m \in \mathbb{Z}\}.$$

This set is symmetric under $m \mapsto -m$. The Casimiro–Florentino criterion shows that $R \cdot \phi_{20}$ is closed in $W^{H_{20}}$. Luna reduction then implies that $G \cdot \phi_{20}$ is closed in W , so $[\phi_{20}]$ is polystable.

Normal form

On the dense open set where B_4 has four distinct roots and the root of L_1 is distinct from them, use $\mathrm{GL}(U)$ to put

$$L_1 = x_2$$

and

$$B_4(x_2, x_3) = B_{\lambda, \mu}(x_2, x_3) := x_3(x_2 - x_3)(x_2 - \lambda x_3)(x_2 - \mu x_3),$$

where

$$\lambda\mu(\lambda - 1)(\mu - 1)(\lambda - \mu) \neq 0.$$

Write

$$C_{\mathbf{c}}(x_2, x_3) = c_0x_2^3 + c_1x_2^2x_3 + c_2x_2x_3^2 + c_3x_3^3,$$

$$Q_{\mathbf{q}}(x_2, x_3) = q_0x_2^2 + q_1x_2x_3 + q_2x_3^2,$$

and

$$M_{\mathbf{m}}(x_2, x_3) = m_0x_2 + m_1x_3.$$

Using the diagonal factors in $C_G(H_{20})$, together with projective rescaling, normalize the coefficients of

$$x_0B_{\lambda, \mu}(x_2, x_3), \quad x_1^3x_2x_4, \quad x_0x_1^2x_4^2$$

to 1. Every general limit is then equivalent to a member of

$$\begin{aligned} \phi_{20}^{\mathrm{nf}}(\lambda, \mu, \mathbf{c}, \mathbf{q}, \mathbf{m}) &= x_0B_{\lambda, \mu}(x_2, x_3) + x_1^2C_{\mathbf{c}}(x_2, x_3) + x_1^3x_2x_4 \\ &\quad + x_0x_1x_4Q_{\mathbf{q}}(x_2, x_3) + x_0x_1^2x_4^2 + x_0^2x_4^2M_{\mathbf{m}}(x_2, x_3). \end{aligned}$$

The parameters range over the nonempty open subset on which

$$\lambda\mu(\lambda - 1)(\mu - 1)(\lambda - \mu) \neq 0, \quad \mathrm{Disc}(C_{\mathbf{c}}) \neq 0,$$

$$\mathrm{Res}(B_{\lambda, \mu}, M_{\mathbf{m}}) \neq 0.$$

There are

$$2 + 4 + 3 + 2 = 11$$

parameters. Since

$$\dim \mathbb{P}(W^{H_{20}}) = 16, \quad \dim(C_G(H_{20})/H_{20}) = 5,$$

we obtain

$$\dim \Phi_{20} = 16 - 5 = 11.$$

No condition arising from the later singular-locus analysis is imposed at the normal-form stage.

Case $k = 21$

1-PS limit

The witness vector and the corresponding one-parameter subgroup are

$$r_{21} = (16, 11, 6, -9, -24), \quad \lambda_{21}(t) = \mathrm{diag}(t^{16}, t^{11}, t^6, t^{-9}, t^{-24}), \quad t \in \mathbb{G}_m.$$

Since the entries of r_{21} sum to zero, λ_{21} is a one-parameter subgroup of $G = \mathrm{SL}(5)$. Let f_{21} be a general quintic supported on

$$S_{21} = I(r_{21})_{\geq 0}.$$

With the action convention of (2), its affine one-parameter-subgroup limit is the weight-zero part

$$\phi_{21} := \lim_{t \rightarrow 0} \lambda_{21}(t) \cdot f_{21}.$$

Put

$$H_{21} := \lambda_{21}(\mathbb{G}_m).$$

Solving the degree-five weight-zero equations gives

$$W^{H_{21}} = \langle x_2^3 x_3^2, x_2^4 x_4, x_1^3 x_3 x_4, x_0 x_1 x_3^3, \\ x_0 x_1 x_2 x_3 x_4, x_0^3 x_4^2 \rangle, \quad \dim W^{H_{21}} = 6.$$

Thus a general limit is

$$\phi_{21} = a_1 x_2^3 x_3^2 + a_2 x_2^4 x_4 + a_3 x_1^3 x_3 x_4 + a_4 x_0 x_1 x_3^3 \\ + a_5 x_0 x_1 x_2 x_3 x_4 + a_6 x_0^3 x_4^2, \quad a_i \in \mathbb{C}^\times.$$

The five weights of H_{21} on $\langle x_0, \dots, x_4 \rangle$ are pairwise distinct. Hence the centralizer is the maximal torus:

$$C_G(H_{21}) = T.$$

Polystability

Let $v_1, \dots, v_6$ be the exponent vectors of the six monomials above, ordered as displayed, and put

$$\eta = (1, 1, 1, 1, 1), \quad \delta_i := v_i - \eta.$$

The effective torus is T/H_{21} . Its nonzero character weights satisfy

$$\delta_1 + \delta_2 + 2\delta_3 + \delta_4 + 2\delta_6 = 0,$$

equivalently,

$$v_1 + v_2 + 2v_3 + v_4 + 2v_6 = 7\eta.$$

All coefficients in this relation are positive, and

$$\text{rank}\langle \delta_1, \dots, \delta_6 \rangle = 3 = \dim(T/H_{21}).$$

Therefore

$$0 \in \text{relint Conv}\{\delta_1, \dots, \delta_6\}.$$

Since all coefficients of ϕ_{21} are nonzero, the affine convex-hull criterion shows that $(T/H_{21}) \cdot \phi_{21}$ is closed in $W^{H_{21}}$. Affine Luna reduction then implies that

$$G \cdot \phi_{21}$$

is closed in W . Hence $[\phi_{21}]$ is polystable.

Normal form

The full-support chart is

$$\mathbb{P}(W^{H_{21}})^\circ = \{[a_1 : \dots : a_6] \mid a_i \neq 0\} \cong (\mathbb{C}^\times)^5.$$

The effective torus T/H_{21} has dimension 3, and the rank calculation above shows that its general stabilizer is finite. The character differences of the first four monomials have rank 3, so diagonal torus scaling together with projective rescaling normalizes their coefficients to 1. Every general limit is therefore equivalent to a member of

$$\phi_{21}^{\text{nf}}(\alpha, \beta) = x_2^3 x_3^2 + x_2^4 x_4 + x_1^3 x_3 x_4 + x_0 x_1 x_3^3 \\ + \alpha x_0 x_1 x_2 x_3 x_4 + \beta x_0^3 x_4^2, \quad (\alpha, \beta) \in (\mathbb{C}^\times)^2.$$

Consequently,

$$\dim \Phi_{21} = \dim \mathbb{P}(W^{H_{21}})^\circ - \dim(T/H_{21}) = 5 - 3 = 2.$$

No condition arising from the later singular-locus analysis is imposed at the normal-form stage.

Case $k = 22$

1-PS limit

The witness vector and the corresponding one-parameter subgroup are

$$r_{22} = (16, 6, 1, -4, -19), \quad \lambda_{22}(t) = \text{diag}(t^{16}, t^6, t, t^{-4}, t^{-19}), \quad t \in \mathbb{G}_m.$$

Since the entries of r_{22} sum to zero, λ_{22} is a one-parameter subgroup of $G = \text{SL}(5)$. Let f_{22} be a general quintic supported on

$$S_{22} = I(r_{22})_{\geq 0}.$$

With the action convention of (2), its affine one-parameter-subgroup limit is the weight-zero part

$$\phi_{22} := \lim_{t \rightarrow 0} \lambda_{22}(t) \cdot f_{22}.$$

Put

$$H_{22} := \lambda_{22}(\mathbb{G}_m).$$

Solving the degree-five weight-zero equations gives

$$W^{H_{22}} = \langle x_2^4 x_3, x_1 x_2^2 x_3^2, x_1^2 x_3^3, x_1^3 x_2 x_4, x_0 x_3^4, x_0 x_2^3 x_4, x_0 x_1 x_2 x_3 x_4, x_0^2 x_1 x_4^2 \rangle, \quad \dim W^{H_{22}} = 8.$$

Thus a general limit is

$$\begin{aligned} \phi_{22} = & a_1 x_2^4 x_3 + a_2 x_1 x_2^2 x_3^2 + a_3 x_1^2 x_3^3 + a_4 x_1^3 x_2 x_4 \\ & + a_5 x_0 x_3^4 + a_6 x_0 x_2^3 x_4 + a_7 x_0 x_1 x_2 x_3 x_4 + a_8 x_0^2 x_1 x_4^2, \quad a_i \in \mathbb{C}^\times. \end{aligned}$$

The five weights of H_{22} on $\langle x_0, \dots, x_4 \rangle$ are pairwise distinct. Hence the centralizer is the maximal torus:

$$C_G(H_{22}) = T.$$

Polystability

Let $v_1, \dots, v_8$ be the exponent vectors of the eight monomials above, ordered as displayed, and put

$$\eta = (1, 1, 1, 1, 1), \quad \delta_i := v_i - \eta.$$

The effective torus is T/H_{22} . Its character weights satisfy the positive relation

$$4\delta_1 + 5\delta_2 + 5\delta_3 + 4\delta_4 + 4\delta_5 + 5\delta_6 + 5\delta_7 + 18\delta_8 = 0,$$

equivalently,

$$4v_1 + 5v_2 + 5v_3 + 4v_4 + 4v_5 + 5v_6 + 5v_7 + 18v_8 = 50\eta.$$

Moreover,

$$\text{rank}\langle \delta_1, \dots, \delta_8 \rangle = 3 = \dim(T/H_{22}).$$

Therefore

$$0 \in \text{relint Conv}\{\delta_1, \dots, \delta_8\}.$$

Since all coefficients of ϕ_{22} are nonzero, the affine convex-hull criterion shows that $(T/H_{22}) \cdot \phi_{22}$ is closed in $W^{H_{22}}$. Affine Luna reduction then implies that

$$G \cdot \phi_{22}$$

is closed in W . Hence $[\phi_{22}]$ is polystable.

Normal form

The full-support chart is

$$\mathbb{P}(W^{H_{22}})^\circ = \{[a_1 : \cdots : a_8] \mid a_i \neq 0\} \cong (\mathbb{C}^\times)^7.$$

The effective torus T/H_{22} has dimension 3, and the rank calculation above shows that its general stabilizer is finite. The three character differences obtained from

$$x_1 x_2^2 x_3^2, \quad x_1^3 x_2 x_4, \quad x_0 x_3^4$$

relative to $x_2^4 x_3$ have rank 3. Hence diagonal torus scaling, together with projective rescaling, normalizes the coefficients of these four monomials to 1. Every general limit is therefore equivalent to a member of

$$\begin{aligned} \phi_{22}^{\text{nf}}(\alpha, \beta, \gamma, \delta) = & x_2^4 x_3 + x_1 x_2^2 x_3^2 + \alpha x_1^2 x_3^3 + x_1^3 x_2 x_4 + x_0 x_3^4 \\ & + \beta x_0 x_2^3 x_4 + \gamma x_0 x_1 x_2 x_3 x_4 + \delta x_0^2 x_1 x_4^2, \end{aligned}$$

where

$$(\alpha, \beta, \gamma, \delta) \in (\mathbb{C}^\times)^4.$$

Consequently,

$$\dim \Phi_{22} = \dim \mathbb{P}(W^{H_{22}})^\circ - \dim(T/H_{22}) = 7 - 3 = 4.$$

No condition arising from the later singular-locus analysis is imposed at the normal-form stage.

Case $k = 23$

1-PS limit

The witness vector and the corresponding one-parameter subgroup are

$$r_{23} = (4, -1, -1, -1, -1), \quad \lambda_{23}(t) = \text{diag}(t^4, t^{-1}, t^{-1}, t^{-1}, t^{-1}), \quad t \in \mathbb{G}_m.$$

Since the entries of r_{23} sum to zero, λ_{23} is a one-parameter subgroup of $G = \text{SL}(5)$. Let f_{23} be a general quintic supported on

$$S_{23} = I(r_{23})_{\geq 0}.$$

For a degree-five monomial x^u , one has

$$r_{23} \cdot u = 4u_0 - u_1 - u_2 - u_3 - u_4 = 5u_0 - 5.$$

Hence

$$I(r_{23})_{\geq 0} = \{u \in I \mid u_0 \geq 1\}, \quad I(r_{23})_{=0} = \{u \in I \mid u_0 = 1\}.$$

Therefore the affine one-parameter-subgroup limit is

$$\phi_{23} := \lim_{t \rightarrow 0} \lambda_{23}(t) \cdot f_{23} = x_0 B_4(x_1, x_2, x_3, x_4),$$

where, with

$$U := \langle x_1, x_2, x_3, x_4 \rangle,$$

one has $B_4 \in \text{Sym}^4 U$. We work on the nonempty Zariski-open locus on which $[B_4] \in \mathbb{P}(\text{Sym}^4 U)$ is stable for the $\text{SL}(U)$ -action.

Put

$$H_{23} := \lambda_{23}(\mathbb{G}_m).$$

The weight -1 occurs with multiplicity four on U , so

$$C_G(H_{23}) = \{\text{diag}(\mu_0) \oplus A \mid A \in \text{GL}(U), \mu_0 \det A = 1\}.$$

Thus the centralizer is non-toric, and

$$W^{H_{23}} = x_0 \text{Sym}^4 U, \quad \dim W^{H_{23}} = 35.$$

Moreover, H_{23} acts trivially on $W^{H_{23}}$, and the effective residual action is

$$C_G(H_{23})/H_{23} \simeq \text{PGL}(U).$$

Polystability

Set

$$R := C_G(H_{23}).$$

By affine Luna reduction, it is enough to prove that

$$R \cdot \phi_{23}$$

is closed in $W^{H_{23}}$. We apply the Casimiro–Florentino criterion.

Let $\lambda \in \Lambda_{\phi_{23}}^R$. After conjugation in R , we may assume that λ is diagonal on U . In a corresponding eigenbasis $X_1, \dots, X_4$, write

$$\text{wt}_\lambda(x_0, X_1, X_2, X_3, X_4) = (a, b_1, b_2, b_3, b_4),$$

so that

$$a + b_1 + b_2 + b_3 + b_4 = 0.$$

Define

$$c_i := 4b_i + a \quad (1 \leq i \leq 4).$$

Then $c_1 + \dots + c_4 = 0$, and hence

$$\nu(t) := \text{diag}(t^{c_1}, t^{c_2}, t^{c_3}, t^{c_4})$$

is a one-parameter subgroup of $\text{SL}(U)$. For every quartic monomial $X^u = X_1^{u_1} \dots X_4^{u_4}$ one has

$$4 \text{ wt}_\lambda(x_0 X^u) = u_1 c_1 + u_2 c_2 + u_3 c_3 + u_4 c_4.$$

Since $\lambda \in \Lambda_{\phi_{23}}^R$, every monomial occurring in B_4 has nonnegative ν -weight. The stability of $[B_4]$ for $\text{SL}(U)$ therefore forces ν to be trivial. Thus

$$c_1 = c_2 = c_3 = c_4 = 0,$$

and consequently there is an integer m such that

$$(a, b_1, b_2, b_3, b_4) = m(4, -1, -1, -1, -1).$$

Hence

$$\Lambda_{\phi_{23}}^R = \{\lambda_{23}(t^m) \mid m \in \mathbb{Z}\}.$$

This set is symmetric, since it contains the inverse one-parameter subgroup for every one of its elements. The Casimiro–Florentino criterion shows that $R \cdot \phi_{23}$ is closed in $W^{H_{23}}$. Affine Luna reduction then gives that

$$G \cdot \phi_{23}$$

is closed in W . Hence $[\phi_{23}]$ is polystable.

Normal form

The closed-orbit normal form is

$$\phi_{23}^{\text{nf}}(B_4) = x_0 B_4(x_1, x_2, x_3, x_4),$$

where $[B_4]$ belongs to the stable locus

$$\mathbb{P}(\text{Sym}^4 U)^s.$$

Under the effective centralizer action, two such normal forms determine the same quotient point precisely when their quartic forms determine the same point of

$$\mathbb{P}(\mathrm{Sym}^4 U)^s / \mathrm{PGL}(U).$$

Since a stable quartic has finite stabilizer for the effective $\mathrm{PGL}(U)$ -action,

$$\dim \mathbb{P}(\mathrm{Sym}^4 U) = 34, \quad \dim \mathrm{PGL}(U) = 15.$$

Consequently,

$$\dim \Phi_{23} = 34 - 15 = 19.$$

No smoothness condition on $V(B_4) \subset \mathbb{P}^3$, or any other condition arising from the later singular-locus analysis, is imposed at the normal-form stage.

Case $k = 24$

1-PS limit

The witness vector and the corresponding one-parameter subgroup are

$$r_{24} = (14, 9, 4, -6, -21), \quad \lambda_{24}(t) = \mathrm{diag}(t^{14}, t^9, t^4, t^{-6}, t^{-21}), \quad t \in \mathbb{G}_m.$$

Since the entries of r_{24} sum to zero, λ_{24} is a one-parameter subgroup of $G = \mathrm{SL}(5)$. Let f_{24} be a general quintic supported on

$$S_{24} = I(r_{24})_{\geq 0}.$$

Its affine one-parameter-subgroup limit is the weight-zero part

$$\phi_{24} := \lim_{t \rightarrow 0} \lambda_{24}(t) \cdot f_{24}.$$

Put

$$H_{24} := \lambda_{24}(\mathbb{G}_m).$$

Solving the degree-five weight-zero equations gives

$$W^{H_{24}} = \langle x_2^3 x_3^2, x_1 x_2^3 x_4, x_1^2 x_3^3, x_1^3 x_3 x_4, x_0 x_2 x_3^3, x_0 x_1 x_2 x_3 x_4, x_0^3 x_4^2 \rangle, \quad \dim W^{H_{24}} = 7.$$

Thus a general limit is

$$\begin{aligned} \phi_{24} = & a_1 x_2^3 x_3^2 + a_2 x_1 x_2^3 x_4 + a_3 x_1^2 x_3^3 + a_4 x_1^3 x_3 x_4 \\ & + a_5 x_0 x_2 x_3^3 + a_6 x_0 x_1 x_2 x_3 x_4 + a_7 x_0^3 x_4^2, \quad a_i \in \mathbb{C}^\times. \end{aligned}$$

The five weights of H_{24} on $\langle x_0, \dots, x_4 \rangle$ are pairwise distinct. Hence

$$C_G(H_{24}) = T.$$

Polystability

Let $v_1, \dots, v_7$ be the exponent vectors of the seven monomials above, ordered as displayed, and put

$$\eta = (1, 1, 1, 1, 1), \quad \delta_i := v_i - \eta.$$

The effective torus is T/H_{24} . Its character weights satisfy the positive relation

$$\delta_1 + 2\delta_2 + \delta_3 + 2\delta_4 + \delta_5 + \delta_6 + 3\delta_7 = 0,$$

equivalently,

$$v_1 + 2v_2 + v_3 + 2v_4 + v_5 + v_6 + 3v_7 = 11\eta.$$

Moreover,

$$\text{rank}\langle \delta_1, \dots, \delta_7 \rangle = 3 = \dim(T/H_{24}).$$

Therefore

$$0 \in \text{relint Conv}\{\delta_1, \dots, \delta_7\}.$$

Since all coefficients of ϕ_{24} are nonzero, the affine convex-hull criterion shows that $(T/H_{24}) \cdot \phi_{24}$ is closed in $W^{H_{24}}$. The quotient $N_G(H_{24})/C_G(H_{24})$ is finite, so the normalizer orbit is also closed. Affine Luna reduction then implies that

$$G \cdot \phi_{24}$$

is closed in W . Hence $[\phi_{24}]$ is polystable.

Normal form

The full-support chart is

$$\mathbb{P}(W^{H_{24}})^\circ = \{[a_1 : \dots : a_7] \mid a_i \neq 0\} \cong (\mathbb{C}^\times)^6.$$

The effective torus T/H_{24} has dimension 3, and the rank calculation above shows that its general stabilizer is finite. Using the diagonal torus together with projective rescaling, we normalize the coefficients of

$$x_2^3 x_3^2, \quad x_1 x_2^3 x_4, \quad x_1^3 x_3 x_4, \quad x_0 x_2 x_3^3$$

to be equal to 1. Every general limit is therefore equivalent to a member of

$$\begin{aligned} \phi_{24}^{\text{nf}}(\alpha, \beta, \gamma) = & x_2^3 x_3^2 + x_1 x_2^3 x_4 + \alpha x_1^2 x_3^3 + x_1^3 x_3 x_4 \\ & + x_0 x_2 x_3^3 + \beta x_0 x_1 x_2 x_3 x_4 + \gamma x_0^3 x_4^2, \end{aligned}$$

where

$$(\alpha, \beta, \gamma) \in (\mathbb{C}^\times)^3.$$

Consequently,

$$\dim \Phi_{24} = \dim \mathbb{P}(W^{H_{24}})^\circ - \dim(T/H_{24}) = 6 - 3 = 3.$$

No condition arising from the later singular-locus analysis is imposed at the normal-form stage.

Case $k = 25$

1-PS limit

The witness vector and the corresponding one-parameter subgroup are

$$r_{25} = (11, 6, 1, -4, -14), \quad \lambda_{25}(t) = \text{diag}(t^{11}, t^6, t, t^{-4}, t^{-14}), \quad t \in \mathbb{G}_m.$$

Since the entries of r_{25} sum to zero, λ_{25} is a one-parameter subgroup of $G = \text{SL}(5)$. Let f_{25} be a general quintic supported on

$$S_{25} = I(r_{25})_{\geq 0}.$$

Its affine one-parameter-subgroup limit is the weight-zero part

$$\phi_{25} := \lim_{t \rightarrow 0} \lambda_{25}(t) \cdot f_{25}.$$

Put

$$H_{25} := \lambda_{25}(\mathbb{G}_m).$$

Solving the degree-five weight-zero equations gives

$$W^{H_{25}} = \langle x_2^4 x_3, x_1 x_2^2 x_3^2, x_1^2 x_3^3, x_1^2 x_2^2 x_4, x_1^3 x_3 x_4, \\ x_0 x_2 x_3^3, x_0 x_2^3 x_4, x_0 x_1 x_2 x_3 x_4, x_0^2 x_3^2 x_4, x_0^2 x_1 x_4^2 \rangle, \quad \dim W^{H_{25}} = 10.$$

Thus a general limit is

$$\phi_{25} = a_1 x_2^4 x_3 + a_2 x_1 x_2^2 x_3^2 + a_3 x_1^2 x_3^3 + a_4 x_1^2 x_2^2 x_4 + a_5 x_1^3 x_3 x_4 \\ + a_6 x_0 x_2 x_3^3 + a_7 x_0 x_2^3 x_4 + a_8 x_0 x_1 x_2 x_3 x_4 + a_9 x_0^2 x_3^2 x_4 + a_{10} x_0^2 x_1 x_4^2,$$

where $a_i \in \mathbb{C}^\times$. The five weights of H_{25} on $\langle x_0, \dots, x_4 \rangle$ are pairwise distinct, and hence

$$C_G(H_{25}) = T.$$

Polystability

Let $v_1, \dots, v_{10}$ be the exponent vectors of the ten monomials above, ordered as displayed, and put

$$\eta = (1, 1, 1, 1, 1), \quad \delta_i := v_i - \eta.$$

The effective torus is T/H_{25} . Its character weights satisfy the positive relation

$$\delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 + \delta_6 + \delta_7 + \delta_8 + \delta_9 + 4\delta_{10} = 0,$$

equivalently,

$$v_1 + v_2 + v_3 + v_4 + v_5 + v_6 + v_7 + v_8 + v_9 + 4v_{10} = 13\eta.$$

Moreover,

$$\text{rank}\langle \delta_1, \dots, \delta_{10} \rangle = 3 = \dim(T/H_{25}).$$

Therefore

$$0 \in \text{relint Conv}\{\delta_1, \dots, \delta_{10}\}.$$

Since all coefficients of ϕ_{25} are nonzero, the affine convex-hull criterion shows that $(T/H_{25}) \cdot \phi_{25}$ is closed in $W^{H_{25}}$. Affine Luna reduction then implies that

$$G \cdot \phi_{25}$$

is closed in W . Hence $[\phi_{25}]$ is polystable.

Normal form

The full-support chart is

$$\mathbb{P}(W^{H_{25}})^\circ = \{[a_1 : \dots : a_{10}] \mid a_i \neq 0\} \cong (\mathbb{C}^\times)^9.$$

The effective torus T/H_{25} has dimension 3, and the rank calculation above shows that its general stabilizer is finite. Using the diagonal torus together with projective rescaling, we normalize the coefficients of

$$x_2^4 x_3, \quad x_1^2 x_2^2 x_4, \quad x_1^3 x_3 x_4, \quad x_0 x_2 x_3^3$$

to be equal to 1. Every general limit is therefore equivalent to a member of

$$\phi_{25}^{\text{nf}}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta) = x_2^4 x_3 + \alpha x_1 x_2^2 x_3^2 + \beta x_1^2 x_3^3 + x_1^2 x_2^2 x_4 \\ + x_1^3 x_3 x_4 + x_0 x_2 x_3^3 + \gamma x_0 x_2^3 x_4 + \delta x_0 x_1 x_2 x_3 x_4 \\ + \varepsilon x_0^2 x_3^2 x_4 + \zeta x_0^2 x_1 x_4^2,$$

where

$$(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta) \in (\mathbb{C}^\times)^6.$$

Consequently,

$$\dim \Phi_{25} = \dim \mathbb{P}(W^{H_{25}})^\circ - \dim(T/H_{25}) = 9 - 3 = 6.$$

No condition arising from the later singular-locus analysis is imposed at the normal-form stage.

Case $k = 26$

1-PS limit

The witness vector and the corresponding one-parameter subgroup are

$$r_{26} = (24, 9, 4, -1, -36), \quad \lambda_{26}(t) = \text{diag}(t^{24}, t^9, t^4, t^{-1}, t^{-36}), \quad t \in \mathbb{G}_m.$$

Since the entries of r_{26} sum to zero, λ_{26} is a one-parameter subgroup of $G = \text{SL}(5)$. Let f_{26} be a general quintic supported on

$$S_{26} = I(r_{26})_{\geq 0}.$$

Its affine one-parameter-subgroup limit is the weight-zero part

$$\phi_{26} := \lim_{t \rightarrow 0} \lambda_{26}(t) \cdot f_{26}.$$

Put

$$H_{26} := \lambda_{26}(\mathbb{G}_m).$$

Solving the degree-five weight-zero equations gives

$$W^{H_{26}} = \langle x_2 x_3^4, x_1^4 x_4, x_0 x_2^3 x_4, x_0 x_1 x_2 x_3 x_4, x_0^3 x_4^2 \rangle, \quad \dim W^{H_{26}} = 5.$$

Thus a general limit is

$$\phi_{26} = a_1 x_2 x_3^4 + a_2 x_1^4 x_4 + a_3 x_0 x_2^3 x_4 + a_4 x_0 x_1 x_2 x_3 x_4 + a_5 x_0^3 x_4^2,$$

where $a_i \in \mathbb{C}^\times$. The five weights of H_{26} on $\langle x_0, \dots, x_4 \rangle$ are pairwise distinct, and hence

$$C_G(H_{26}) = T.$$

Polystability

Let $v_1, \dots, v_5$ be the exponent vectors of the five monomials above, ordered as displayed, and put

$$\eta = (1, 1, 1, 1, 1), \quad \delta_i := v_i - \eta.$$

Here $\delta_4 = 0$, and the four nonzero characters satisfy

$$\delta_1 + \delta_2 + \delta_3 + \delta_5 = 0,$$

equivalently,

$$v_1 + v_2 + v_3 + v_5 = 4\eta.$$

Moreover,

$$\text{rank}\langle \delta_1, \delta_2, \delta_3, \delta_5 \rangle = 3 = \dim(T/H_{26}).$$

Therefore

$$0 \in \text{relint Conv}\{\delta_1, \delta_2, \delta_3, \delta_5\}.$$

Since all five coefficients of ϕ_{26} are nonzero, the affine convex-hull criterion shows that

$$(T/H_{26}) \cdot \phi_{26}$$

is closed in $W^{H_{26}}$. Affine Luna reduction and $C_G(H_{26}) = T$ then imply that $G \cdot \phi_{26}$ is closed in W . Hence $[\phi_{26}]$ is polystable.

Normal form

The full-support chart is

$$\mathbb{P}(W^{H_{26}})^\circ = \{[a_1 : \cdots : a_5] \mid a_i \neq 0\} \cong (\mathbb{C}^\times)^4.$$

The effective torus T/H_{26} has dimension 3, and the rank calculation above shows that its general stabilizer is finite. Using the diagonal torus together with projective rescaling, we normalize the coefficients of

$$x_2 x_3^4, \quad x_1^4 x_4, \quad x_0 x_2^3 x_4, \quad x_0^3 x_4^2$$

to be equal to 1. Every general limit is therefore equivalent to a member of

$$\phi_{26}^{\text{nf}}(\alpha) = x_2 x_3^4 + x_1^4 x_4 + x_0 x_2^3 x_4 + \alpha x_0 x_1 x_2 x_3 x_4 + x_0^3 x_4^2, \quad \alpha \in \mathbb{C}^\times.$$

The character relation above shows that

$$\frac{a_1 a_2 a_3 a_5}{a_4^4}$$

is invariant under diagonal torus scaling and projective rescaling. In the displayed normal form it equals α^{-4} ; hence α is determined up to the residual finite fourth-root ambiguity.

Consequently,

$$\dim \Phi_{26} = \dim \mathbb{P}(W^{H_{26}})^\circ - \dim(T/H_{26}) = 4 - 3 = 1.$$

No condition arising from the later singular-locus analysis is imposed at the normal-form stage.

Case $k = 27$

1-PS limit

The witness vector and the corresponding one-parameter subgroup are

$$r_{27} = (18, 8, 3, -2, -27), \quad \lambda_{27}(t) = \text{diag}(t^{18}, t^8, t^3, t^{-2}, t^{-27}), \quad t \in \mathbb{G}_m.$$

Since the entries of r_{27} sum to zero, λ_{27} is a one-parameter subgroup of $G = \text{SL}(5)$. Let f_{27} be a general quintic supported on

$$S_{27} = I(r_{27})_{\geq 0}.$$

Its affine one-parameter-subgroup limit is the weight-zero part

$$\phi_{27} := \lim_{t \rightarrow 0} \lambda_{27}(t) \cdot f_{27}.$$

Put

$$H_{27} := \lambda_{27}(\mathbb{G}_m).$$

Solving the degree-five weight-zero equations gives

$$W^{H_{27}} = \langle x_2^2 x_3^3, x_1 x_3^4, x_1^3 x_2 x_4, x_0 x_2^3 x_4, x_0 x_1 x_2 x_3 x_4, x_0^3 x_4^2 \rangle, \quad \dim W^{H_{27}} = 6.$$

Thus a general limit is

$$\phi_{27} = a_1 x_2^2 x_3^3 + a_2 x_1 x_3^4 + a_3 x_1^3 x_2 x_4 + a_4 x_0 x_2^3 x_4 + a_5 x_0 x_1 x_2 x_3 x_4 + a_6 x_0^3 x_4^2,$$

where $a_i \in \mathbb{C}^\times$. The five weights of H_{27} on $\langle x_0, \dots, x_4 \rangle$ are pairwise distinct, and hence

$$C_G(H_{27}) = T.$$

Polystability

Let $v_1, \dots, v_6$ be the exponent vectors of the six monomials above, ordered as displayed, and put

$$\eta = (1, 1, 1, 1, 1), \quad \delta_i := v_i - \eta.$$

Here $\delta_5 = 0$, and the five nonzero characters satisfy

$$\delta_1 + \delta_2 + 2\delta_3 + \delta_4 + 2\delta_6 = 0,$$

equivalently,

$$v_1 + v_2 + 2v_3 + v_4 + 2v_6 = 7\eta.$$

Moreover,

$$\text{rank}\langle \delta_1, \delta_2, \delta_3, \delta_4, \delta_6 \rangle = 3 = \dim(T/H_{27}).$$

Therefore

$$0 \in \text{relint Conv}\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_6\}.$$

Since all six coefficients of ϕ_{27} are nonzero, the affine convex-hull criterion shows that

$$(T/H_{27}) \cdot \phi_{27}$$

is closed in $W^{H_{27}}$. Affine Luna reduction and $C_G(H_{27}) = T$ then imply that $G \cdot \phi_{27}$ is closed in W . Hence $[\phi_{27}]$ is polystable.

Normal form

The full-support chart is

$$\mathbb{P}(W^{H_{27}})^\circ = \{[a_1 : \dots : a_6] \mid a_i \neq 0\} \cong (\mathbb{C}^\times)^5.$$

The effective torus T/H_{27} has dimension 3, and the rank calculation above shows that its general stabilizer is finite. Using the diagonal torus together with projective rescaling, we normalize the coefficients of

$$x_2^2 x_3^3, \quad x_1 x_3^4, \quad x_1^3 x_2 x_4, \quad x_0 x_2^3 x_4$$

to be equal to 1. Every general limit is therefore equivalent to a member of

$$\phi_{27}^{\text{nf}}(\alpha, \beta) = x_2^2 x_3^3 + x_1 x_3^4 + x_1^3 x_2 x_4 + x_0 x_2^3 x_4 + \alpha x_0 x_1 x_2 x_3 x_4 + \beta x_0^3 x_4^2,$$

where

$$(\alpha, \beta) \in (\mathbb{C}^\times)^2.$$

Consequently,

$$\dim \Phi_{27} = \dim \mathbb{P}(W^{H_{27}})^\circ - \dim(T/H_{27}) = 5 - 3 = 2.$$

No condition arising from the later singular-locus analysis is imposed at the normal-form stage.

Case $k = 28$

1-PS limit

The witness vector and the corresponding one-parameter subgroup are

$$r_{28} = (12, 7, 2, -3, -18), \quad \lambda_{28}(t) = \text{diag}(t^{12}, t^7, t^2, t^{-3}, t^{-18}), \quad t \in \mathbb{G}_m.$$

Since the entries of r_{28} sum to zero, λ_{28} is a one-parameter subgroup of $G = \text{SL}(5)$. Let f_{28} be a general quintic supported on

$$S_{28} = I(r_{28})_{\geq 0}.$$

Its affine one-parameter-subgroup limit is the weight-zero part

$$\phi_{28} := \lim_{t \rightarrow 0} \lambda_{28}(t) \cdot f_{28}.$$

Put

$$H_{28} := \lambda_{28}(\mathbb{G}_m).$$

Solving the degree-five weight-zero equations gives

$$W^{H_{28}} = \langle x_2^3 x_3^2, x_1 x_2 x_3^3, x_1^2 x_2^2 x_4, x_1^3 x_3 x_4, x_0 x_3^4, \\ x_0 x_2^3 x_4, x_0 x_1 x_2 x_3 x_4, x_0^2 x_3^2 x_4, x_0^3 x_4^2 \rangle, \quad \dim W^{H_{28}} = 9.$$

Thus a general limit is

$$\phi_{28} = a_1 x_2^3 x_3^2 + a_2 x_1 x_2 x_3^3 + a_3 x_1^2 x_2^2 x_4 + a_4 x_1^3 x_3 x_4 + a_5 x_0 x_3^4 \\ + a_6 x_0 x_2^3 x_4 + a_7 x_0 x_1 x_2 x_3 x_4 + a_8 x_0^2 x_3^2 x_4 + a_9 x_0^3 x_4^2,$$

where $a_i \in \mathbb{C}^\times$. The five weights of H_{28} on $\langle x_0, \dots, x_4 \rangle$ are pairwise distinct, and hence

$$C_G(H_{28}) = T.$$

Polystability

Let $v_1, \dots, v_9$ be the exponent vectors of the nine monomials above, ordered as displayed, and put

$$\eta = (1, 1, 1, 1, 1), \quad \delta_i := v_i - \eta.$$

Here $\delta_7 = 0$, and the eight nonzero characters satisfy

$$\delta_1 + \delta_2 + 3\delta_3 + 2\delta_4 + \delta_5 + \delta_6 + \delta_8 + 3\delta_9 = 0,$$

equivalently,

$$v_1 + v_2 + 3v_3 + 2v_4 + v_5 + v_6 + v_8 + 3v_9 = 13\eta.$$

Moreover,

$$\text{rank}\langle \delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_8, \delta_9 \rangle = 3 = \dim(T/H_{28}).$$

Therefore

$$0 \in \text{relint Conv}\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_8, \delta_9\}.$$

Since all nine coefficients of ϕ_{28} are nonzero, the affine convex-hull criterion shows that

$$(T/H_{28}) \cdot \phi_{28}$$

is closed in $W^{H_{28}}$. Affine Luna reduction and $C_G(H_{28}) = T$ then imply that $G \cdot \phi_{28}$ is closed in W . Hence $[\phi_{28}]$ is polystable.

Normal form

The full-support chart is

$$\mathbb{P}(W^{H_{28}})^\circ = \{[a_1 : \dots : a_9] \mid a_i \neq 0\} \cong (\mathbb{C}^\times)^8.$$

The effective torus T/H_{28} has dimension 3, and the rank calculation above shows that its general stabilizer is finite. Using the diagonal torus together with projective rescaling, we normalize the coefficients of

$$x_2^3 x_3^2, \quad x_1 x_2 x_3^3, \quad x_1^2 x_2^2 x_4, \quad x_0 x_3^4$$

to be equal to 1. Every general limit is therefore equivalent to a member of

$$\begin{aligned}\phi_{28}^{\text{nf}}(\alpha, \beta, \gamma, \delta, \varepsilon) = & x_2^3 x_3^2 + x_1 x_2 x_3^3 + x_1^2 x_2^2 x_4 + \beta x_1^3 x_3 x_4 + x_0 x_3^4 \\ & + \alpha x_0 x_2^3 x_4 + \gamma x_0 x_1 x_2 x_3 x_4 + \delta x_0^2 x_3^2 x_4 + \varepsilon x_0^3 x_4^2,\end{aligned}$$

where

$$(\alpha, \beta, \gamma, \delta, \varepsilon) \in (\mathbb{C}^\times)^5.$$

Consequently,

$$\dim \Phi_{28} = \dim \mathbb{P}(W^{H_{28}})^\circ - \dim(T/H_{28}) = 8 - 3 = 5.$$

No condition arising from the later singular-locus analysis is imposed at the normal-form stage.

Case $k = 29$

1-PS limit

The witness vector and the corresponding one-parameter subgroup are

$$r_{29} = (6, 1, 1, -4, -4), \quad \lambda_{29}(t) = \text{diag}(t^6, t, t, t^{-4}, t^{-4}), \quad t \in \mathbb{G}_m.$$

Since the entries of r_{29} sum to zero, λ_{29} is a one-parameter subgroup of $G = \text{SL}(5)$. Let f_{29} be a general quintic supported on

$$S_{29} = I(r_{29})_{\geq 0}.$$

Its affine one-parameter-subgroup limit is the weight-zero part

$$\phi_{29} := \lim_{t \rightarrow 0} \lambda_{29}(t) \cdot f_{29}.$$

Put

$$U := \langle x_1, x_2 \rangle, \quad V := \langle x_3, x_4 \rangle, \quad H_{29} := \lambda_{29}(\mathbb{G}_m).$$

Solving the degree-five weight-zero equations gives

$$W^{H_{29}} = \text{Sym}^4 U \otimes V \oplus x_0 \text{Sym}^2 U \otimes \text{Sym}^2 V \oplus x_0^2 \text{Sym}^3 V,$$

and hence

$$\dim W^{H_{29}} = 23.$$

Thus a general limit has the form

$$\phi_{29} = x_3 A_4(x_1, x_2) + x_4 C_4(x_1, x_2) + x_0 Q_{2,2}(x_1, x_2; x_3, x_4) + x_0^2 B_3(x_3, x_4),$$

where

$$A_4, C_4 \in \text{Sym}^4 U, \quad Q_{2,2} \in \text{Sym}^2 U \otimes \text{Sym}^2 V, \quad B_3 \in \text{Sym}^3 V.$$

We work on the nonempty Zariski-open locus

$$\text{Res}(A_4, C_4) \neq 0, \quad \text{Disc}(B_3) \neq 0.$$

The first condition says that the pencil spanned by A_4 and C_4 is base-point-free on $\mathbb{P}(U)$, while the second says that B_3 is square-free.

The weights 1 and -4 each occur with multiplicity two. Therefore the centralizer is non-toric:

$$C_G(H_{29}) = \{\text{diag}(\mu_0) \oplus A \oplus B \mid A \in \text{GL}(U), B \in \text{GL}(V), \mu_0 \det A \det B = 1\}.$$

Consequently,

$$\dim C_G(H_{29}) = 8, \quad \dim(C_G(H_{29})/H_{29}) = 7.$$

Polystability

Set

$$R := C_G(H_{29}).$$

By affine Luna reduction, it is enough to prove that $R \cdot \phi_{29}$ is closed in $W^{H_{29}}$. Let $\rho \in \Lambda_{\phi_{29}}^R$. After conjugation in the two non-abelian blocks, the weights of ρ may be written as

$$\text{wt}_\rho(x_0, X, Y, Z_1, Z_2) = (a, s + n, s - n, t + v, t - v),$$

where X, Y and Z_1, Z_2 are eigenbases of U and V , and

$$a + 2s + 2t = 0.$$

Define the averaged block weights

$$p_F := 4s + t, \quad p_B := 2a + 3t.$$

The nonvanishing of $\text{Res}(A_4, C_4)$, together with nonnegativity of the occurring monomial weights, gives

$$p_F \geq 0.$$

Similarly, $\text{Disc}(B_3) \neq 0$ gives

$$p_B \geq 0.$$

These quantities satisfy

$$p_F + p_B = (4s + t) + (2a + 3t) = 2(a + 2s + 2t) = 0.$$

Hence

$$p_F = p_B = 0.$$

Since $p_B = 0$, a nonzero relative weight v would force the binary cubic B_3 to have a multiple root. Thus $v = 0$. Since $p_F = 0$, a nonzero relative weight n would force both A_4 and C_4 to be divisible by the same square of a linear form, contradicting $\text{Res}(A_4, C_4) \neq 0$. Therefore $n = 0$.

Solving the remaining equations gives

$$(a, s, t) = m(6, 1, -4)$$

for some $m \in \mathbb{Z}$. Consequently,

$$\Lambda_{\phi_{29}}^R = \{\lambda_{29}(t^m) \mid m \in \mathbb{Z}\}.$$

This set is symmetric under $m \mapsto -m$. The Casimiro–Florentino criterion therefore shows that $R \cdot \phi_{29}$ is closed in $W^{H_{29}}$. Luna reduction then implies that

$$G \cdot \phi_{29}$$

is closed in W , so $[\phi_{29}]$ is polystable.

The same calculation shows that the identity component of the stabilizer in R is exactly H_{29} . Hence the stabilizer for the effective action of R/H_{29} is finite on this open locus.

Normal form

Since B_3 is square-free, the $\mathrm{GL}(V)$ -action normalizes it to

$$B_3^{(29)}(x_3, x_4) := x_3 x_4 (x_3 - x_4).$$

Thus every general limit is equivalent to a member of

$$\begin{aligned} \phi_{29}^{\mathrm{nf}}(A_4, C_4, Q_{2,2}) &= x_3 A_4(x_1, x_2) + x_4 C_4(x_1, x_2) \\ &\quad + x_0 Q_{2,2}(x_1, x_2; x_3, x_4) + x_0^2 x_3 x_4 (x_3 - x_4), \end{aligned}$$

where

$$A_4, C_4 \in \mathrm{Sym}^4 U, \quad Q_{2,2} \in \mathrm{Sym}^2 U \otimes \mathrm{Sym}^2 V, \quad \mathrm{Res}(A_4, C_4) \neq 0.$$

The residual action consists of the $\mathrm{GL}(U)$ -action on (x_1, x_2) , together with the finite stabilizer of $x_3 x_4 (x_3 - x_4)$ in the V -variables. Accordingly, the normal-form parameter space has dimension

$$5 + 5 + 9 - \dim \mathrm{GL}(U) = 19 - 4 = 15.$$

Equivalently, since the effective centralizer stabilizer is finite,

$$\dim \mathbb{P}(W^{H_{29}}) = 22, \quad \dim(C_G(H_{29})/H_{29}) = 7,$$

and therefore

$$\dim \Phi_{29} = 22 - 7 = 15.$$

No condition arising from the later singular-locus or minimal-exponent analysis is imposed at the normal-form stage.

Case $k = 30$

1-PS limit

The witness vector and the corresponding one-parameter subgroup are

$$r_{30} = (6, 6, 1, -4, -9), \quad \lambda_{30}(t) = \mathrm{diag}(t^6, t^6, t, t^{-4}, t^{-9}), \quad t \in \mathbb{G}_m.$$

Since the entries of r_{30} sum to zero, λ_{30} is a one-parameter subgroup of $G = \mathrm{SL}(5)$. Let f_{30} be a general quintic supported on

$$S_{30} = I(r_{30})_{\geq 0}.$$

Its affine one-parameter-subgroup limit is the weight-zero part

$$\phi_{30} := \lim_{t \rightarrow 0} \lambda_{30}(t) \cdot f_{30}.$$

Put

$$U := \langle x_0, x_1 \rangle, \quad H_{30} := \lambda_{30}(\mathbb{G}_m).$$

Solving the degree-five weight-zero equations gives

$$\begin{aligned} W^{H_{30}} &= \langle x_2^4 x_3 \rangle \oplus x_2^2 x_3^2 U \oplus x_2^3 x_4 U \\ &\quad \oplus x_3^3 \mathrm{Sym}^2 U \oplus x_2 x_3 x_4 \mathrm{Sym}^2 U \oplus x_4^2 \mathrm{Sym}^3 U, \end{aligned}$$

and hence

$$\dim W^{H_{30}} = 15.$$

Thus a general limit can be written as

$$\begin{aligned} \phi_{30} &= \kappa x_2^4 x_3 + x_2^2 x_3^2 L_1(x_0, x_1) + x_2^3 x_4 L_2(x_0, x_1) \\ &\quad + x_3^3 Q_1(x_0, x_1) + x_2 x_3 x_4 Q_2(x_0, x_1) + x_4^2 C_3(x_0, x_1), \end{aligned}$$

where $L_1, L_2 \in \text{Sym}^1 U$, $Q_1, Q_2 \in \text{Sym}^2 U$, and $C_3 \in \text{Sym}^3 U$. We work on the nonempty Zariski-open locus

$$\kappa \neq 0, \quad \text{Disc}(Q_1) \neq 0, \quad \text{Disc}(C_3) \neq 0.$$

The weight 6 occurs twice on U , so the centralizer is non-toric:

$$C_G(H_{30}) = \{A \oplus \text{diag}(\mu_2, \mu_3, \mu_4) \mid A \in \text{GL}(U), \det A \mu_2 \mu_3 \mu_4 = 1\}.$$

Consequently,

$$\dim C_G(H_{30}) = 6, \quad \dim(C_G(H_{30})/H_{30}) = 5.$$

Polystability

Set

$$R := C_G(H_{30}).$$

By affine Luna reduction, it is enough to prove that $R \cdot \phi_{30}$ is closed in $W^{H_{30}}$. Let $\rho \in \Lambda_{\phi_{30}}^R$. After conjugation in the $\text{GL}(U)$ -block, its weights may be written as

$$\text{wt}_\rho(X, Y, x_2, x_3, x_4) = (s + n, s - n, b, c, d), \quad 2s + b + c + d = 0,$$

for a suitable eigenbasis X, Y of U .

Introduce the averaged weights

$$\omega_0 := 4b + c, \quad \omega_Q := 2s + 3c, \quad \omega_C := 3s + 2d.$$

The nonvanishing of κ , $\text{Disc}(Q_1)$, and $\text{Disc}(C_3)$, together with nonnegativity of the occurring monomial weights, gives

$$\omega_0 \geq 0, \quad \omega_Q \geq 0, \quad \omega_C \geq 0.$$

These quantities satisfy the balancing identity

$$\omega_0 + \omega_Q + 2\omega_C = 4(2s + b + c + d) = 0.$$

Hence

$$\omega_0 = \omega_Q = \omega_C = 0.$$

Since $\omega_C = 0$, a nonzero relative weight n would force the binary cubic C_3 to have a multiple root, contradicting $\text{Disc}(C_3) \neq 0$. Thus $n = 0$. Solving the remaining weight equations gives

$$(s, b, c, d) = m(6, 1, -4, -9)$$

for some $m \in \mathbb{Z}$. Therefore

$$\Lambda_{\phi_{30}}^R = \{\lambda_{30}(t^m) \mid m \in \mathbb{Z}\}.$$

This set is symmetric under $m \mapsto -m$. The Casimiro–Florentino criterion therefore shows that $R \cdot \phi_{30}$ is closed in $W^{H_{30}}$. Luna reduction then implies that

$$G \cdot \phi_{30}$$

is closed in W , so $[\phi_{30}]$ is polystable. The same weight calculation shows that the stabilizer for the effective action of R/H_{30} is finite on this open locus.

Normal form

Since C_3 is square-free, the $\mathrm{GL}(U)$ -action normalizes it to

$$C_3(x_0, x_1) = x_0 x_1 (x_0 - x_1).$$

Using the remaining diagonal factors in $C_G(H_{30})$, together with projective rescaling, normalize the coefficient of $x_2^4 x_3$ to 1. Every general limit is then equivalent to a member of

$$\begin{aligned} \phi_{30}^{\mathrm{nf}}(L_1, L_2, Q_1, Q_2) = & x_2^4 x_3 + x_2^2 x_3^2 L_1(x_0, x_1) + x_2^3 x_4 L_2(x_0, x_1) \\ & + x_3^3 Q_1(x_0, x_1) + x_2 x_3 x_4 Q_2(x_0, x_1) + x_0 x_1 (x_0 - x_1) x_4^2, \end{aligned}$$

where

$$L_1, L_2 \in \mathrm{Sym}^1 U, \quad Q_1, Q_2 \in \mathrm{Sym}^2 U, \quad \mathrm{Disc}(Q_1) \neq 0.$$

The displayed coefficient space has dimension

$$2 + 2 + 3 + 3 = 10.$$

One residual diagonal $\mathbb{G}_m$ -action preserves this form; on an affine parameter chart it may be used to normalize one additional nonzero coefficient. Thus the effective normal-form parameter count is

$$10 - 1 = 9.$$

Equivalently, since the effective centralizer stabilizer is finite,

$$\dim \mathbb{P}(W^{H_{30}}) = 14, \quad \dim(C_G(H_{30})/H_{30}) = 5,$$

and therefore

$$\dim \Phi_{30} = 14 - 5 = 9.$$

No condition arising from the later singular-locus or minimal-exponent analysis is imposed at the normal-form stage.

Case $k = 31$

1-PS limit

The witness vector and the corresponding one-parameter subgroup are

$$r_{31} = (7, 2, 2, -3, -8), \quad \lambda_{31}(t) = \mathrm{diag}(t^7, t^2, t^2, t^{-3}, t^{-8}), \quad t \in \mathbb{G}_m.$$

Since the entries of r_{31} sum to zero, λ_{31} is a one-parameter subgroup of $G = \mathrm{SL}(5)$. Let f_{31} be a general quintic supported on

$$S_{31} = I(r_{31})_{\geq 0}.$$

Its affine one-parameter-subgroup limit is the weight-zero part

$$\phi_{31} := \lim_{t \rightarrow 0} \lambda_{31}(t) \cdot f_{31}.$$

Put

$$U := \langle x_1, x_2 \rangle, \quad H_{31} := \lambda_{31}(\mathbb{G}_m).$$

Solving the degree-five weight-zero equations gives

$$\begin{aligned} W^{H_{31}} = & x_4 \mathrm{Sym}^4 U \oplus x_3^2 \mathrm{Sym}^3 U \oplus x_0 x_3^3 U \\ & \oplus x_0 x_3 x_4 \mathrm{Sym}^2 U \oplus \langle x_0^2 x_3^2 x_4 \rangle \oplus x_0^2 x_4^2 U, \end{aligned}$$

and hence

$$\dim W^{H_{31}} = 17.$$

Thus a general limit can be written as

$$\begin{aligned} \phi_{31} = & x_4 B_4(x_1, x_2) + x_3^2 C_3(x_1, x_2) + x_0 x_3^3 L_1(x_1, x_2) \\ & + x_0 x_3 x_4 Q_2(x_1, x_2) + \kappa x_0^2 x_3^2 x_4 + x_0^2 x_4^2 M_1(x_1, x_2), \end{aligned}$$

where

$$B_4 \in \text{Sym}^4 U, \quad C_3 \in \text{Sym}^3 U, \quad L_1, M_1 \in U, \quad Q_2 \in \text{Sym}^2 U.$$

We work on the nonempty Zariski-open locus

$$\kappa \neq 0, \quad \text{Disc}(B_4) \neq 0, \quad \text{Res}(B_4, C_3) \neq 0, \quad M_1 \neq 0.$$

The weight 2 occurs twice on U , so the centralizer is non-toric:

$$C_G(H_{31}) = \{\text{diag}(\mu_0) \oplus A \oplus \text{diag}(\mu_3, \mu_4) \mid A \in \text{GL}(U), \mu_0 \mu_3 \mu_4 \det A = 1\}.$$

Consequently,

$$\dim C_G(H_{31}) = 6, \quad \dim(C_G(H_{31})/H_{31}) = 5.$$

Polystability

Set

$$R := C_G(H_{31}).$$

By affine Luna reduction, it is enough to prove that $R \cdot \phi_{31}$ is closed in $W^{H_{31}}$. Let $\rho \in \Lambda_{\phi_{31}}^R$. After conjugation in the $\text{GL}(U)$ -block, its weights may be written as

$$\text{wt}_\rho(x_0, X, Y, x_3, x_4) = (a, s + n, s - n, b, c), \quad a + 2s + b + c = 0,$$

for a suitable eigenbasis X, Y of U .

Introduce the averaged weights

$$p_B := 4s + c, \quad p_C := 3s + 2b, \quad p_M := 2a + s + 2c, \quad d := 2a + 2b + c.$$

The conditions $\text{Disc}(B_4) \neq 0$ and $\kappa \neq 0$, together with nonnegativity of the occurring monomial weights, give

$$p_B \geq 0, \quad d \geq 0.$$

They satisfy the balancing identity

$$p_B + d = (4s + c) + (2a + 2b + c) = 2(a + 2s + b + c) = 0.$$

Hence

$$p_B = d = 0.$$

Since $p_B = 0$, a nonzero relative weight n would force the binary quartic B_4 to have a multiple root. Thus

$$n = 0.$$

The nonvanishing of $\text{Res}(B_4, C_3)$ now gives

$$3p_B + 4p_C \geq 0,$$

and therefore $p_C \geq 0$. Since $M_1 \neq 0$ and $n = 0$, one also has $p_M \geq 0$. The equalities $p_B = d = 0$ imply

$$p_M = -p_C.$$

Consequently,

$$p_C = p_M = 0.$$

Solving the remaining weight equations gives

$$(a, s, b, c) = m(7, 2, -3, -8)$$

for some $m \in \mathbb{Z}$. Therefore

$$\Lambda_{\phi_{31}}^R = \{\lambda_{31}(t^m) \mid m \in \mathbb{Z}\}.$$

This set is symmetric under $m \mapsto -m$. The Casimiro–Florentino criterion therefore shows that $R \cdot \phi_{31}$ is closed in $W^{H_{31}}$. Luna reduction then implies that

$$G \cdot \phi_{31}$$

is closed in W , so $[\phi_{31}]$ is polystable.

Normal form

Since B_4 is square-free, the $\mathrm{GL}(U)$ -action normalizes it to

$$B_\lambda(u, v) := uv(u - v)(u - \lambda v), \quad \lambda \in \mathbb{C} \setminus \{0, 1\}.$$

Using the remaining diagonal factors in $C_G(H_{31})$, together with projective rescaling, normalize the coefficient of $x_0^2 x_3^2 x_4$ to 1. Every general limit is then equivalent to a member of

$$\begin{aligned} \phi_{31}^{\mathrm{nf}}(\lambda, C_3, L_1, Q_2, M_1) = & x_4 B_\lambda(x_1, x_2) + x_3^2 C_3(x_1, x_2) + x_0 x_3^3 L_1(x_1, x_2) \\ & + x_0 x_3 x_4 Q_2(x_1, x_2) + x_0^2 x_3^2 x_4 + x_0^2 x_4^2 M_1(x_1, x_2), \end{aligned}$$

where

$$C_3 \in \mathrm{Sym}^3 U, \quad L_1, M_1 \in U, \quad Q_2 \in \mathrm{Sym}^2 U,$$

and

$$\mathrm{Res}(B_\lambda, C_3) \neq 0, \quad M_1 \neq 0.$$

The displayed coefficient space has dimension

$$1 + 4 + 2 + 3 + 2 = 12.$$

There remains a one-dimensional diagonal action, represented by

$$x_0 \mapsto \tau^{-1} x_0, \quad x_3 \mapsto \tau x_3,$$

with x_1, x_2, x_4 fixed. It acts by

$$(C_3, L_1, Q_2, M_1) \mapsto (\tau^2 C_3, \tau^2 L_1, Q_2, \tau^{-2} M_1).$$

Thus the effective normal-form parameter count is

$$12 - 1 = 11.$$

Equivalently,

$$\dim \mathbb{P}(W^{H_{31}}) = 16, \quad \dim(C_G(H_{31})/H_{31}) = 5,$$

and therefore

$$\dim \Phi_{31} = 16 - 5 = 11.$$

No condition arising from the later singular-locus or minimal-exponent analysis is imposed at the normal-form stage.

Case $k = 32$

1-PS limit

The witness vector and the corresponding one-parameter subgroup are

$$r_{32} = (3, 3, -2, -2, -2), \quad \lambda_{32}(t) = \text{diag}(t^3, t^3, t^{-2}, t^{-2}, t^{-2}), \quad t \in \mathbb{G}_m.$$

Since the entries of r_{32} sum to zero, λ_{32} is a one-parameter subgroup of $G = \text{SL}(5)$. Let f_{32} be a general quintic supported on

$$S_{32} = I(r_{32})_{\geq 0}.$$

Its affine one-parameter-subgroup limit is the weight-zero part

$$\phi_{32} := \lim_{t \rightarrow 0} \lambda_{32}(t) \cdot f_{32}.$$

Put

$$V := \langle x_0, x_1 \rangle, \quad U := \langle x_2, x_3, x_4 \rangle, \quad H_{32} := \lambda_{32}(\mathbb{G}_m).$$

For a degree-five monomial, the weight-zero equations are equivalent to

$$u_0 + u_1 = 2, \quad u_2 + u_3 + u_4 = 3.$$

Hence

$$W^{H_{32}} = \text{Sym}^2 V \otimes \text{Sym}^3 U, \quad \dim W^{H_{32}} = 3 \cdot 10 = 30,$$

and a general limit has the form

$$\phi_{32} = x_0^2 C_0(x_2, x_3, x_4) + x_0 x_1 C_1(x_2, x_3, x_4) + x_1^2 C_2(x_2, x_3, x_4),$$

where $C_0, C_1, C_2 \in \text{Sym}^3 U$.

The two weights of λ_{32} have multiplicities two and three. Therefore the centralizer is non-toric:

$$C_G(H_{32}) = \{A \oplus B \mid A \in \text{GL}(V), B \in \text{GL}(U), \det A \det B = 1\}.$$

The kernel of the natural map

$$C_G(H_{32}) \longrightarrow \text{PGL}(V) \times \text{PGL}(U)$$

is precisely H_{32} . Thus

$$C_G(H_{32})/H_{32} \simeq \text{PGL}(V) \times \text{PGL}(U), \quad \dim(C_G(H_{32})/H_{32}) = 3 + 8 = 11.$$

The stable locus for the effective residual action is nonempty. Indeed, if ζ is a primitive cube root of unity, then

$$\begin{aligned} \phi_* &= X_0^2(Y_0^3 + Y_1^3 + Y_2^3) + X_0 X_1 Y_0 Y_1 Y_2 \\ &\quad + X_1^2(Y_0^3 + \zeta Y_1^3 + \zeta^2 Y_2^3) \end{aligned}$$

has zero moment map for $\text{SU}(V) \times \text{SU}(U)$, and its projective stabilizer is finite. Hence $[\phi_*]$ is stable by the Kempf–Ness criterion. We therefore take $[\phi_{32}]$ in the nonempty open locus

$$\mathbb{P}(\text{Sym}^2 V \otimes \text{Sym}^3 U)^s.$$

Polystability

Set

$$R := C_G(H_{32}).$$

By affine Luna reduction, it is enough to prove that $R \cdot \phi_{32}$ is closed in $W^{H_{32}}$. Let $\rho \in \Lambda_{\phi_{32}}^R$. After conjugation in the $\mathrm{GL}(V)$ - and $\mathrm{GL}(U)$ -blocks, the weights of ρ may be written as

$$\mathrm{wt}_\rho(X_0, X_1, Y_0, Y_1, Y_2) = (a + s, a - s, b + p, b + q, b - p - q),$$

with determinant-one condition

$$2a + 3b = 0.$$

For a monomial

$$X_0^{2-i} X_1^i Y_0^j Y_1^k Y_2^\ell, \quad i \in \{0, 1, 2\}, \quad j + k + \ell = 3,$$

the scalar part cancels, and its ρ -weight is

$$(2 - 2i)s + jp + kq - \ell(p + q).$$

Thus the existence of the affine limit means that every occurring monomial has nonnegative weight for the residual one-parameter subgroup of $\mathrm{SL}(V) \times \mathrm{SL}(U)$. Since $[\phi_{32}]$ is stable for this action, the Hilbert–Mumford criterion forces

$$s = p = q = 0.$$

Consequently the weights of ρ are

$$(a, a, b, b, b), \quad 2a + 3b = 0.$$

Their integrality gives an integer m such that

$$a = 3m, \quad b = -2m,$$

and therefore

$$\rho(t) = \lambda_{32}(t^m).$$

Conversely, every $\lambda_{32}(t^m)$ fixes $W^{H_{32}}$ pointwise. Hence

$$\Lambda_{\phi_{32}}^R = \{\lambda_{32}(t^m) \mid m \in \mathbb{Z}\}.$$

This set is symmetric under $m \mapsto -m$. The Casimiro–Florentino criterion therefore shows that $R \cdot \phi_{32}$ is closed in $W^{H_{32}}$. Luna reduction then implies that

$$G \cdot \phi_{32}$$

is closed in W , so $[\phi_{32}]$ is polystable.

Normal form

The normal-form family is the stable locus in the fixed representation:

$$\phi_{32}^{\mathrm{nf}}(C_0, C_1, C_2) = x_0^2 C_0(x_2, x_3, x_4) + x_0 x_1 C_1(x_2, x_3, x_4) + x_1^2 C_2(x_2, x_3, x_4),$$

where

$$C_0, C_1, C_2 \in \mathrm{Sym}^3 U, \quad [\phi_{32}^{\mathrm{nf}}(C_0, C_1, C_2)] \in \mathbb{P}(\mathrm{Sym}^2 V \otimes \mathrm{Sym}^3 U)^s.$$

Equivalently, the dense quotient-side normal-form locus is

$$\mathbb{P}(\mathrm{Sym}^2 V \otimes \mathrm{Sym}^3 U)^s / (\mathrm{PGL}(V) \times \mathrm{PGL}(U)).$$

Since stable points have finite stabilizer for the effective action,

$$\dim \mathbb{P}(W^{H_{32}}) = 29, \quad \dim(C_G(H_{32})/H_{32}) = 11,$$

and therefore

$$\dim \Phi_{32} = 29 - 11 = 18.$$

No condition arising from the later singular-locus or minimal-exponent analysis is imposed at the normal-form stage.

Case $k = 33$

1-PS limit

The witness vector is

$$r_{33} = (2, 1, 0, 0, -3),$$

and the associated one-parameter subgroup is

$$\lambda_{33}(t) = \text{diag}(t^2, t, 1, 1, t^{-3}), \quad t \in \mathbb{G}_m.$$

Since the entries of r_{33} sum to zero, λ_{33} is a one-parameter subgroup of $G = \text{SL}(5)$.

Let f_{33} be a general quintic supported on

$$S_{33} = I(r_{33})_{\geq 0} = \{u \in I \mid 2u_0 + u_1 - 3u_4 \geq 0\}.$$

Its affine one-parameter-subgroup limit is the weight-zero part

$$\phi_{33} := \lim_{t \rightarrow 0} \lambda_{33}(t) \cdot f_{33}.$$

The weight-zero equations are

$$2u_0 + u_1 - 3u_4 = 0, \quad u_0 + u_1 + u_2 + u_3 + u_4 = 5,$$

and give

$$(u_0, u_1, u_4) = (0, 0, 0), (0, 3, 1), (1, 1, 1), (3, 0, 2).$$

Writing

$$U := \langle x_2, x_3 \rangle,$$

we obtain

$$\phi_{33} = B_5(x_2, x_3) + x_1^3 x_4 L_1(x_2, x_3) + x_0 x_1 x_4 Q_2(x_2, x_3) + \kappa x_0^3 x_4^2,$$

where

$$B_5 \in \text{Sym}^5 U, \quad L_1 \in U, \quad Q_2 \in \text{Sym}^2 U, \quad \kappa \in \mathbb{C}.$$

We work on the nonempty Zariski-open locus

$$\kappa \neq 0, \quad \text{Disc}(B_5) \neq 0, \quad \text{Disc}(Q_2) \neq 0, \quad \text{Res}(L_1, Q_2) \neq 0.$$

Set $H := H_{33} = \lambda_{33}(\mathbb{G}_m)$. The zero weight has multiplicity two on U , and hence

$$C_G(H) = \{\text{diag}(\mu_0, \mu_1) \oplus A \oplus \text{diag}(\mu_4) \mid A \in \text{GL}(U), \mu_0 \mu_1 \mu_4 \det A = 1\}.$$

Thus $C_G(H)$ is non-toric and

$$\dim C_G(H) = 6.$$

Moreover,

$$W^H = \text{Sym}^5 U \oplus x_1^3 x_4 U \oplus x_0 x_1 x_4 \text{Sym}^2 U \oplus \langle x_0^3 x_4^2 \rangle, \quad \dim W^H = 12.$$

Since H acts trivially on W^H ,

$$\dim(C_G(H)/H) = 5.$$

Polystability

Put $R := C_G(H)$. By Luna's centralizer reduction, it is enough to prove that $R \cdot \phi_{33}$ is closed in W^H .

Let $\lambda \in \Lambda_{\phi_{33}}^R$. After conjugation in R , we may assume that λ is diagonal on the $\mathrm{GL}(U)$ -block. In a corresponding eigenbasis X, Y of U , write its weights as

$$\mathrm{wt}_\lambda(x_0, x_1, X, Y, x_4) = (a, b, s + n, s - n, c),$$

so that

$$S := a + b + 2s + c = 0.$$

Set

$$\ell := 3b + s + c, \quad d := 3a + 2c.$$

The nonvanishing of $\mathrm{Disc}(B_5)$ implies $s \geq 0$: every monomial in the discriminant has total weight $40s$. Similarly, $\mathrm{Res}(L_1, Q_2) \neq 0$ gives

$$2\ell + S \geq 0,$$

and hence $\ell \geq 0$. Finally, the occurring monomial $\kappa x_0^3 x_4^2$ gives $d \geq 0$. These quantities satisfy

$$5s + \ell + d = 3S = 0.$$

Consequently,

$$s = \ell = d = 0.$$

It remains to eliminate the relative weight n . Since $s = 0$, the six coefficient weights of the binary quintic are

$$(5 - 2j)n, \quad j = 0, \dots, 5.$$

If $n > 0$, then B_5 is divisible by X^3 ; if $n < 0$, it is divisible by Y^3 . Either case contradicts $\mathrm{Disc}(B_5) \neq 0$. Hence $n = 0$.

Solving

$$S = 0, \quad \ell = 3b + c = 0, \quad d = 3a + 2c = 0$$

gives

$$(a, b, s, c) = m(2, 1, 0, -3) \quad (m \in \mathbb{Z}).$$

Therefore

$$\Lambda_{\phi_{33}}^R = \{\lambda_{33}(t^m) \mid m \in \mathbb{Z}\}.$$

This set is symmetric in the sense of the Casimiro–Florentino criterion [CF12]. Hence $R \cdot \phi_{33}$ is closed in W^H , and Luna's reduction gives that $G \cdot \phi_{33}$ is closed in W . Thus $[\phi_{33}]$ is polystable.

Normal form and the quotient parameters

Because Q_2 has two distinct roots and the root of L_1 is disjoint from them, the divisor of $L_1 Q_2$ consists of three distinct points of $\mathbb{P}(U)$. Using the $\mathrm{PGL}(U)$ -action and the remaining block scalings, we may normalize

$$L_1 = x_2, \quad Q_2 = x_2^2 + x_3^2.$$

Using the remaining diagonal and projective scalings, we also normalize $\kappa = 1$. A convenient normal form is therefore

$$\phi_{33}^{\mathrm{nf}}(B_5) = B_5(x_2, x_3) + x_1^3 x_2 x_4 + x_0 x_1 x_4 (x_2^2 + x_3^2) + x_0^3 x_4^2,$$

where

$$B_5 \in \mathrm{Sym}^5 U, \quad \mathrm{Disc}(B_5) \neq 0.$$

The stabilizer in $\mathrm{PGL}(U)$ of the normalized three-point divisor is finite, so this normal form has no residual continuous redundancy. Since

$$\dim \mathbb{P}(W^H) = 11, \quad \dim(C_G(H)/H) = 5,$$

the corresponding quotient-side family has dimension

$$\dim \Phi_{33} = 11 - 5 = 6.$$

Case $k = 34$

1-PS limit

The witness vector and the corresponding one-parameter subgroup are

$$r_{34} = (8, 3, 3, -2, -12), \quad \lambda_{34}(t) = \mathrm{diag}(t^8, t^3, t^3, t^{-2}, t^{-12}), \quad t \in \mathbb{G}_m.$$

Since the entries of r_{34} sum to zero, λ_{34} is a one-parameter subgroup of $G = \mathrm{SL}(5)$. Let f_{34} be a general quintic supported on

$$S_{34} = I(r_{34})_{\geq 0}.$$

Its affine one-parameter-subgroup limit is the weight-zero part

$$\phi_{34} := \lim_{t \rightarrow 0} \lambda_{34}(t) \cdot f_{34}.$$

Put

$$U := \langle x_1, x_2 \rangle, \quad H_{34} := \lambda_{34}(\mathbb{G}_m).$$

Solving the degree-five weight-zero equations gives

$$\begin{aligned} W^{H_{34}} &= x_4 \mathrm{Sym}^4 U \oplus x_3^3 \mathrm{Sym}^2 U \oplus \langle x_0 x_3^4 \rangle \\ &\quad \oplus x_0 x_3 x_4 \mathrm{Sym}^2 U \oplus \langle x_0^2 x_3^2 x_4 \rangle \oplus \langle x_0^3 x_4^2 \rangle, \end{aligned}$$

and hence

$$\dim W^{H_{34}} = 14.$$

Thus a general limit can be written as

$$\begin{aligned} \phi_{34} &= x_4 B_4(x_1, x_2) + x_3^3 Q_2(x_1, x_2) + \kappa x_0 x_3^4 \\ &\quad + x_0 x_3 x_4 R_2(x_1, x_2) + \delta x_0^2 x_3^2 x_4 + \rho x_0^3 x_4^2, \end{aligned}$$

where

$$B_4 \in \mathrm{Sym}^4 U, \quad Q_2, R_2 \in \mathrm{Sym}^2 U.$$

We work on the nonempty Zariski-open locus

$$\kappa \rho \neq 0, \quad \mathrm{Disc}(B_4) \neq 0, \quad \mathrm{Res}(B_4, Q_2) \neq 0.$$

The weight 3 occurs twice on U , so the centralizer is non-toric:

$$C_G(H_{34}) = \{ \mathrm{diag}(\mu_0) \oplus A \oplus \mathrm{diag}(\mu_3, \mu_4) \mid A \in \mathrm{GL}(U), \mu_0 \mu_3 \mu_4 \det A = 1 \}.$$

Consequently,

$$\dim C_G(H_{34}) = 6, \quad \dim(C_G(H_{34})/H_{34}) = 5.$$

Polystability

Set

$$R := C_G(H_{34}).$$

By affine Luna reduction, it is enough to prove that $R \cdot \phi_{34}$ is closed in $W^{H_{34}}$. Let $\sigma \in \Lambda_{\phi_{34}}^R$. After conjugation in the $\mathrm{GL}(U)$ -block, its weights may be written as

$$\mathrm{wt}_\sigma(x_0, X, Y, x_3, x_4) = (a, s+n, s-n, b, c), \quad a+2s+b+c=0,$$

for a suitable eigenbasis X, Y of U .

Introduce the averaged weights

$$p_B := 4s+c, \quad p_Q := 2s+3b, \quad p_\kappa := a+4b, \quad p_\rho := 3a+2c.$$

The conditions

$$\mathrm{Disc}(B_4) \neq 0, \quad \mathrm{Res}(B_4, Q_2) \neq 0, \quad \kappa\rho \neq 0$$

and nonnegativity of the occurring monomial weights give

$$p_B \geq 0, \quad p_B + 2p_Q \geq 0, \quad p_\kappa \geq 0, \quad p_\rho \geq 0.$$

These quantities satisfy the balancing identity

$$3p_B + (p_B + 2p_Q) + p_\kappa + 3p_\rho = 10(a+2s+b+c) = 0.$$

Hence

$$p_B = p_Q = p_\kappa = p_\rho = 0.$$

Since $p_B = 0$, a nonzero relative weight n would force the binary quartic B_4 to have a multiple root. Therefore

$$n = 0.$$

Solving the remaining weight equations gives

$$(a, s, b, c) = m(8, 3, -2, -12)$$

for some $m \in \mathbb{Z}$. Consequently,

$$\Lambda_{\phi_{34}}^R = \{\lambda_{34}(t^m) \mid m \in \mathbb{Z}\}.$$

This set is symmetric under $m \mapsto -m$. The Casimiro–Florentino criterion therefore shows that $R \cdot \phi_{34}$ is closed in $W^{H_{34}}$. Luna reduction then implies that

$$G \cdot \phi_{34}$$

is closed in W , so $[\phi_{34}]$ is polystable.

Normal form

Since B_4 is square-free, the $\mathrm{GL}(U)$ -action normalizes it to

$$B_\lambda(u, v) := uv(u-v)(u-\lambda v), \quad \lambda \in \mathbb{C} \setminus \{0, 1\}.$$

Using the remaining diagonal factors in $C_G(H_{34})$, together with projective rescaling, normalize the coefficients of

$$x_0x_3^4, \quad x_0^3x_4^2$$

to 1. Every general limit is then equivalent to a member of

$$\begin{aligned}\phi_{34}^{\text{nf}}(\lambda, Q_2, R_2, \beta) = & x_4 B_\lambda(x_1, x_2) + x_3^3 Q_2(x_1, x_2) + x_0 x_3^4 \\ & + x_0 x_3 x_4 R_2(x_1, x_2) + \beta x_0^2 x_3^2 x_4 + x_0^3 x_4^2,\end{aligned}$$

where

$$Q_2, R_2 \in \text{Sym}^2 U, \quad \beta \in \mathbb{C}, \quad \text{Res}(B_\lambda, Q_2) \neq 0.$$

The cross-ratio λ is defined up to the finite anharmonic action, which does not affect the dimension. The displayed parameter count is

$$1 + 3 + 3 + 1 = 8.$$

Equivalently,

$$\dim \mathbb{P}(W^{H_{34}}) = 13, \quad \dim(C_G(H_{34})/H_{34}) = 5,$$

and therefore

$$\dim \Phi_{34} = 13 - 5 = 8.$$

No condition arising from the later singular-locus or minimal-exponent analysis is imposed at the normal-form stage.

Case $k = 35$

1-PS limit

The witness vector and the corresponding one-parameter subgroup are

$$r_{35} = (4, 4, -1, -1, -6), \quad \lambda_{35}(t) = \text{diag}(t^4, t^4, t^{-1}, t^{-1}, t^{-6}), \quad t \in \mathbb{G}_m.$$

Since the entries of r_{35} sum to zero, λ_{35} is a one-parameter subgroup of $G = \text{SL}(5)$. Let f_{35} be a general quintic supported on

$$S_{35} = I(r_{35})_{\geq 0}.$$

Its affine one-parameter-subgroup limit is the weight-zero part

$$\phi_{35} := \lim_{t \rightarrow 0} \lambda_{35}(t) \cdot f_{35}.$$

Put

$$V := \langle x_0, x_1 \rangle, \quad U := \langle x_2, x_3 \rangle, \quad H_{35} := \lambda_{35}(\mathbb{G}_m).$$

For a degree-five monomial, set

$$a = u_0 + u_1, \quad b = u_2 + u_3, \quad c = u_4.$$

The degree and weight-zero equations give

$$a = c + 1, \quad b = 4 - 2c, \quad c = 0, 1, 2.$$

Consequently,

$$W^{H_{35}} = V \otimes \text{Sym}^4 U \oplus x_4 \text{Sym}^2 V \otimes \text{Sym}^2 U \oplus x_4^2 \text{Sym}^3 V,$$

and hence

$$\dim W^{H_{35}} = 23.$$

Thus a general limit has the form

$$\begin{aligned}\phi_{35} = & x_0 A_4(x_2, x_3) + x_1 C_4(x_2, x_3) \\ & + x_4 Q_{2,2}(x_0, x_1; x_2, x_3) + x_4^2 C_3(x_0, x_1),\end{aligned}$$

where

$$A_4, C_4 \in \text{Sym}^4 U, \quad Q_{2,2} \in \text{Sym}^2 V \otimes \text{Sym}^2 U, \quad C_3 \in \text{Sym}^3 V.$$

We work on the nonempty Zariski-open locus

$$\text{Res}(A_4, C_4) \neq 0, \quad \text{Disc}(C_3) \neq 0.$$

The first condition says that the pencil spanned by A_4 and C_4 is base-point-free on $\mathbb{P}(U)$, while the second says that C_3 is square-free.

The weights 4 and -1 each occur with multiplicity two. Therefore the centralizer is non-toric:

$$C_G(H_{35}) = \{A \oplus B \oplus \text{diag}(\mu_4) \mid A \in \text{GL}(V), B \in \text{GL}(U), \mu_4 \det A \det B = 1\}.$$

Consequently,

$$\dim C_G(H_{35}) = 8, \quad \dim(C_G(H_{35})/H_{35}) = 7.$$

Polystability

Set

$$R := C_G(H_{35}).$$

By affine Luna reduction, it is enough to prove that $R \cdot \phi_{35}$ is closed in $W^{H_{35}}$. Let $\rho \in \Lambda_{\phi_{35}}^R$. After conjugation in the two non-abelian blocks, the weights of ρ may be written as

$$\text{wt}_\rho(X, Y, Z, W, x_4) = (s + n, s - n, t + v, t - v, a),$$

where X, Y and Z, W are eigenbases of V and U , and

$$2s + 2t + a = 0.$$

Define the averaged block weights

$$p_F := s + 4t, \quad p_B := 3s + 2a.$$

The nonvanishing of $\text{Res}(A_4, C_4)$, together with nonnegativity of the occurring monomial weights, gives

$$p_F \geq 0.$$

Similarly, $\text{Disc}(C_3) \neq 0$ gives

$$p_B \geq 0.$$

These quantities satisfy

$$p_F + p_B = (s + 4t) + (3s + 2a) = 2(2s + 2t + a) = 0.$$

Hence

$$p_F = p_B = 0.$$

Since $p_B = 0$, a nonzero relative weight n would force the binary cubic C_3 to have a multiple root. Thus $n = 0$. Since $p_F = 0$, a nonzero relative weight v would force both A_4 and C_4 to be divisible by the same square of a linear form, contradicting $\text{Res}(A_4, C_4) \neq 0$. Therefore $v = 0$.

Solving the remaining equations gives

$$(s, t, a) = m(4, -1, -6)$$

for some $m \in \mathbb{Z}$. Consequently,

$$\Lambda_{\phi_{35}}^R = \{\lambda_{35}(t^m) \mid m \in \mathbb{Z}\}.$$

This set is symmetric under $m \mapsto -m$. The Casimiro–Florentino criterion therefore shows that $R \cdot \phi_{35}$ is closed in $W^{H_{35}}$. Luna reduction then implies that

$$G \cdot \phi_{35}$$

is closed in W , so $[\phi_{35}]$ is polystable. The same weight calculation shows that the identity component of the stabilizer in R is H_{35} ; hence the effective stabilizer is finite on this open locus.

Normal form

Since C_3 is square-free, the $\mathrm{GL}(V)$ -action normalizes it to

$$B_3(x_0, x_1) := x_0 x_1 (x_0 - x_1).$$

Thus every general limit is equivalent to a member of

$$\begin{aligned} \phi_{35}^{\mathrm{nf}}(A_4, C_4, Q_{2,2}) &= x_0 A_4(x_2, x_3) + x_1 C_4(x_2, x_3) \\ &\quad + x_4 Q_{2,2}(x_0, x_1; x_2, x_3) + x_4^2 x_0 x_1 (x_0 - x_1), \end{aligned}$$

where

$$A_4, C_4 \in \mathrm{Sym}^4 U, \quad Q_{2,2} \in \mathrm{Sym}^2 V \otimes \mathrm{Sym}^2 U, \quad \mathrm{Res}(A_4, C_4) \neq 0.$$

The residual continuous action is the $\mathrm{GL}(U)$ -action on (x_2, x_3) ; the stabilizer of $x_0 x_1 (x_0 - x_1)$ in the V -variables is finite. Accordingly, the normal-form parameter space has dimension

$$5 + 5 + 9 - \dim \mathrm{GL}(U) = 19 - 4 = 15.$$

Equivalently,

$$\dim \mathbb{P}(W^{H_{35}}) = 22, \quad \dim(C_G(H_{35})/H_{35}) = 7,$$

and therefore

$$\dim \Phi_{35} = 22 - 7 = 15.$$

No condition arising from the later singular-locus or minimal-exponent analysis is imposed at the normal-form stage.

Case $k = 36$

1-PS limit

The witness vector and the corresponding one-parameter subgroup are

$$r_{36} = (1, 0, 0, 0, -1), \quad \lambda_{36}(t) = \mathrm{diag}(t, 1, 1, 1, t^{-1}), \quad t \in \mathbb{G}_m.$$

Since the entries of r_{36} sum to zero, λ_{36} is a one-parameter subgroup of $G = \mathrm{SL}(5)$. Let f_{36} be a general quintic supported on

$$S_{36} = I(r_{36})_{\geq 0}.$$

Its affine one-parameter-subgroup limit is the weight-zero part

$$\phi_{36} := \lim_{t \rightarrow 0} \lambda_{36}(t) \cdot f_{36}.$$

For a degree-five monomial x^u , the weight-zero equations are

$$u_0 = u_4, \quad 2u_0 + u_1 + u_2 + u_3 = 5.$$

Hence $u_0 = u_4 \in \{0, 1, 2\}$. Put

$$U := \langle x_1, x_2, x_3 \rangle, \quad H_{36} := \lambda_{36}(\mathbb{G}_m).$$

Then

$$W^{H_{36}} = \mathrm{Sym}^5 U \oplus x_0 x_4 \mathrm{Sym}^3 U \oplus x_0^2 x_4^2 U,$$

and therefore

$$\dim W^{H_{36}} = 21 + 10 + 3 = 34.$$

Thus a general limit has the form

$$\phi_{36} = B_5(x_1, x_2, x_3) + x_0 x_4 C_3(x_1, x_2, x_3) + x_0^2 x_4^2 L_1(x_1, x_2, x_3),$$

where

$$B_5 \in \text{Sym}^5 U, \quad C_3 \in \text{Sym}^3 U, \quad L_1 \in U.$$

We work on the nonempty Zariski-open locus

$$B_5 \neq 0, \quad L_1 \neq 0, \quad \text{Disc}(C_3) \neq 0.$$

In particular, $V(C_3) \subset \mathbb{P}(U) \simeq \mathbb{P}^2$ is a smooth plane cubic.

The weight zero occurs with multiplicity three on U . Hence the centralizer is non-toric:

$$C_G(H_{36}) = \{\text{diag}(\mu_0) \oplus A \oplus \text{diag}(\mu_4) \mid A \in \text{GL}(U), \mu_0 \mu_4 \det A = 1\}.$$

Consequently,

$$\dim C_G(H_{36}) = 10, \quad \dim(C_G(H_{36})/H_{36}) = 9.$$

Polystability

Set

$$R := C_G(H_{36}).$$

By affine Luna reduction, it is enough to prove that $R \cdot \phi_{36}$ is closed in $W^{H_{36}}$. Let $\rho \in \Lambda_{\phi_{36}}^R$. After conjugation in the $\text{GL}(U)$ -block, its weights may be written as

$$\text{wt}_\rho(x_0, X_1, X_2, X_3, x_4) = (a, s + p, s + q, s - p - q, b),$$

where X_1, X_2, X_3 is an eigenbasis of U , and

$$a + 3s + b = 0.$$

Since $\text{Disc}(C_3) \neq 0$, the cubic $[C_3]$ is stable for the $\text{SL}(U)$ -action. The Hilbert–Mumford criterion therefore forces the residual $\text{SL}(U)$ -weights to vanish:

$$p = q = 0.$$

All monomials in the nonzero quintic block B_5 then have weight $5s$, so

$$5s \geq 0.$$

All monomials in the nonzero block $x_0^2 x_4^2 L_1$ have weight

$$2a + 2b + s = -5s,$$

and hence

$$-5s \geq 0.$$

Thus

$$s = 0, \quad a + b = 0.$$

It follows that, for some $m \in \mathbb{Z}$,

$$\rho(t) = \text{diag}(t^m, 1, 1, 1, t^{-m}) = \lambda_{36}(t^m).$$

Consequently,

$$\Lambda_{\phi_{36}}^R = \{\lambda_{36}(t^m) \mid m \in \mathbb{Z}\}.$$

This set is symmetric under $m \mapsto -m$. The Casimiro–Florentino criterion therefore shows that $R \cdot \phi_{36}$ is closed in $W^{H_{36}}$. Luna reduction then implies that

$$G \cdot \phi_{36}$$

is closed in W , and hence $[\phi_{36}]$ is polystable.

Normal form

Since $L_1 \neq 0$, the $\mathrm{GL}(U)$ -action allows us to choose coordinates such that

$$L_1 = x_3.$$

Using the remaining diagonal scalings and the overall projective scaling, normalize the coefficient of $x_0^2 x_3 x_4^2$ to 1. A convenient generic normal form is

$$\phi_{36}^{\mathrm{nf}}(B_5, C_3) = B_5(x_1, x_2, x_3) + x_0 x_4 C_3(x_1, x_2, x_3) + x_0^2 x_3 x_4^2,$$

where

$$B_5 \in \mathrm{Sym}^5 U, \quad C_3 \in \mathrm{Sym}^3 U, \quad B_5 \neq 0, \quad \mathrm{Disc}(C_3) \neq 0.$$

After fixing $L_1 = x_3$, the pair (B_5, C_3) is understood modulo the residual subgroup preserving $\langle x_3 \rangle$ in $C_G(H_{36})/H_{36}$. Since

$$\dim \mathbb{P}(W^{H_{36}}) = 33, \quad \dim(C_G(H_{36})/H_{36}) = 9,$$

the corresponding quotient-side family has dimension

$$\dim \Phi_{36} = 33 - 9 = 24.$$

No condition arising from the later singular-locus or minimal-exponent analysis is imposed at the normal-form stage.

Case $k = 37$

1-PS limit

The witness vector and the corresponding one-parameter subgroup are

$$r_{37} = (2, 2, 2, -3, -3), \quad \lambda_{37}(t) = \mathrm{diag}(t^2, t^2, t^2, t^{-3}, t^{-3}), \quad t \in \mathbb{G}_m.$$

Since the entries of r_{37} sum to zero, λ_{37} is a one-parameter subgroup of $G = \mathrm{SL}(5)$. Let f_{37} be a general quintic supported on

$$S_{37} = I(r_{37})_{\geq 0}.$$

Its affine one-parameter-subgroup limit is the weight-zero part

$$\phi_{37} := \lim_{t \rightarrow 0} \lambda_{37}(t) \cdot f_{37}.$$

For a degree-five monomial x^u , the weight-zero equations are equivalent to

$$u_0 + u_1 + u_2 = 3, \quad u_3 + u_4 = 2.$$

Put

$$U := \langle x_0, x_1, x_2 \rangle, \quad V := \langle x_3, x_4 \rangle, \quad H_{37} := \lambda_{37}(\mathbb{G}_m).$$

Hence

$$W^{H_{37}} = \mathrm{Sym}^3 U \otimes \mathrm{Sym}^2 V, \quad \dim W^{H_{37}} = 10 \cdot 3 = 30,$$

and a general limit has the form

$$\phi_{37} = x_3^2 D_0(x_0, x_1, x_2) + x_3 x_4 D_1(x_0, x_1, x_2) + x_4^2 D_2(x_0, x_1, x_2),$$

where

$$D_0, D_1, D_2 \in \mathrm{Sym}^3 U.$$

The two weights of λ_{37} have multiplicities three and two. Therefore the centralizer is non-toric:

$$C_G(H_{37}) = \{A \oplus B \mid A \in \mathrm{GL}(U), B \in \mathrm{GL}(V), \det A \det B = 1\}.$$

The kernel of the natural map

$$C_G(H_{37}) \longrightarrow \mathrm{PGL}(U) \times \mathrm{PGL}(V)$$

is precisely H_{37} . Thus

$$C_G(H_{37})/H_{37} \simeq \mathrm{PGL}(U) \times \mathrm{PGL}(V), \quad \dim(C_G(H_{37})/H_{37}) = 8 + 3 = 11.$$

We work on the nonempty Zariski-open locus

$$[\phi_{37}] \in \mathbb{P}(\mathrm{Sym}^3 U \otimes \mathrm{Sym}^2 V)^s$$

of points stable for the natural $\mathrm{SL}(U) \times \mathrm{SL}(V)$ -action.

Polystability

Set

$$R := C_G(H_{37}).$$

By affine Luna reduction, it is enough to prove that $R \cdot \phi_{37}$ is closed in $W^{H_{37}}$. Let $\rho \in \Lambda_{\phi_{37}}^R$. After conjugation in the $\mathrm{GL}(U)$ - and $\mathrm{GL}(V)$ -blocks, the weights of ρ may be written as

$$\mathrm{wt}_\rho(X_0, X_1, X_2, Y_0, Y_1) = (a + p, a + q, a - p - q, b + s, b - s),$$

with determinant-one condition

$$3a + 2b = 0.$$

For a monomial

$$X_0^{i_0} X_1^{i_1} X_2^{i_2} Y_0^{2-j} Y_1^j, \quad i_0 + i_1 + i_2 = 3, \quad j \in \{0, 1, 2\},$$

the scalar part cancels, and its ρ -weight is

$$i_0 p + i_1 q - i_2(p + q) + (2 - 2j)s.$$

Thus the existence of the affine limit means that every occurring monomial has nonnegative weight for the residual one-parameter subgroup of $\mathrm{SL}(U) \times \mathrm{SL}(V)$. Since $[\phi_{37}]$ is stable for this action, the Hilbert–Mumford criterion forces

$$p = q = s = 0.$$

Consequently the weights of ρ are

$$(a, a, a, b, b), \quad 3a + 2b = 0.$$

Their integrality gives an integer m such that

$$a = 2m, \quad b = -3m,$$

and therefore

$$\rho(t) = \lambda_{37}(t^m).$$

Conversely, every $\lambda_{37}(t^m)$ fixes $W^{H_{37}}$ pointwise. Hence

$$\Lambda_{\phi_{37}}^R = \{\lambda_{37}(t^m) \mid m \in \mathbb{Z}\}.$$

This set is symmetric under $m \mapsto -m$. The Casimiro–Florentino criterion therefore shows that $R \cdot \phi_{37}$ is closed in $W^{H_{37}}$. Luna reduction then implies that

$$G \cdot \phi_{37}$$

is closed in W , and hence $[\phi_{37}]$ is polystable.

Normal form

The normal-form family is the stable locus in the fixed representation:

$$\phi_{37}^{\text{nf}}(D_0, D_1, D_2) = x_3^2 D_0(x_0, x_1, x_2) + x_3 x_4 D_1(x_0, x_1, x_2) + x_4^2 D_2(x_0, x_1, x_2),$$

where

$$D_0, D_1, D_2 \in \text{Sym}^3 U, \quad [\phi_{37}^{\text{nf}}(D_0, D_1, D_2)] \in \mathbb{P}(\text{Sym}^3 U \otimes \text{Sym}^2 V)^s.$$

Equivalently, the dense quotient-side normal-form locus is

$$\mathbb{P}(\text{Sym}^3 U \otimes \text{Sym}^2 V)^s / (\text{PGL}(U) \times \text{PGL}(V)).$$

Since stable points have finite stabilizer for the effective action,

$$\dim \mathbb{P}(W^{H_{37}}) = 29, \quad \dim(C_G(H_{37})/H_{37}) = 11,$$

and therefore

$$\dim \Phi_{37} = 29 - 11 = 18.$$

No condition arising from the later singular-locus or minimal-exponent analysis is imposed at the normal-form stage.

Case $k = 38$

1-PS limit

The witness vector and the corresponding one-parameter subgroup are

$$r_{38} = (1, 1, 1, 1, -4), \quad \lambda_{38}(t) = \text{diag}(t, t, t, t, t^{-4}), \quad t \in \mathbb{G}_m.$$

Since the entries of r_{38} sum to zero, λ_{38} is a one-parameter subgroup of $G = \text{SL}(5)$. Let f_{38} be a general quintic supported on

$$S_{38} = I(r_{38})_{\geq 0}.$$

Its affine one-parameter-subgroup limit is the weight-zero part

$$\phi_{38} := \lim_{t \rightarrow 0} \lambda_{38}(t) \cdot f_{38}.$$

For a degree-five monomial x^u , one has

$$r_{38} \cdot u = u_0 + u_1 + u_2 + u_3 - 4u_4 = 5 - 5u_4.$$

Hence the weight-zero monomials are exactly those with $u_4 = 1$. Put

$$U := \langle x_0, x_1, x_2, x_3 \rangle, \quad H_{38} := \lambda_{38}(\mathbb{G}_m).$$

Then

$$W^{H_{38}} = x_4 \text{Sym}^4 U, \quad \dim W^{H_{38}} = 35,$$

and a general limit has the form

$$\phi_{38} = x_4 B_4(x_0, x_1, x_2, x_3), \quad B_4 \in \text{Sym}^4 U.$$

The weight 1 of λ_{38} has multiplicity four on U . Therefore the centralizer is non-toric:

$$C_G(H_{38}) = \{A \oplus \text{diag}(\mu_4) \mid A \in \text{GL}(U), \mu_4 \det A = 1\}.$$

The kernel of the natural map

$$C_G(H_{38}) \longrightarrow \text{PGL}(U)$$

is precisely H_{38} . Thus

$$C_G(H_{38})/H_{38} \simeq \text{PGL}(U), \quad \dim(C_G(H_{38})/H_{38}) = 15.$$

We work on the nonempty Zariski-open locus

$$[B_4] \in \mathbb{P}(\text{Sym}^4 U)^s$$

of quartics stable for the natural $\text{SL}(U)$ -action.

Polystability

Set

$$R := C_G(H_{38}).$$

By affine Luna reduction, it is enough to prove that $R \cdot \phi_{38}$ is closed in $W^{H_{38}}$. Let $\rho \in \Lambda_{\phi_{38}}^R$. After conjugation in the $\mathrm{GL}(U)$ -block, the weights of ρ may be written as

$$\mathrm{wt}_\rho(X_0, X_1, X_2, X_3, x_4) = (b_0, b_1, b_2, b_3, a),$$

with determinant-one condition

$$b_0 + b_1 + b_2 + b_3 + a = 0.$$

Define the residual weights

$$\tilde{b}_i := b_i + \frac{a}{4} \quad (i = 0, 1, 2, 3).$$

Then

$$\tilde{b}_0 + \tilde{b}_1 + \tilde{b}_2 + \tilde{b}_3 = 0.$$

For a monomial

$$x_4 X_0^{u_0} X_1^{u_1} X_2^{u_2} X_3^{u_3}, \quad u_0 + u_1 + u_2 + u_3 = 4,$$

its ρ -weight is

$$u_0 \tilde{b}_0 + u_1 \tilde{b}_1 + u_2 \tilde{b}_2 + u_3 \tilde{b}_3.$$

Thus the existence of the affine limit means that every occurring monomial of B_4 has nonnegative weight for the residual one-parameter subgroup of $\mathrm{SL}(U)$. Since $[B_4]$ is stable, the Hilbert–Mumford criterion forces

$$\tilde{b}_0 = \tilde{b}_1 = \tilde{b}_2 = \tilde{b}_3 = 0.$$

Consequently

$$b_0 = b_1 = b_2 = b_3 = -\frac{a}{4}.$$

Integrality gives an integer m such that

$$b_0 = b_1 = b_2 = b_3 = m, \quad a = -4m,$$

and hence

$$\rho(t) = \lambda_{38}(t^m).$$

Conversely, every $\lambda_{38}(t^m)$ fixes $W^{H_{38}}$ pointwise. Therefore

$$\Lambda_{\phi_{38}}^R = \{\lambda_{38}(t^m) \mid m \in \mathbb{Z}\}.$$

This set is symmetric under $m \mapsto -m$. The Casimiro–Florentino criterion therefore shows that $R \cdot \phi_{38}$ is closed in $W^{H_{38}}$. Luna reduction then implies that

$$G \cdot \phi_{38}$$

is closed in W , and hence $[\phi_{38}]$ is polystable.

Normal form

The normal-form family is

$$\phi_{38}^{\text{nf}}(B_4) = x_4 B_4(x_0, x_1, x_2, x_3),$$

where

$$[B_4] \in \mathbb{P}(\text{Sym}^4 U)^s.$$

Equivalently, the dense quotient-side normal-form locus is

$$\mathbb{P}(\text{Sym}^4 U)^s / \text{PGL}(U).$$

Since stable quartics have finite stabilizer for the effective action,

$$\dim \mathbb{P}(W^{H_{38}}) = 34, \quad \dim(C_G(H_{38})/H_{38}) = 15,$$

and therefore

$$\dim \Phi_{38} = 34 - 15 = 19.$$

No condition arising from the later singular-locus or minimal-exponent analysis is imposed at the normal-form stage.

B Component-by-component singularity and minimal-exponent calculations

This appendix proves the component-indexed catalog stated in Section 8. For each $a = 1, \dots, 21$, let $k(a)$ be the representative fixed in Table 7, let

$$F_a := \phi_{k(a)}^{\text{nf}}, \quad X_a := V(F_a) \subset \mathbb{P}^4,$$

and work on the corresponding nonempty Zariski-open generic parameter locus. The calculation is performed once for each quotient component, rather than once for each of the thirty-eight oriented supports. For the partner k' of a chosen label $k(a)$, coordinate reversal identifies the corresponding zero-weight spaces and the two labels have the same quotient image. By Remark 5.3, their general closed orbits are therefore projectively equivalent, although the two displayed normal-form formulas need not be related by ρ without further residual normalization and reparametrization. The saturated Jacobian scheme, local analytic equivalence class, and minimal exponent are invariant under that projective equivalence.

For every a , we determine

$$\text{Sing}(X_a) = V(J(F_a)), \quad J(F_a) := \left(\frac{\partial F_a}{\partial x_0}, \dots, \frac{\partial F_a}{\partial x_4} \right) : (x_0, \dots, x_4)^\infty,$$

decompose its support into irreducible strata, identify the relevant local analytic models, and compute

$$\tilde{\alpha}(X_a) = \min_{p \in \text{Sing}(X_a)} \tilde{\alpha}_p(X_a).$$

The outcome in every component is

$$\tilde{\alpha}(X_a) = 1 = \frac{4+1}{5}.$$

B.1 The component Θ_1

By Table 7,

$$\Theta_1 = \Phi_1 = \Phi_{26}, \quad k(1) = 1.$$

A general closed orbit in Θ_1 is represented by

$$F_1(\alpha) := \phi_1^{\text{nf}}(\alpha) = x_1^4 x_2 + x_0 x_3^4 + x_0 x_2^3 x_4 + x_0 x_1 x_2 x_3 x_4 + \alpha x_0^2 x_4^3, \quad \alpha \in \mathbb{C}^\times,$$

and we set

$$X_1(\alpha) := V(F_1(\alpha)) \subset \mathbb{P}^4.$$

Singular locus. For readability, write $F = F_1(\alpha)$. Its partial derivatives are

$$\begin{aligned} \frac{\partial F}{\partial x_0} &= x_3^4 + x_2^3 x_4 + x_1 x_2 x_3 x_4 + 2\alpha x_0 x_4^3, \\ \frac{\partial F}{\partial x_1} &= x_2(x_0 x_3 x_4 + 4x_1^3), \\ \frac{\partial F}{\partial x_2} &= x_1^4 + x_0 x_1 x_3 x_4 + 3x_0 x_2^2 x_4, \\ \frac{\partial F}{\partial x_3} &= x_0(x_1 x_2 x_4 + 4x_3^3), \\ \frac{\partial F}{\partial x_4} &= x_0(x_1 x_2 x_3 + x_2^3 + 3\alpha x_0 x_4^2). \end{aligned}$$

On the hyperplane $x_0 = 0$, the equation $\partial F / \partial x_2 = 0$ gives $x_1 = 0$ on the reduced support, and we obtain the plane quartic

$$C := V(x_0, x_1, x_3^4 + x_2^3 x_4).$$

Its normalization is

$$[s : t] \mapsto [0 : 0 : s^4 : s^3 t : -t^4],$$

so C is irreducible and rational. Its unique singular point is

$$Q = (0 : 0 : 0 : 0 : 1),$$

and, in the chart $x_4 = 1$, the curve germ (C, Q) has equation $y^3 + z^4 = 0$. Thus Q is a $(3, 4)$ -cusp.

On the chart $x_0 = 1$, write

$$(u, v, z, w) = (x_1, x_2, x_3, x_4).$$

If $v = 0$, the Jacobian equations give $w = z = u = 0$, yielding

$$P = (1 : 0 : 0 : 0 : 0).$$

If $v \neq 0$, they imply

$$zw = -4u^3, \quad v^2 w = u^4, \quad u^8 = 256v^7, \quad \alpha u^8 = v^7.$$

Consequently this branch occurs only for $\alpha = 1/256$. At that value an additional rational singular curve appears. Hence, on the nonempty open set

$$\alpha \in \mathbb{C}^\times \setminus \left\{ \frac{1}{256} \right\},$$

the reduced singular locus is

$$\text{Sing}(X_1(\alpha))_{\text{red}} = C \sqcup \{P\}.$$

We impose this genericity condition from now on.

The isolated point P . The local equation at P is

$$f_P = u^4 v + z^4 + v^3 w + uvzw + \alpha w^3.$$

It has an isolated critical point and is weighted homogeneous of degree 36 with weights

$$\text{wt}(u, v, z, w) = (7, 8, 9, 12).$$

After a diagonal rescaling, this germ is analytically equivalent to

$$U^4 V + Z^4 + V^3 W + \tau UVZW + W^3, \quad \tau \in \mathbb{C}^\times, \quad \tau^4 = \alpha^{-1}.$$

The genericity condition $\alpha \neq 1/256$ is equivalent to $\tau^4 \neq 256$. Hence, in the Yonemura-number notation of Section 8,

$$(X_1(\alpha), P) \sim_{\text{an}} \mathbf{S}_{87}^{\text{Y52}}(\tau).$$

In particular, the isolated singularity has Milnor number 87 and Yonemura number 52. Therefore

$$\tilde{\alpha}_P(X_1(\alpha)) = \frac{7 + 8 + 9 + 12}{36} = 1.$$

Smooth points of C . Every point of $C \setminus \{Q\}$ lies in the chart $x_2 = 1$. Put

$$\eta = x_4 + x_3^4.$$

Then

$$F = x_1^4 + x_0\Theta, \quad \Theta = \eta + x_1x_3(\eta - x_3^4) + \alpha x_0(\eta - x_3^4)^3.$$

Since $\partial\Theta/\partial\eta = 1$ along C , the function $\eta' = \Theta$ is an analytic coordinate, and the local equation becomes

$$x_0\eta' + x_1^4 = 0$$

together with the smooth parameter x_3 . Thus the transverse singularity is of type A_3 , and

$$\tilde{\alpha}_p(X_1(\alpha)) = \frac{2+2+1}{4} = \frac{5}{4} \quad (p \in C \setminus \{Q\}).$$

The transverse Jacobian ideal is (η', x_0, x_1^3) ; in particular, the saturated Jacobian scheme is nonreduced along C , with generic transverse length 3.

The cusp point Q . In the chart $x_4 = 1$, with

$$(u, v, y, z) = (x_0, x_1, x_2, x_3),$$

the local equation is

$$f_Q = \alpha u^2 + u(y^3 + z^4 + vyz) + v^4y.$$

Completing the square in u gives

$$f_Q = \alpha u'^2 + g(v, y, z),$$

where

$$g(v, y, z) = v^4y - \frac{1}{4\alpha}(y^3 + z^4 + vyz)^2.$$

The germ g is weighted homogeneous of degree 24 with weights

$$\text{wt}(v, y, z) = (5, 4, 3).$$

A direct solution of $\nabla g = 0$ gives

$$\text{Sing}(V(g))_{\text{red}} = V(v, y^3 + z^4).$$

At every nonzero point of this curve, g is analytically equivalent to $V^4 - H^2$ together with one smooth parameter, and hence has local log canonical threshold $3/4$. Away from the critical locus the threshold is 1. Thus $(\mathbb{C}^3, \frac{1}{2}V(g))$ is log canonical away from the origin. The weighted-blow-up criterion gives

$$\text{lct}_0(g) = \frac{5+4+3}{24} = \frac{1}{2};$$

see [Pae24, Proposition 3.2]. Since

$$\text{lct}_0(g) = \min\{\tilde{\alpha}_0(g), 1\},$$

it follows that

$$\tilde{\alpha}_0(g) = \frac{1}{2}.$$

Applying the Euler-homogeneous Thom–Sebastiani theorem [Sai94, Theorem 0.8], we obtain

$$\tilde{\alpha}_Q(X_1(\alpha)) = \tilde{\alpha}_0(\alpha u'^2) + \tilde{\alpha}_0(g) = \frac{1}{2} + \frac{1}{2} = 1.$$

Global minimal exponent. These strata exhaust the singular locus. Therefore

$$\tilde{\alpha}(X_1(\alpha)) = \min \left\{ 1, 1, \frac{5}{4} \right\} = 1.$$

Thus a general closed-orbit representative of Θ_1 has global minimal exponent

$$\tilde{\alpha}(X_1(\alpha)) = 1 = \frac{4+1}{5}.$$

B.2 The component Θ_2

By Table 7,

$$\Theta_2 = \Phi_2 = \Phi_{27}, \quad k(2) = 2.$$

A general closed orbit in Θ_2 is represented by

$$\begin{aligned} F_2(\alpha, \beta) := \phi_2^{\text{nf}}(\alpha, \beta) &= x_1^3 x_2^2 + x_1^4 x_3 + x_0 x_2 x_3^3 + x_0 x_2^3 x_4 \\ &\quad + \alpha x_0 x_1 x_2 x_3 x_4 + \beta x_0^2 x_4^3, \end{aligned} \quad (\alpha, \beta) \in (\mathbb{C}^\times)^2,$$

and we set

$$X_2(\alpha, \beta) := V(F_2(\alpha, \beta)) \subset \mathbb{P}^4.$$

The required genericity condition is

$$\mathcal{D}_2(\alpha, \beta) \neq 0,$$

where

$$\mathcal{D}_2(\alpha, \beta) = \alpha^6 - 2\alpha^5 + \alpha^4 + 54\alpha^3\beta - 378\alpha^2\beta + 576\alpha\beta + 729\beta^2 - 256\beta.$$

Singular locus. For readability, write $F = F_2(\alpha, \beta)$. Its partial derivatives are

$$\begin{aligned} \frac{\partial F}{\partial x_0} &= x_2 x_3^3 + x_2^3 x_4 + \alpha x_1 x_2 x_3 x_4 + 2\beta x_0 x_4^3, \\ \frac{\partial F}{\partial x_1} &= 3x_1^2 x_2^2 + 4x_1^3 x_3 + \alpha x_0 x_2 x_3 x_4, \\ \frac{\partial F}{\partial x_2} &= 2x_1^3 x_2 + x_0 x_3^3 + 3x_0 x_2^2 x_4 + \alpha x_0 x_1 x_3 x_4, \\ \frac{\partial F}{\partial x_3} &= x_1^4 + 3x_0 x_2 x_3^2 + \alpha x_0 x_1 x_2 x_4, \\ \frac{\partial F}{\partial x_4} &= x_0(x_2^3 + \alpha x_1 x_2 x_3 + 3\beta x_0 x_4^2). \end{aligned}$$

On the hyperplane $x_0 = 0$, these equations reduce on the support to

$$x_1 = 0, \quad x_2(x_3^3 + x_2^2 x_4) = 0.$$

Hence the singular support in this hyperplane is $L \cup C$, where

$$L := V(x_0, x_1, x_2)$$

is a line and

$$C := V(x_0, x_1, x_3^3 + x_2^2 x_4)$$

is an irreducible cuspidal plane cubic. A parametrization of C is

$$[s : t] \mapsto [0 : 0 : s^3 : -s^2 t : t^3].$$

Its unique cusp is

$$Q = (0 : 0 : 0 : 0 : 1), \quad L \cap C = \{Q\}.$$

On the chart $x_0 = 1$, put

$$(u, v, z, w) = (x_1, x_2, x_3, x_4).$$

The affine equation is

$$f = u^3v^2 + u^4z + vz^3 + v^3w + \alpha uvzw + \beta w^3.$$

If one of u, v, z, w vanishes, the Jacobian equations force $u = v = z = w = 0$, giving the isolated point

$$P = (1 : 0 : 0 : 0 : 0).$$

If $uvzw \neq 0$, multiplying the four affine Jacobian equations by u, v, z, w , respectively, and eliminating the occurring monomials yields a parameter t satisfying

$$\alpha(t-1)^2 = 1 - 4t, \quad \beta(t-1)^6 = t^2.$$

Eliminating t gives $\mathcal{D}_2(\alpha, \beta) = 0$. Thus, under the genericity condition above, there are no further affine singular points, and

$$\text{Sing}(X_2(\alpha, \beta))_{\text{red}} = (L \cup C) \sqcup \{P\}.$$

The isolated point P . The local equation f is weighted homogeneous of degree 27 with weights

$$\text{wt}(u, v, z, w) = (5, 6, 7, 9).$$

The condition $\mathcal{D}_2(\alpha, \beta) \neq 0$ ensures that its critical point is isolated. With

$$(X_1, X_2, X_3, X_4) = (u, v, z, w),$$

the local equation is precisely the defining normal form of the family $\mathbf{S}_{88}^{\text{Y84}}(\alpha, \beta)$. Hence, in the Yonemura-number notation of Section 8,

$$(X_2(\alpha, \beta), P) \sim_{\text{an}} \mathbf{S}_{88}^{\text{Y84}}(\alpha, \beta).$$

In particular, the isolated singularity has Milnor number 88 and Yonemura number 84. Therefore

$$\tilde{\alpha}_P(X_2(\alpha, \beta)) = \frac{5 + 6 + 7 + 9}{27} = 1.$$

Smooth points of the line L . Let $p \in L \setminus \{Q\}$. Since $x_3(p) \neq 0$, work in the chart $x_3 = 1$ and write

$$u = x_0, \quad v = x_1, \quad y = x_2, \quad x_4 = q + \zeta,$$

where $q = x_4(p)/x_3(p)$. The quadratic part in the transverse variables (u, y) is

$$uy + \beta q^3 u^2,$$

which is nondegenerate for every q , while

$$F(0, v, 0, \zeta) = v^4.$$

By the holomorphic splitting lemma, the germ is analytically equivalent to

$$U_1^2 + U_2^2 + v^4$$

together with the smooth parameter ζ . Thus the transverse type is A_3 , and

$$\tilde{\alpha}_p(X_2(\alpha, \beta)) = \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{5}{4} \quad (p \in L \setminus \{Q\}).$$

The generic transverse Jacobian ideal is (U_1, U_2, v^3) , so the saturated Jacobian scheme has generic transverse length 3 along L .

Smooth points of the cubic C . Let $p \in C \setminus \{Q\}$. Then $x_2(p) \neq 0$. On the chart $x_2 = 1$, put

$$\eta = x_4 + x_3^3.$$

The local equation becomes

$$F = x_1^3 + x_1^4 x_3 + x_0 \eta + \alpha x_0 x_1 x_3 (\eta - x_3^3) + \beta x_0^2 (\eta - x_3^3)^3.$$

The quadratic form in (x_0, η) has rank 2. Its critical section is

$$x_0 = 0, \quad \eta = \frac{\alpha x_1 x_3^4}{1 + \alpha x_1 x_3},$$

and the restriction of F to this section is

$$x_1^3(1 + x_1 x_3).$$

Since the second factor is a unit, the transverse germ is analytically equivalent to

$$U_1^2 + U_2^2 + x_1^3.$$

Hence the transverse type is A_2 , and

$$\tilde{\alpha}_p(X_2(\alpha, \beta)) = \frac{1}{2} + \frac{1}{2} + \frac{1}{3} = \frac{4}{3} \quad (p \in C \setminus \{Q\}).$$

The generic transverse Jacobian ideal is (U_1, U_2, x_1^2) , so the saturated Jacobian scheme has generic transverse length 2 along C .

The intersection point Q . In the chart $x_4 = 1$, with

$$(u, v, y, z) = (x_0, x_1, x_2, x_3),$$

the local equation is

$$f_Q = \beta u^2 + u(y^3 + yz^3 + \alpha vyz) + v^3 y^2 + v^4 z.$$

Completing the square in u gives

$$f_Q = \beta u'^2 + g(v, y, z),$$

where

$$g(v, y, z) = v^3 y^2 + v^4 z - \frac{1}{4\beta} (y^3 + yz^3 + \alpha vyz)^2.$$

The germ g is weighted homogeneous of degree 18 with weights

$$\text{wt}(v, y, z) = (4, 3, 2).$$

The face with zero v -exponent is proportional to $y^2(y^2 + z^3)^2$, so g is Newton degenerate. We therefore use the weighted-cone criterion instead. Its reduced critical locus is the union

$$\Sigma_L = V(v, y), \quad \Sigma_C = V(v, y^2 + z^3).$$

At a nonzero point of Σ_L , the transverse type is $Y^2 + V^4$, whose log canonical threshold is $3/4$; at a nonzero point of Σ_C , it is $Y^2 + V^3$, whose log canonical threshold is $5/6$. Away from these curves the divisor $V(g)$ is smooth. Hence $(\mathbb{C}^3, \frac{1}{2}V(g))$ is log canonical away from the origin. The weighted-cone criterion [Pae24, Proposition 3.2] gives

$$\text{lct}_0(g) = \frac{4 + 3 + 2}{18} = \frac{1}{2}.$$

By the general relation [MP20],

$$\text{lct}_0(g) = \min\{\tilde{\alpha}_0(g), 1\},$$

so

$$\tilde{\alpha}_0(g) = \frac{1}{2}.$$

The Euler-homogeneous Thom–Sebastiani theorem [Sai94, Proposition 0.7 and Theorem 0.8], together with [CDNM24, Theorem 1.2], then yields

$$\tilde{\alpha}_Q(X_2(\alpha, \beta)) = \tilde{\alpha}_0(\beta u'^2) + \tilde{\alpha}_0(g) = \frac{1}{2} + \frac{1}{2} = 1.$$

Global minimal exponent. The preceding strata exhaust the singular locus. Therefore

$$\tilde{\alpha}(X_2(\alpha, \beta)) = \min\left\{1, 1, \frac{5}{4}, \frac{4}{3}\right\} = 1 = \frac{4+1}{5}.$$

Thus a general closed-orbit representative of Θ_2 has global minimal exponent equal to 1.

B.3 The component Θ_3

By Table 7,

$$\Theta_3 = \Phi_3 = \Phi_{33}, \quad k(3) = 3.$$

Write

$$B_5(x_1, x_2) = b_0x_1^5 + b_1x_1^4x_2 + b_2x_1^3x_2^2 + b_3x_1^2x_2^3 + b_4x_1x_2^4 + b_5x_2^5.$$

A general closed orbit in Θ_3 is represented by

$$\begin{aligned} F_3(B_5) &:= \phi_3^{\text{nf}}(B_5) = B_5(x_1, x_2) + x_0x_2x_3^3 \\ &\quad + x_0(x_1^2 + x_2^2)x_3x_4 + x_0^2x_4^3, \end{aligned}$$

and we set

$$X_3(B_5) := V(F_3(B_5)) \subset \mathbb{P}^4.$$

We work on the nonempty Zariski-open set

$$b_0b_2 \text{Disc}(B_5)\mathcal{A}_3(B_5) \neq 0,$$

where, writing $B_{5,i} = \partial B_5 / \partial x_i$,

$$\mathcal{A}_3(B_5) := \text{Res}_\rho \left(9B_{5,1}(\rho, 1) + 2\rho(\rho^2 + 1)^2, 27B_{5,2}(\rho, 1) + (\rho^2 + 1)^2(5 - \rho^2) \right).$$

Singular locus. On the hyperplane $x_0 = 0$, the Jacobian equations include

$$B_{5,1}(x_1, x_2) = B_{5,2}(x_1, x_2) = 0.$$

Since B_5 is square-free, these equations force $x_1 = x_2 = 0$. Thus the singular support in this hyperplane is the line

$$L := V(x_0, x_1, x_2).$$

On the chart $x_0 = 1$, put

$$(u, v, z, w) = (x_1, x_2, x_3, x_4).$$

The affine equation is

$$f = B_5(u, v) + vz^3 + (u^2 + v^2)zw + w^3,$$

and its critical-point equations are

$$\begin{aligned} B_{5,1}(u, v) + 2uzw &= 0, \\ B_{5,2}(u, v) + z^3 + 2vzw &= 0, \\ 3vz^2 + (u^2 + v^2)w &= 0, \\ (u^2 + v^2)z + 3w^2 &= 0. \end{aligned}$$

If $z = 0$ or $w = 0$, these equations give $u = v = z = w = 0$, hence the isolated point

$$P = (1 : 0 : 0 : 0 : 0).$$

If $zw \neq 0$, then $v \neq 0$. Using the weighted action of weights $(3, 3, 4, 5)$, normalize $v = 1$ and put $\rho = u/v$. The last two critical equations yield

$$zw = \frac{(\rho^2 + 1)^2}{9}, \quad z^3 = -\frac{(\rho^2 + 1)^3}{27},$$

while the first two become the two equations occurring in the definition of $\mathcal{A}_3(B_5)$. Hence $\mathcal{A}_3(B_5) \neq 0$ excludes all such points. Therefore

$$\text{Sing}(X_3(B_5))_{\text{red}} = L \sqcup \{P\}.$$

The distinguished point on L is

$$Q = (0 : 0 : 0 : 0 : 1).$$

The isolated point P . The local equation at P is the weighted-homogeneous polynomial

$$f_P = B_5(u, v) + vz^3 + (u^2 + v^2)zw + w^3$$

of degree 15 with weights

$$\text{wt}(u, v, z, w) = (3, 3, 4, 5).$$

The genericity assumptions make its critical point isolated. With

$$(X_1, X_2, X_3, X_4) = (u, v, z, w),$$

the local equation is precisely the defining normal form of the family $\mathbf{S}_{88}^{\text{Y15}}(B_5)$. Hence, in the Yonemura-number notation of Section 8,

$$(X_3(B_5), P) \sim_{\text{an}} \mathbf{S}_{88}^{\text{Y15}}(B_5).$$

In particular, the isolated singularity has Milnor number 88 and Yonemura number 15. Therefore

$$\tilde{\alpha}_P(X_3(B_5)) = \frac{3 + 3 + 4 + 5}{15} = 1.$$

Points of $L \setminus \{Q\}$. Let $p \in L \setminus \{Q\}$. In the chart $x_3 = 1$, write

$$u = x_0, \quad v = x_1, \quad y = x_2, \quad x_4 = q + \zeta.$$

Then

$$F_3 = B_5(v, y) + uy + u(v^2 + y^2)(q + \zeta) + u^2(q + \zeta)^3.$$

The quadratic part in (u, y) is $uy + q^3u^2$, which has rank 2. The splitting lemma therefore gives

$$F_3 \sim_{\text{an}} U_1^2 + U_2^2 + g(v, \zeta).$$

Solving the critical-section equations in (u, y) gives

$$y = -(q + \zeta)v^2 + O(v^4), \quad u = O(v^4),$$

and consequently

$$g(v, \zeta) = b_0 v^5 + O(v^6).$$

Since $b_0 \neq 0$, an analytic change of the coordinate v yields

$$F_3 \sim_{\text{an}} U_1^2 + U_2^2 + V^5$$

together with the smooth parameter ζ . Thus the transverse type is A_4 , and

$$\tilde{\alpha}_p(X_3(B_5)) = \frac{1}{2} + \frac{1}{2} + \frac{1}{5} = \frac{6}{5} \quad (p \in L \setminus \{Q\}).$$

The generic transverse Jacobian ideal is (U_1, U_2, V^4) ; hence the saturated Jacobian scheme has generic transverse length 4 along L .

The distinguished point Q . In the chart $x_4 = 1$, with

$$(u, v, y, z) = (x_0, x_1, x_2, x_3),$$

the local equation is

$$f_Q = u^2 + B_5(v, y) + uyz^3 + u(v^2 + y^2)z.$$

Set

$$A(v, y, z) := z(v^2 + y^2 + yz^2), \quad U := u + \frac{1}{2}A(v, y, z).$$

Completing the square gives

$$f_Q = U^2 + h(v, y, z),$$

where

$$h(v, y, z) = B_5(v, y) - \frac{1}{4}z^2(v^2 + y^2 + yz^2)^2.$$

To compute its log canonical threshold, introduce

$$G(v, y, t) := B_5(v, y) - \frac{1}{4}t(v^2 + y^2 + yt)^2, \quad h(v, y, z) = G(v, y, z^2).$$

For the double cover

$$\pi: (v, y, z) \mapsto (v, y, z^2),$$

branched along $T = V(t)$, and $D = V(G)$, the finite-cover criterion gives

$$\text{lct}_0(h) = \sup \left\{ c > 0 \mid \left(\mathbb{C}_{v,y,t}^3, \frac{1}{2}T + cD \right) \text{ is log canonical at } 0 \right\}.$$

Since $\text{mult}_0 T = 1$ and $\text{mult}_0 D = 5$, the exceptional divisor of the ordinary blow-up has log discrepancy

$$3 - \frac{1}{2} - 5c.$$

Thus log canonicity forces $c \leq 1/2$, and hence

$$\text{lct}_0(h) \leq \frac{1}{2}.$$

For $c = 1/2$, inversion of adjunction reduces log canonicity to that of

$$\left(\mathbb{P}^2, \frac{1}{2}\ell_\infty + \frac{1}{2}C \right),$$

where

$$\ell_\infty = V(t), \quad C = V(G) \subset \mathbb{P}_{[v:y:t]}^2.$$

The curve C meets ℓ_∞ transversely at the five distinct roots of B_5 . The resultant condition excludes singularities of C away from

$$R = (0 : 0 : 1).$$

At R , the splitting lemma gives

$$(C, R) \sim_{\text{an}} V(Y^2 + V^5), \quad \text{lct}_R(C) = \frac{1}{2} + \frac{1}{5} = \frac{7}{10} > \frac{1}{2}.$$

It follows that the displayed pair on $\mathbb{P}^2$ is log canonical, and hence

$$\text{lct}_0(h) = \frac{1}{2}.$$

Using

$$\text{lct}_0(h) = \min\{\tilde{\alpha}_0(h), 1\},$$

we obtain $\tilde{\alpha}_0(h) = 1/2$; see [MP20]. The Thom–Sebastiani theorem then yields

$$\tilde{\alpha}_Q(X_3(B_5)) = \tilde{\alpha}_0(U^2) + \tilde{\alpha}_0(h) = \frac{1}{2} + \frac{1}{2} = 1;$$

see [Sai94, CDN24].

Global minimal exponent. The preceding strata exhaust the singular locus. Therefore

$$\tilde{\alpha}(X_3(B_5)) = \min\left\{1, 1, \frac{6}{5}\right\} = 1 = \frac{4+1}{5}.$$

Thus a general closed-orbit representative of Θ_3 has global minimal exponent equal to 1.

B.4 The component Θ_4

By Table 7,

$$\Theta_4 = \Phi_4 = \Phi_{21}, \quad k(4) = 4.$$

A general closed orbit in Θ_4 is represented by

$$\begin{aligned} F_4(\alpha, \beta) := \phi_4^{\text{nf}}(\alpha, \beta) &= x_1^2 x_2^3 + x_1^3 x_3 x_4 + x_0 x_2^4 + x_0 x_1 x_3^3 \\ &\quad + \alpha x_0 x_1 x_2 x_3 x_4 + \beta x_0^2 x_4^3, \end{aligned}$$

where $(\alpha, \beta) \in (\mathbb{C}^\times)^2$, and we set

$$X_4(\alpha, \beta) := V(F_4(\alpha, \beta)) \subset \mathbb{P}^4.$$

We work on the nonempty Zariski-open set

$$\mathcal{D}_4(\alpha, \beta) \neq 0,$$

where

$$\mathcal{D}_4(\alpha, \beta) = \alpha^6 - 2\alpha^5 + \alpha^4 + 54\alpha^3\beta - 378\alpha^2\beta + 576\alpha\beta + 729\beta^2 - 256\beta.$$

Singular locus. On the hyperplane $x_0 = 0$, the Jacobian equations give either $x_1 = x_2 = 0$, which yields the line

$$L := V(x_0, x_1, x_2),$$

or the single point

$$R = (0 : 1 : 0 : 0 : 0).$$

On the chart $x_0 = 1$, put

$$(u, v, z, w) = (x_1, x_2, x_3, x_4).$$

The affine equation is

$$f = u^2v^3 + u^3zw + v^4 + uz^3 + \alpha uvzw + \beta w^3.$$

Its critical equations vanish identically along $v = z = w = 0$, giving the affine part of the line

$$M := V(x_2, x_3, x_4).$$

Outside M , all four coordinates u, v, z, w are nonzero. The weighted $\mathbb{G}_m$ -action of weights $(3, 6, 7, 8)$ allows us to normalize $u = 1$, after which the critical equations imply

$$\alpha v^2 + 4v + 3 = 0, \quad (1 + \alpha v)^2 = 9\beta v^4.$$

Their resultant is $9\mathcal{D}_4(\alpha, \beta)$. Hence the genericity condition excludes every additional singular point. Since $R \in M$, we obtain

$$\text{Sing}(X_4(\alpha, \beta))_{\text{red}} = M \sqcup L,$$

and the two lines are disjoint. In particular, there are no isolated singular points.

Points of $M \setminus \{P\}$. Let

$$P = (1 : 0 : 0 : 0 : 0).$$

For $p \in M \setminus \{P\}$, work in the chart $x_1 = 1$, write $s = x_0$, $y = x_2$, $z = x_3$, and $w = x_4$, and put $s = s_0 + \sigma$. The local equation is

$$F_4 = y^3 + zw + sy^4 + sz^3 + \alpha syzw + \beta s^2w^3.$$

The quadratic form in (z, w) is nondegenerate, and $z = w = 0$ is the exact critical section. Its critical value is

$$y^3(1 + (s_0 + \sigma)y),$$

so the splitting lemma gives

$$F_4 \sim_{\text{an}} U_1^2 + U_2^2 + Y^3$$

with σ a smooth parameter. Thus the transverse type is A_2 , and

$$\tilde{\alpha}_p(X_4(\alpha, \beta)) = \frac{1}{2} + \frac{1}{2} + \frac{1}{3} = \frac{4}{3} \quad (p \in M \setminus \{P\}).$$

The generic transverse Jacobian ideal is (U_1, U_2, Y^2) , so its generic length along M is 2.

The distinguished point P . In the chart $x_0 = 1$, the local equation at P is

$$f_P = u^2v^3 + u^3zw + v^4 + uz^3 + \alpha uvzw + \beta w^3.$$

It is weighted homogeneous of degree 24 for

$$\text{wt}(u, v, z, w) = (3, 6, 7, 8).$$

Every proper Newton face is nondegenerate, while a torus critical point of the principal face would determine a singular point of $X_4(\alpha, \beta)$ outside $M \sqcup L$, which has already been excluded. Thus f_P is Newton nondegenerate. Since the monomial $uvzw$ occurs and all support exponents lie on the weight hyperplane of degree 24, its Newton distance is 1. Consequently

$$\text{lct}_0(f_P) = 1$$

by the Newton-polyhedron formula [How03]. Using $\text{lct}_0(f_P) = \min\{\tilde{\alpha}_0(f_P), 1\}$ and the weighted upper bound for the minimal exponent, we obtain

$$1 \leq \tilde{\alpha}_0(f_P) \leq \frac{3+6+7+8}{24} = 1;$$

see [MP20, CDM25]. Therefore

$$\tilde{\alpha}_P(X_4(\alpha, \beta)) = 1.$$

Points of $L \setminus \{Q\}$. Let

$$Q = (0 : 0 : 0 : 0 : 1).$$

For $p \in L \setminus \{Q\}$, work in the chart $x_3 = 1$, write

$$u = x_0, \quad v = x_1, \quad y = x_2, \quad x_4 = q + \zeta,$$

and set

$$A = 1 + \alpha y(q + \zeta), \quad B = q + \zeta.$$

Then

$$F_4 = Auv + \beta B^3u^2 + uy^4 + y^3v^2 + Bv^3.$$

The quadratic form in (u, v) has rank 2. If $(u_*(y, \zeta), v_*(y, \zeta))$ denotes its critical section, then

$$v_* = -\frac{y^4}{A} + O_y(y^7), \quad u_* = \frac{2y^7}{A^2} + O_y(y^8),$$

and substitution gives

$$F_4(u_*, v_*, y, \zeta) = \frac{y^{11}}{A^2} + O_y(y^{12}) = y^{11} \cdot \text{unit}.$$

Hence

$$F_4 \sim_{\text{an}} U_1^2 + U_2^2 + Y^{11}$$

with ζ a smooth parameter. The transverse type is A_{10} , and

$$\tilde{\alpha}_p(X_4(\alpha, \beta)) = \frac{1}{2} + \frac{1}{2} + \frac{1}{11} = \frac{12}{11} \quad (p \in L \setminus \{Q\}).$$

The generic transverse Jacobian ideal is (U_1, U_2, Y^{10}) , so its generic length along L is 10.

The distinguished point Q . In the chart $x_4 = 1$, with

$$(u, v, y, z) = (x_0, x_1, x_2, x_3),$$

the local equation is

$$f_Q = \beta u^2 + v^2 y^3 + v^3 z + u y^4 + u v z^3 + \alpha u v y z.$$

It is weighted homogeneous of degree 16 for

$$\text{wt}(u, v, y, z) = (8, 5, 2, 1).$$

The same Newton argument as at P shows that f_Q is nondegenerate and has Newton distance 1. Hence

$$\text{lct}_0(f_Q) = 1, \quad 1 \leq \tilde{\alpha}_0(f_Q) \leq \frac{8 + 5 + 2 + 1}{16} = 1,$$

and therefore

$$\tilde{\alpha}_Q(X_4(\alpha, \beta)) = 1.$$

Global minimal exponent. The preceding strata exhaust the singular locus. Therefore

$$\tilde{\alpha}(X_4(\alpha, \beta)) = \min \left\{ \frac{4}{3}, 1, \frac{12}{11}, 1 \right\} = 1 = \frac{4 + 1}{5}.$$

Thus a general closed-orbit representative of Θ_4 has global minimal exponent equal to 1.

B.5 The component Θ_5

By Table 7,

$$\Theta_5 = \Phi_5 = \Phi_{24}, \quad k(5) = 5.$$

A general closed orbit in Θ_5 is represented by

$$\begin{aligned} F_5(\alpha, \beta, \gamma) := \phi_5^{\text{nf}}(\alpha, \beta, \gamma) &= x_1^2 x_2^3 + x_1^3 x_3^2 + x_1^3 x_2 x_4 + x_0 x_2^3 x_3 \\ &\quad + \alpha x_0 x_1 x_3^3 + \beta x_0 x_1 x_2 x_3 x_4 + \gamma x_0^2 x_4^3, \end{aligned}$$

where $(\alpha, \beta, \gamma) \in (\mathbb{C}^\times)^3$, and we set

$$X_5(\alpha, \beta, \gamma) := V(F_5(\alpha, \beta, \gamma)) \subset \mathbb{P}^4.$$

We work on the nonempty Zariski-open set

$$\alpha \neq 1, \quad \mathcal{R}_5(\alpha, \beta, \gamma) \neq 0,$$

where

$$\mathcal{R}_5(\alpha, \beta, \gamma) := \text{Res}_s(A_5(s), B_5(s)),$$

with

$$\begin{aligned} A_5(s) &= 81\alpha\gamma s^2(s-1)^2 + \beta^3(s-1)^2(2s+1) + 3\beta^2(s^2-1) \\ &\quad + 3\beta + 54\gamma s^2(s-1) - 1, \end{aligned}$$

and

$$\begin{aligned} B_5(s) &= 27\alpha\gamma s^2(s-1)^3 + \beta^3(s-1)^3(3s-2) + 3\beta^2(s-1)^2(5s-2) \\ &\quad + 3\beta(s-1)(7s-2) + 81\gamma s^2(s-1)^2 + 9s - 2. \end{aligned}$$

The resultant contains the factor $(\beta - 1)^6$; hence its nonvanishing also implies $\beta \neq 1$.

Singular locus. On the hyperplane $x_0 = 0$, the Jacobian equations give the two lines

$$L := V(x_0, x_1, x_2), \quad M := V(x_0, x_1, x_3),$$

together with the point $(0 : 1 : 0 : 0 : 0)$, which lies on the line

$$N := V(x_2, x_3, x_4).$$

On the chart $x_0 = 1$, put

$$(u, v, z, w) = (x_1, x_2, x_3, x_4).$$

The affine equation is

$$f = u^2v^3 + u^3z^2 + u^3vw + v^3z + \alpha uz^3 + \beta uvzw + \gamma w^3.$$

Its critical equations vanish identically along $v = z = w = 0$, giving the remaining affine part of N . Away from N , one has $u \neq 0$, and the weighted action of weights $(3, 5, 6, 7)$ allows us to normalize $u = 1$. The critical equations then force $vw \neq 0$. Writing

$$w = tv, \quad s = \gamma t^3,$$

the equations $f_v = f_w = 0$ give

$$z = s - 1, \quad v = -\frac{1 - \beta + \beta s}{3\gamma t^2}, \quad w = -\frac{1 - \beta + \beta s}{3\gamma t},$$

and the remaining two equations become

$$A_5(s) = B_5(s) = 0.$$

Thus $\mathcal{R}_5 \neq 0$ excludes every additional affine singular point. Consequently

$$\text{Sing}(X_5(\alpha, \beta, \gamma))_{\text{red}} = N \sqcup (L \cup M).$$

Here N is disjoint from $L \cup M$, while

$$L \cap M = \{Q\}, \quad Q = (0 : 0 : 0 : 0 : 1).$$

In particular, there are no isolated singular points. Scheme-theoretically, N and M are generically reduced, whereas the generic transverse length along L is 8.

Points of $N \setminus \{P\}$. Let

$$P = (1 : 0 : 0 : 0 : 0).$$

For $p \in N \setminus \{P\}$, work in the chart $x_1 = 1$ and write

$$(s, y, z, w) = (x_0, x_2, x_3, x_4).$$

The local equation is

$$y^3 + z^2 + yw + sy^3z + \alpha sz^3 + \beta syzw + \gamma s^2w^3.$$

Its quadratic part $z^2 + yw$ has rank 3. Hence the transverse germ is a three-variable Morse singularity, with s as a smooth parameter, and

$$\tilde{\alpha}_p(X_5(\alpha, \beta, \gamma)) = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2} \quad (p \in N \setminus \{P\}).$$

The distinguished point P . At P , the local equation is

$$f_P = u^2v^3 + u^3z^2 + u^3vw + v^3z + \alpha uz^3 + \beta uvzw + \gamma w^3.$$

It is weighted homogeneous of degree 21 for

$$\text{wt}(u, v, z, w) = (3, 5, 6, 7).$$

Every proper simplex face of its Newton boundary is nondegenerate. The only nonsimplicial proper face has face polynomial

$$u^2v^3 + u^3z^2 + v^3z + \alpha uz^3,$$

and its logarithmic critical equations have a torus solution precisely when $\alpha = 1$. A torus critical point of the full compact face would give an affine singular point outside N , which is excluded by $\mathcal{R}_5 \neq 0$. Thus f_P is Newton nondegenerate.

Since the monomial $uvzw$ occurs, every Newton ratio is at least 1, and equality is attained by the weight vector $(3, 5, 6, 7)$. Hence

$$\text{lct}_0(f_P) = 1$$

by the Newton-polyhedron formula [How03], and therefore $\tilde{\alpha}_P(X_5) \geq 1$ by [MP20]. On the other hand, the weighted blow-up of weight $(3, 5, 6, 7)$ has log discrepancy

$$3 + 5 + 6 + 7 - 21 = 0$$

over the hypersurface germ, so the germ is not canonical. Since it is a normal Gorenstein hypersurface, it cannot be rational. Saito's rationality criterion [Sai94, MP20] gives $\tilde{\alpha}_P(X_5) \leq 1$. Therefore

$$\tilde{\alpha}_P(X_5(\alpha, \beta, \gamma)) = 1.$$

Points of $L \setminus \{Q\}$. For $p \in L \setminus \{Q\}$, work in the chart $x_3 = 1$, write

$$(u, v, y) = (x_0, x_1, x_2), \quad x_4 = q + \zeta,$$

and regard ζ as the parameter along L . The quadratic part in (u, v) is

$$\alpha uv + \gamma q^3 u^2,$$

which has rank 2. Eliminating these two Morse variables gives the transverse critical value

$$\frac{\alpha - 1}{\alpha^3} y^9 + O(y^{10}).$$

Since $\alpha \neq 1$, the splitting lemma yields

$$F_5 \sim_{\text{an}} U_1^2 + U_2^2 + Y^9$$

with one smooth parameter. Thus the transverse type is A_8 , and

$$\tilde{\alpha}_p(X_5(\alpha, \beta, \gamma)) = \frac{1}{2} + \frac{1}{2} + \frac{1}{9} = \frac{10}{9} \quad (p \in L \setminus \{Q\}).$$

The generic transverse Jacobian ideal is (U_1, U_2, Y^8) , explaining the generic transverse length 8 along L .

Points of $M \setminus \{Q\}$. For $p \in M \setminus \{Q\}$, work in the chart $x_2 = 1$, write

$$(u, v, z) = (x_0, x_1, x_3), \quad x_4 = q + \zeta.$$

The quadratic part in the transverse variables is

$$v^2 + uz + \gamma q^3 u^2,$$

which has rank 3. Hence the transverse germ is again a three-variable Morse singularity, and

$$\tilde{\alpha}_p(X_5(\alpha, \beta, \gamma)) = \frac{3}{2} \quad (p \in M \setminus \{Q\}).$$

The intersection point Q . In the chart $x_4 = 1$, with

$$(u, v, y, z) = (x_0, x_1, x_2, x_3),$$

the local equation is

$$f_Q = \gamma u^2 + v^2 y^3 + v^3 z^2 + v^3 y + uy^3 z + \alpha uvz^3 + \beta uvyz.$$

It is weighted homogeneous of degree 14 for

$$\text{wt}(u, v, y, z) = (7, 4, 2, 1).$$

As at P , the only nonsimplicial proper Newton face has face polynomial

$$v^2 y^3 + v^3 z^2 + uy^3 z + \alpha uvz^3,$$

whose torus critical equations force $\alpha = 1$. The condition $\mathcal{R}_5 \neq 0$ excludes a critical point of the full compact face. Thus f_Q is Newton nondegenerate. Since the monomial $uvyz$ occurs and

$$7 + 4 + 2 + 1 = 14,$$

the Newton-polyhedron formula gives

$$\text{lct}_0(f_Q) = 1.$$

The weighted blow-up of weight $(7, 4, 2, 1)$ has log discrepancy zero over the hypersurface germ, so the germ is not canonical. Since it is normal Gorenstein, the same rationality argument as at P gives

$$\tilde{\alpha}_Q(X_5(\alpha, \beta, \gamma)) = 1.$$

Global minimal exponent. The preceding strata exhaust the singular locus. Therefore

$$\tilde{\alpha}(X_5(\alpha, \beta, \gamma)) = \min \left\{ 1, \frac{3}{2}, \frac{10}{9}, \frac{3}{2}, 1 \right\} = 1 = \frac{4+1}{5}.$$

Thus a general closed-orbit representative of Θ_5 has global minimal exponent equal to 1.

B.6 The component Θ_6

By Table 7,

$$\Theta_6 = \Phi_6 = \Phi_{28}, \quad k(6) = 6.$$

A general closed orbit in Θ_6 is represented by

$$\begin{aligned} F_6(\alpha, \beta, \gamma, \delta, \epsilon) := \phi_6^{\text{nf}}(\alpha, \beta, \gamma, \delta, \epsilon) = & x_1^2 x_2^3 + x_1^3 x_2 x_3 + x_1^4 x_4 + x_0 x_2^2 x_3^2 + \alpha x_0 x_2^3 x_4 \\ & + \beta x_0 x_1 x_3^3 + \gamma x_0 x_1 x_2 x_3 x_4 + \delta x_0 x_1^2 x_4^2 + \epsilon x_0^2 x_4^3, \end{aligned}$$

where $(\alpha, \beta, \gamma, \delta, \epsilon) \in (\mathbb{C}^\times)^5$, and we set

$$X_6(\alpha, \beta, \gamma, \delta, \epsilon) := V(F_6(\alpha, \beta, \gamma, \delta, \epsilon)) \subset \mathbb{P}^4.$$

Let

$$\begin{aligned} f_P(u, v, z, w) = & u^2 v^3 + u^3 v z + u^4 w + v^2 z^2 + \alpha v^3 w + \beta u z^3 \\ & + \gamma u v z w + \delta u^2 w^2 + \epsilon w^3. \end{aligned}$$

This is weighted homogeneous of degree 18 for the weights $(3, 4, 5, 6)$. Denote by $\mathcal{D}_6$ the closed locus of parameters for which $\nabla f_P = 0$ has a solution in $\mathbb{P}(3, 4, 5, 6)$, and choose a polynomial $\mathcal{R}_6$ such that

$$\mathcal{D}_6 \subseteq V(\mathcal{R}_6), \quad \mathcal{R}_6(1, 2, 1, 1, 1) \neq 0.$$

We work on the nonempty Zariski-open set

$$\beta \neq 1, \quad \mathcal{R}_6(\alpha, \beta, \gamma, \delta, \epsilon) \neq 0.$$

At $(\alpha, \beta, \gamma, \delta, \epsilon) = (1, 2, 1, 1, 1)$, a Gröbner-basis computation gives a Jacobi algebra of dimension 91. Weighted homogeneity then excludes every nonzero critical point, so this open set is nonempty.

Singular locus. On the hyperplane $x_0 = 0$, the Jacobian equations reduce on the support to

$$x_1 = 0, \quad x_2^2(x_3^2 + \alpha x_2 x_4) = 0.$$

Hence they give the line

$$L := V(x_0, x_1, x_2)$$

and the smooth conic

$$C := V(x_0, x_1, x_3^2 + \alpha x_2 x_4).$$

These curves meet only at

$$Q = (0 : 0 : 0 : 0 : 1).$$

On the chart $x_0 = 1$, the affine equation is precisely f_P . The condition $\mathcal{R}_6 \neq 0$ excludes every nonzero critical point, so this chart contributes only

$$P = (1 : 0 : 0 : 0 : 0).$$

Consequently

$$\text{Sing}(X_6(\alpha, \beta, \gamma, \delta, \epsilon))_{\text{red}} = (L \cup C) \sqcup \{P\}.$$

Scheme-theoretically, the generic transverse length is 6 along L , whereas C is generically reduced.

The isolated point P . The local equation at P is f_P . By $\mathcal{R}_6 \neq 0$, its critical point is isolated. With

$$(X_1, X_2, X_3, X_4) = (u, v, z, w),$$

the local equation is precisely the defining normal form of the family $\mathbf{S}_{91}^{Y53}(\alpha, \beta, \gamma, \delta, \epsilon)$. Hence, in the Yonemura-number notation of Section 8,

$$(X_6(\alpha, \beta, \gamma, \delta, \epsilon), P) \sim_{\text{an}} \mathbf{S}_{91}^{Y53}(\alpha, \beta, \gamma, \delta, \epsilon).$$

In particular, the isolated singularity has Milnor number 91 and Yonemura number 53. Since f_P is weighted homogeneous of degree 18 with weights $(3, 4, 5, 6)$,

$$\tilde{\alpha}_P(X_6(\alpha, \beta, \gamma, \delta, \epsilon)) = \frac{3+4+5+6}{18} = 1.$$

Points of $L \setminus \{Q\}$. Let $p \in L \setminus \{Q\}$. In the chart $x_3 = 1$, write

$$u = x_0, \quad v = x_1, \quad y = x_2, \quad x_4 = q + \zeta,$$

where ζ is a parameter along L . The Hessian in (u, v) has determinant $-\beta^2$, so the splitting lemma gives

$$F_6 \sim_{\text{an}} U_1^2 + U_2^2 + h(y, \zeta).$$

Solving the critical-section equations in (u, v) yields

$$h(y, \zeta) = \frac{\beta - 1}{\beta^3} y^7 + O(y^8).$$

Since $\beta \neq 1$, the transverse type is A_6 . Therefore

$$\tilde{\alpha}_p(X_6(\alpha, \beta, \gamma, \delta, \epsilon)) = \frac{1}{2} + \frac{1}{2} + \frac{1}{7} = \frac{8}{7} \quad (p \in L \setminus \{Q\}).$$

The transverse Jacobian ideal is analytically equivalent to (U_1, U_2, y^6) , which gives the generic transverse length 6 along L .

Points of $C \setminus \{Q\}$. Let $p \in C \setminus \{Q\}$. Since $x_2(p) \neq 0$, work in the chart $x_2 = 1$ and write

$$u = x_0, \quad v = x_1, \quad x_3 = q + z, \quad x_4 = r + \zeta, \quad q^2 + \alpha r = 0.$$

The quadratic part has rank 3; explicitly, it is

$$v^2 + u(2qz + \alpha\zeta) + \epsilon r^3 u^2.$$

The splitting lemma, together with the fact that the whole conic lies in the critical locus, gives a three-variable Morse singularity times one smooth parameter. Hence

$$\tilde{\alpha}_p(X_6(\alpha, \beta, \gamma, \delta, \epsilon)) = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2} \quad (p \in C \setminus \{Q\}).$$

In particular, the saturated Jacobian scheme is generically reduced along C .

The intersection point Q . In the chart $x_4 = 1$, with

$$(u, v, y, z) = (x_0, x_1, x_2, x_3),$$

the local equation is

$$\begin{aligned} f_Q = & v^2 y^3 + v^3 y z + v^4 + u y^2 z^2 + \alpha u y^3 + \beta u v z^3 \\ & + \gamma u v y z + \delta u v^2 + \epsilon u^2. \end{aligned}$$

It is weighted homogeneous of degree 12 for the weights $(6, 3, 2, 1)$, but its singularity is non-isolated. Away from the origin, the two singular branches are the punctured germs of L and C , where the local minimal exponents are respectively $8/7$ and $3/2$. Thus $(\mathbb{C}^4, V(f_Q))$ is log canonical away from the origin. The weighted-blow-up criterion then gives

$$\text{lct}_0(f_Q) = \frac{6 + 3 + 2 + 1}{12} = 1,$$

and hence

$$\tilde{\alpha}_Q(X_6(\alpha, \beta, \gamma, \delta, \epsilon)) \geq 1;$$

see [Pae24, Proposition 3.2] and [MP20].

For the reverse inequality, complete the square in u . Set

$$A = y^2 z^2 + \alpha y^3 + \beta v z^3 + \gamma v y z + \delta v^2, \quad B = v^2 y^3 + v^3 y z + v^4.$$

Then

$$f_Q = \epsilon \left(u + \frac{A}{2\epsilon} \right)^2 + g, \quad g := B - \frac{A^2}{4\epsilon}.$$

The germ $g(v, y, z)$ is weighted homogeneous of degree 12 for the weights $(3, 2, 1)$. The corresponding monomial valuation gives

$$\text{lct}_0(g) \leq \frac{3 + 2 + 1}{12} = \frac{1}{2}.$$

Since this bound is below 1, the relation between the log canonical threshold and the minimal exponent gives $\tilde{\alpha}_0(g) = \text{lct}_0(g) \leq 1/2$. Thom–Sebastiani therefore yields

$$\tilde{\alpha}_0(f_Q) = \frac{1}{2} + \tilde{\alpha}_0(g) \leq 1;$$

see [Sai94, Proposition 0.7 and Theorem 0.8] and [CDNM24, Theorem 1.2]. Combining the two inequalities gives

$$\tilde{\alpha}_Q(X_6(\alpha, \beta, \gamma, \delta, \epsilon)) = 1.$$

Global minimal exponent. The preceding strata exhaust the singular locus. Therefore

$$\tilde{\alpha}(X_6(\alpha, \beta, \gamma, \delta, \epsilon)) = \min \left\{ 1, 1, \frac{8}{7}, \frac{3}{2} \right\} = 1 = \frac{4+1}{5}.$$

Thus a general closed-orbit representative of Θ_6 has global minimal exponent equal to 1.

B.7 The component Θ_7

By Table 7,

$$\Theta_7 = \Phi_7 = \Phi_{18}, \quad k(7) = 7.$$

A general closed orbit in Θ_7 is represented by

$$\begin{aligned} F_7(\alpha, \beta) := \phi_7^{\text{nf}}(\alpha, \beta) &= x_2^5 + x_1^2 x_2^2 x_3 + x_1^4 x_4 + x_0 x_1 x_3^3 \\ &\quad + \alpha x_0 x_1 x_2 x_3 x_4 + \beta x_0^2 x_3 x_4^2, \end{aligned}$$

where $(\alpha, \beta) \in (\mathbb{C}^\times)^2$, and we set

$$X_7(\alpha, \beta) := V(F_7(\alpha, \beta)) \subset \mathbb{P}^4.$$

Let $\mathcal{R}_7 := \mathcal{R}_{96}$ denote the elimination condition for

$$f_P(u, v, z, w) = v^5 + u^2 v^2 z + u^4 w + u z^3 + \alpha u v z w + \beta z w^2$$

to have a nonzero critical point in $\mathbb{P}(4, 5, 7, 9)$. We work on the Zariski-open locus

$$\mathcal{R}_7(\alpha, \beta) \neq 0.$$

Singular locus. On the hyperplane $x_0 = 0$, the equations $\partial F_7 / \partial x_4 = 0$ and $\partial F_7 / \partial x_2 = 0$ give successively

$$x_1 = 0, \quad x_2 = 0.$$

Thus the singular support in this hyperplane is the line

$$L := V(x_0, x_1, x_2).$$

On the chart $x_0 = 1$, the affine equation is f_P . The condition $\mathcal{R}_7 \neq 0$ excludes every nonzero critical point, so this chart contributes only

$$P = (1 : 0 : 0 : 0 : 0).$$

Consequently

$$\text{Sing}(X_7(\alpha, \beta))_{\text{red}} = L \sqcup \{P\}.$$

The distinguished point on L is

$$Q = (0 : 0 : 0 : 0 : 1).$$

Scheme-theoretically, the generic transverse length along L is 4.

The isolated point P . The local equation at P is f_P . It is weighted homogeneous of degree 25 for the weights

$$\text{wt}(u, v, z, w) = (4, 5, 7, 9),$$

and $\mathcal{R}_7 \neq 0$ makes the critical point isolated. With

$$(X_1, X_2, X_3, X_4) = (u, v, z, w),$$

the local equation is precisely the defining normal form of the family $\mathbf{S}_{96}^{\text{Y86}}(\alpha, \beta)$. Hence, in the Yonemura-number notation of Section 8,

$$(X_7(\alpha, \beta), P) \sim_{\text{an}} \mathbf{S}_{96}^{\text{Y86}}(\alpha, \beta).$$

In particular, the isolated singularity has Milnor number 96 and Yonemura number 86. Therefore

$$\tilde{\alpha}_P(X_7(\alpha, \beta)) = \frac{4 + 5 + 7 + 9}{25} = 1.$$

Points of $L \setminus \{Q\}$. Let $p \in L \setminus \{Q\}$. In the chart $x_3 = 1$, write

$$u = x_0, \quad v = x_1, \quad y = x_2, \quad x_4 = q + \zeta.$$

With $s = q + \zeta$, the local equation is

$$F_7 = y^5 + v^2 y^2 + v^4 s + uv + \alpha u v y s + \beta u^2 s^2.$$

The Hessian in (u, v) is nondegenerate along the critical section $u = v = 0$. The parametric splitting lemma therefore gives

$$F_7 \sim_{\text{an}} U_1^2 + U_2^2 + y^5,$$

with ζ as a smooth parameter. Thus the transverse type is A_4 , and

$$\tilde{\alpha}_p(X_7(\alpha, \beta)) = \frac{1}{2} + \frac{1}{2} + \frac{1}{5} = \frac{6}{5} \quad (p \in L \setminus \{Q\}).$$

The transverse Jacobian ideal is analytically equivalent to (U_1, U_2, y^4) , which gives the generic transverse length 4 along L .

The distinguished point Q . In the chart $x_4 = 1$, with

$$(u, v, y, z) = (x_0, x_1, x_2, x_3),$$

the local equation is

$$f_Q = y^5 + v^2 y^2 z + v^4 + uvz^3 + \alpha u v y z + \beta u^2 z.$$

It is weighted homogeneous of degree 20 for

$$\text{wt}(u, v, y, z) = (9, 5, 4, 2).$$

A direct Jacobian calculation gives

$$\text{Sing}(V(f_Q)) = V(u, v, y).$$

Hence this three-dimensional hypersurface germ is normal: its singular locus has codimension two, and a hypersurface is Cohen–Macaulay.

A direct examination of the proper compact Newton faces shows that they are nondegenerate. A torus critical point of the full compact face would produce an additional singular point of $X_7(\alpha, \beta)$ in the chart $x_0 \neq 0$, which is excluded by $\mathcal{R}_7 \neq 0$. Thus f_Q is Newton nondegenerate. Since the monomial $uvyz$ occurs, every toric valuation determined by a primitive vector $a = (a_u, a_v, a_y, a_z) \in \mathbb{Z}_{\geq 0}^4 \setminus \{0\}$ satisfies

$$\nu_a(f_Q) \leq a_u + a_v + a_y + a_z.$$

The weight vector $(9, 5, 4, 2)$ attains equality, and the Newton-polyhedron formula therefore gives

$$\text{lct}_0(f_Q) = \frac{9 + 5 + 4 + 2}{20} = 1;$$

see [How03, Theorem 12]. Since $\text{lct}_0(f_Q) = \min\{\tilde{\alpha}_0(f_Q), 1\}$, it follows that

$$\tilde{\alpha}_Q(X_7(\alpha, \beta)) \geq 1;$$

see [MP20, Section 1].

For the reverse inequality, consider the weighted blow-up of weights $(9, 5, 4, 2)$. The corresponding divisor has discrepancy over the local hypersurface

$$(9 + 5 + 4 + 2 - 1) - 20 = -1.$$

Thus the normal Gorenstein germ $V(f_Q)$ is not canonical. Since a normal Gorenstein rational singularity is canonical [KM98, Corollary 5.24], the germ is not rational. Saito's rationality criterion [Sai93, Theorem 0.4] then yields

$$\tilde{\alpha}_Q(X_7(\alpha, \beta)) \leq 1.$$

Combining the two inequalities gives

$$\tilde{\alpha}_Q(X_7(\alpha, \beta)) = 1.$$

Global minimal exponent. The preceding strata exhaust the singular locus. Therefore

$$\tilde{\alpha}(X_7(\alpha, \beta)) = \min \left\{ 1, 1, \frac{6}{5} \right\} = 1 = \frac{4+1}{5}.$$

Thus a general closed-orbit representative of Θ_7 has global minimal exponent equal to 1.

B.8 The component Θ_8

By Table 7,

$$\Theta_8 = \Phi_8 = \Phi_{16}, \quad k(8) = 8.$$

A general closed orbit in Θ_8 is represented by

$$\begin{aligned} F_8(\alpha, \beta) := \phi_8^{\text{nf}}(\alpha, \beta) &= x_1^2 x_2^2 x_3 + x_1^3 x_4^2 + x_0 x_2^4 + x_0 x_1 x_3^3 \\ &\quad + \alpha x_0 x_1 x_2 x_3 x_4 + \beta x_0^2 x_3 x_4^2, \end{aligned}$$

where $(\alpha, \beta) \in (\mathbb{C}^\times)^2$, and we set

$$X_8(\alpha, \beta) := V(F_8(\alpha, \beta)) \subset \mathbb{P}^4.$$

We work on the nonempty Zariski-open set

$$\mathcal{R}_8(\alpha, \beta) \neq 0,$$

where

$$\mathcal{R}_8(\alpha, \beta) = \alpha^6 + (4 - 12\beta)\alpha^4 + 16\beta(3\beta - 20)\alpha^2 - 64\beta(\beta + 4)^2.$$

Indeed, $\mathcal{R}_8(1, 1) = -1879$, so this open set is nonempty.

Singular locus. On the hyperplane $x_0 = 0$, the Jacobian equations give the line

$$L := V(x_0, x_1, x_2)$$

and the point $(0 : 1 : 0 : 0 : 0)$, which lies on the line

$$N := V(x_2, x_3, x_4).$$

On the chart $x_0 = 1$, with

$$(u, v, z, w) = (x_1, x_2, x_3, x_4),$$

the affine equation is

$$f = u^2 v^2 z + u^3 w^2 + v^4 + u z^3 + \alpha u v z w + \beta z w^2.$$

Its critical equations vanish identically on $v = z = w = 0$, giving the remaining affine part of N . Outside this line one necessarily has $uvzw \neq 0$. Set

$$T_5 := \alpha uvzw, \quad T_6 := \beta zw^2, \quad r := T_5/T_6.$$

The logarithmic critical equations then reduce to

$$A_8(r) = 0, \quad B_8(r) = 0,$$

where

$$A_8(r) = \alpha^2(r+2)^3 + 4r^2(r+4), \quad B_8(r) = \beta(r+2)^3 + 2(r+4)^2.$$

Since

$$\text{Res}_r(A_8, B_8) = 512 \mathcal{R}_8(\alpha, \beta),$$

the genericity condition excludes every additional affine critical point. Consequently

$$\text{Sing}(X_8(\alpha, \beta))_{\text{red}} = N \sqcup L,$$

where N and L are disjoint lines. In particular, there are no isolated singular points. Scheme-theoretically, the generic transverse Jacobian lengths along N and L are respectively 4 and 9.

Points of $N \setminus \{P\}$. Let

$$P = (1 : 0 : 0 : 0 : 0).$$

At a point of $N \setminus \{P\}$ with $x_0 \neq 0$, write $x_1 = s + \xi$, $x_2 = y$, $x_3 = z$, and $x_4 = w$, where $s \neq 0$. Completing the square in w gives

$$F_8 \sim_{\text{an}} W^2 + g(\xi, y, z),$$

where g has order 3 along the smooth ξ -axis $V(y, z)$. Blowing up this axis resolves the germ: the exceptional divisor has log discrepancy 2, the pullback of g has order 3, and the strict transform is smooth and transverse to the exceptional divisor. Thus

$$\text{lct}_0(g) = \tilde{\alpha}_0(g) = \frac{2}{3}.$$

The same calculation in the chart $x_1 = 1$ treats the remaining point $(0 : 1 : 0 : 0 : 0) \in N \setminus \{P\}$. Thom–Sebastiani therefore gives

$$\tilde{\alpha}_p(X_8(\alpha, \beta)) = \frac{1}{2} + \frac{2}{3} = \frac{7}{6} \quad (p \in N \setminus \{P\});$$

see [CDNM24, MP20]. At a general point of N , the residual plane curve is an ordinary triple point, which gives Jacobian length 4.

The distinguished point P . The local equation at P is

$$f_P = u^2 v^2 z + u^3 w^2 + v^4 + uz^3 + \alpha uvzw + \beta zw^2.$$

It is weighted homogeneous of degree 20 for

$$\text{wt}(u, v, z, w) = (2, 5, 6, 7),$$

but its singularity is non-isolated. Every proper compact Newton face is nondegenerate, while a torus critical point of the full compact face would satisfy $A_8(r) = B_8(r) = 0$. Hence $\mathcal{R}_8 \neq 0$ implies that f_P is Newton nondegenerate. Since the support contains the monomial $uvzw$ and

lies on the weight hyperplane of degree 20, the Newton polyhedron has diagonal distance 1. Therefore

$$\text{lct}_0(f_P) = 1$$

by the Newton-polyhedron formula [How03], and hence

$$\tilde{\alpha}_P(X_8(\alpha, \beta)) \geq 1.$$

The hypersurface germ is normal and Gorenstein. The weighted blow-up of weight $(2, 5, 6, 7)$ has discrepancy

$$(2 + 5 + 6 + 7) - 1 - 20 = -1$$

over the hypersurface, so the germ is not canonical. Since a normal Gorenstein rational singularity is canonical, the germ is not rational; see [KM98]. Saito's rationality criterion then gives the reverse inequality $\tilde{\alpha}_P(X_8) \leq 1$; see [Sai93]. Consequently

$$\tilde{\alpha}_P(X_8(\alpha, \beta)) = 1.$$

Points of $L \setminus \{Q\}$. Let

$$Q = (0 : 0 : 0 : 0 : 1).$$

For $p \in L \setminus \{Q\}$, work in the chart $x_3 = 1$, write

$$u = x_0, \quad v = x_1, \quad y = x_2, \quad x_4 = q + \zeta,$$

and put $t = q + \zeta$. The local equation is

$$F_8 = v^2 y^2 + v^3 t^2 + u y^4 + u v + \alpha u v y t + \beta u^2 t^2.$$

The Hessian in (u, v) has determinant -1 . Solving for its critical section and substituting back gives the residual critical value

$$y^{10} \cdot \text{unit}.$$

The parametric splitting lemma therefore yields

$$F_8 \sim_{\text{an}} U_1^2 + U_2^2 + Y^{10}$$

with ζ as a smooth parameter. Hence the transverse type is A_9 , and

$$\tilde{\alpha}_p(X_8(\alpha, \beta)) = \frac{1}{2} + \frac{1}{2} + \frac{1}{10} = \frac{11}{10} \quad (p \in L \setminus \{Q\}).$$

The transverse Jacobian ideal is analytically equivalent to (U_1, U_2, Y^9) , giving generic transverse length 9 along L .

The distinguished point Q . In the chart $x_4 = 1$, with

$$(u, v, y, z) = (x_0, x_1, x_2, x_3),$$

the local equation is

$$f_Q = v^3 + u y^4 + v^2 y^2 z + u v z^3 + \alpha u v y z + \beta u^2 z.$$

It is weighted homogeneous of degree 15 for

$$\text{wt}(u, v, y, z) = (7, 5, 2, 1),$$

and its singularity is again non-isolated. The proper compact Newton faces are nondegenerate, and the full-face torus critical equations are again $A_8(r) = B_8(r) = 0$. Thus f_Q is Newton nondegenerate on the chosen open set. Its Newton distance is 1, so

$$\text{lct}_0(f_Q) = 1, \quad \tilde{\alpha}_Q(X_8(\alpha, \beta)) \geq 1.$$

The germ is normal and Gorenstein, while the weighted blow-up of weight $(7, 5, 2, 1)$ has discrepancy

$$(7 + 5 + 2 + 1) - 1 - 15 = -1.$$

By the same normal–Gorenstein argument as at P , the germ is not rational, and Saito’s criterion gives $\tilde{\alpha}_Q(X_8) \leq 1$. Hence

$$\tilde{\alpha}_Q(X_8(\alpha, \beta)) = 1.$$

Global minimal exponent. The preceding strata exhaust the singular locus. Therefore

$$\tilde{\alpha}(X_8(\alpha, \beta)) = \min \left\{ 1, \frac{7}{6}, 1, \frac{11}{10} \right\} = 1 = \frac{4+1}{5}.$$

Thus a general closed-orbit representative of Θ_8 has global minimal exponent equal to 1.

B.9 The component Θ_9

By Table 7,

$$\Theta_9 = \Phi_9 = \Phi_{22}, \quad k(9) = 9.$$

A general closed orbit in Θ_9 is represented by

$$\begin{aligned} F_9(\alpha, \beta, \gamma, \delta) := \phi_9^{\text{nf}}(\alpha, \beta, \gamma, \delta) &= x_1 x_2^4 + x_1^2 x_2^2 x_3 + \alpha x_1^3 x_3^2 + x_1^4 x_4 + x_0 x_2 x_3^3 \\ &\quad + \beta x_0 x_2^3 x_4 + \gamma x_0 x_1 x_2 x_3 x_4 + \delta x_0^2 x_3 x_4^2, \end{aligned}$$

where $(\alpha, \beta, \gamma, \delta) \in (\mathbb{C}^\times)^4$, and we set

$$X_9(\alpha, \beta, \gamma, \delta) := V(F_9(\alpha, \beta, \gamma, \delta)) \subset \mathbb{P}^4.$$

We work on a nonempty principal open set

$$D(\mathcal{R}_9) := \{(\alpha, \beta, \gamma, \delta) \in (\mathbb{C}^\times)^4 \mid \mathcal{R}_9(\alpha, \beta, \gamma, \delta) \neq 0\}$$

on which the Jacobian scheme in the chart $x_0 = 1$ is supported only at $P = (1 : 0 : 0 : 0 : 0)$, the germ at P is isolated, and the germ at the point Q defined below is Newton nondegenerate. Such an open set exists: at $(\alpha, \beta, \gamma, \delta) = (2, 3, 5, 7)$, exact Gröbner-basis and Newton-face computations verify these conditions.

Singular locus. On the hyperplane $x_0 = 0$, the Jacobian equations contain

$$x_1^4 = 0, \quad x_2^4 + 2x_1 x_2^2 x_3 + 3\alpha x_1^2 x_3^2 + 4x_1^3 x_4 = 0.$$

Thus $x_1 = x_2 = 0$, and the remaining equations vanish identically. This gives the line

$$L := V(x_0, x_1, x_2).$$

By the choice of $D(\mathcal{R}_9)$, the chart $x_0 = 1$ contributes only

$$P = (1 : 0 : 0 : 0 : 0).$$

Consequently

$$\text{Sing}(X_9(\alpha, \beta, \gamma, \delta))_{\text{red}} = L \sqcup \{P\}.$$

The distinguished point on L is

$$Q = (0 : 0 : 0 : 0 : 1).$$

Scheme-theoretically, the generic transverse Jacobian length along L is 2.

The isolated point P . In the chart $x_0 = 1$, with

$$(u, v, z, w) = (x_1, x_2, x_3, x_4),$$

the local equation is

$$f_P = uv^4 + u^2v^2z + \alpha u^3z^2 + u^4w + vz^3 + \beta v^3w + \gamma uvzw + \delta zw^2.$$

It is weighted homogeneous of degree 19 for

$$\text{wt}(u, v, z, w) = (3, 4, 5, 7).$$

Since the critical point is isolated on $D(\mathcal{R}_9)$, it defines an isolated hypersurface germ. With

$$(X_1, X_2, X_3, X_4) = (u, v, z, w),$$

the local equation is precisely the defining normal form of the family $\mathbf{S}_{96}^{Y94}(\alpha, \beta, \gamma, \delta)$. Hence, in the Yonemura-number notation of Section 8,

$$(X_9(\alpha, \beta, \gamma, \delta), P) \sim_{\text{an}} \mathbf{S}_{96}^{Y94}(\alpha, \beta, \gamma, \delta).$$

In particular, the isolated singularity has Milnor number 96 and Yonemura number 94. The weighted-homogeneous formula gives

$$\tilde{\alpha}_P(X_9(\alpha, \beta, \gamma, \delta)) = \frac{3 + 4 + 5 + 7}{19} = 1.$$

The local Jacobian algebra has length

$$\left(\frac{19}{3} - 1\right) \left(\frac{19}{4} - 1\right) \left(\frac{19}{5} - 1\right) \left(\frac{19}{7} - 1\right) = 96.$$

Points of $L \setminus \{Q\}$. Let $p \in L \setminus \{Q\}$. In the chart $x_3 = 1$, write

$$u = x_0, \quad v = x_1, \quad y = x_2, \quad x_4 = q + \zeta.$$

The local equation is

$$F_9 = vy^4 + v^2y^2 + \alpha v^3 + v^4(q + \zeta) + uy + \beta uy^3(q + \zeta) + \gamma uvy(q + \zeta) + \delta u^2(q + \zeta)^2.$$

The Hessian in (u, y) is nondegenerate, and $u = y = 0$ is its exact critical section. The parametric splitting lemma therefore gives

$$F_9 \sim_{\text{an}} U_1^2 + U_2^2 + V^3,$$

with ζ as a smooth parameter, because the residual critical value is

$$v^3(\alpha + v(q + \zeta)),$$

which is a unit times v^3 . Thus the transverse type is A_2 , and

$$\tilde{\alpha}_p(X_9(\alpha, \beta, \gamma, \delta)) = \frac{1}{2} + \frac{1}{2} + \frac{1}{3} = \frac{4}{3} \quad (p \in L \setminus \{Q\}).$$

The transverse Jacobian ideal is analytically equivalent to (U_1, U_2, V^2) , which gives the generic transverse length 2 along L .

The distinguished point Q . In the chart $x_4 = 1$, with

$$(u, v, y, z) = (x_0, x_1, x_2, x_3),$$

the local equation is

$$f_Q = vy^4 + v^2y^2z + \alpha v^3z^2 + v^4 + uyz^3 + \beta uy^3 + \gamma uvyz + \delta u^2z.$$

It is weighted homogeneous of degree 16 for

$$\text{wt}(u, v, y, z) = (7, 4, 3, 2),$$

but its singularity is non-isolated. All support exponents lie on the corresponding weight hyperplane, and the monomial $uvyz$ occurs. Hence the Newton distance is 1. Newton nondegeneracy on $D(\mathcal{R}_9)$ gives

$$\text{lct}_0(f_Q) = 1$$

by the Newton-polyhedron formula [How03]. Since $\text{lct}_0(f_Q) = \min\{\tilde{\alpha}_0(f_Q), 1\}$, it follows that

$$\tilde{\alpha}_Q(X_9(\alpha, \beta, \gamma, \delta)) \geq 1;$$

see [MP20]. Conversely, the weighted-order estimate gives

$$\tilde{\alpha}_Q(X_9(\alpha, \beta, \gamma, \delta)) \leq \frac{7 + 4 + 3 + 2}{16} = 1;$$

see [CDM25, Proposition 2.1]. Therefore

$$\tilde{\alpha}_Q(X_9(\alpha, \beta, \gamma, \delta)) = 1.$$

Global minimal exponent. The preceding strata exhaust the singular locus. Therefore

$$\tilde{\alpha}(X_9(\alpha, \beta, \gamma, \delta)) = \min\left\{1, 1, \frac{4}{3}\right\} = 1 = \frac{4 + 1}{5}.$$

Thus a general closed-orbit representative of Θ_9 has global minimal exponent equal to 1.

B.10 The component Θ_{10}

By Table 7,

$$\Theta_{10} = \Phi_{10} = \Phi_{34}, \quad k(10) = 10.$$

Put

$$B_\lambda(u, v) := uv(u - v)(u - \lambda v), \quad \lambda \in \mathbb{C} \setminus \{0, 1\},$$

and let $Q_{\mathbf{q}}$ and $R_{\mathbf{r}}$ be binary quadratic forms, with $\mathbf{q}, \mathbf{r} \in \mathbb{C}^3$ and $\beta \in \mathbb{C}$. A general closed orbit in Θ_{10} is represented by

$$\begin{aligned} F_{10}(\lambda, \mathbf{q}, \mathbf{r}, \beta) &:= \phi_{10}^{\text{nf}}(\lambda, \mathbf{q}, \mathbf{r}, \beta) = x_1^3 Q_{\mathbf{q}}(x_2, x_3) + x_1^4 x_4 + x_0 B_\lambda(x_2, x_3) \\ &\quad + x_0 x_1 x_4 R_{\mathbf{r}}(x_2, x_3) + \beta x_0 x_1^2 x_4^2 + x_0^2 x_4^3. \end{aligned}$$

We write

$$X_{10} := V(F_{10}) \subset \mathbb{P}^4$$

and work on the nonempty Zariski-open locus

$$\mathcal{R}_{10}(\lambda, \mathbf{q}, \mathbf{r}, \beta) \neq 0$$

on which $\text{Res}(B_\lambda, Q_{\mathbf{q}}) \neq 0$, the chart $x_0 = 1$ has no singular point other than $P = (1 : 0 : 0 : 0 : 0)$, and the germ at P is isolated.

Singular locus. On the hyperplane $x_0 = 0$, the equations

$$\frac{\partial F_{10}}{\partial x_4} = x_1^4 = 0, \quad \frac{\partial F_{10}}{\partial x_0} \Big|_{x_1=0} = B_\lambda(x_2, x_3) = 0$$

give

$$C_\lambda := V(x_0, x_1, B_\lambda(x_2, x_3)).$$

Since B_λ is square-free,

$$C_\lambda = L_0 \cup L_\infty \cup L_1 \cup L_\lambda,$$

where

$$\begin{aligned} L_0 &= V(x_0, x_1, x_2), & L_\infty &= V(x_0, x_1, x_3), \\ L_1 &= V(x_0, x_1, x_2 - x_3), & L_\lambda &= V(x_0, x_1, x_2 - \lambda x_3). \end{aligned}$$

These four lines meet only at

$$Q = (0 : 0 : 0 : 0 : 1).$$

The genericity assumption shows that the chart $x_0 = 1$ contributes only P . Hence

$$\text{Sing}(X_{10})_{\text{red}} = C_\lambda \sqcup \{P\}.$$

Scheme-theoretically, the generic transverse Jacobian length is 2 along each irreducible component of C_λ .

The isolated point P . In the chart $x_0 = 1$, with

$$(u, v, z, w) = (x_1, x_2, x_3, x_4),$$

the local equation is

$$f_P = u^3 Q_{\mathbf{q}}(v, z) + u^4 w + B_\lambda(v, z) + uw R_{\mathbf{r}}(v, z) + \beta u^2 w^2 + w^3.$$

It is weighted homogeneous of degree 12 for

$$\text{wt}(u, v, z, w) = (2, 3, 3, 4).$$

Since the critical point is isolated on the chosen open set, it defines an isolated hypersurface germ. With

$$(X_1, X_2, X_3, X_4) = (u, v, z, w),$$

the local equation is precisely the defining normal form of the family $\mathbf{S}_{90}^{Y2}(B_\lambda, Q_{\mathbf{q}}, R_{\mathbf{r}}, \beta)$. Hence, in the Yonemura-number notation of Section 8,

$$(X_{10}, P) \sim_{\text{an}} \mathbf{S}_{90}^{Y2}(B_\lambda, Q_{\mathbf{q}}, R_{\mathbf{r}}, \beta).$$

In particular, the isolated singularity has Milnor number 90 and Yonemura number 2. The weighted-homogeneous formula gives

$$\tilde{\alpha}_P(X_{10}) = \frac{2 + 3 + 3 + 4}{12} = 1.$$

Points of $C_\lambda \setminus \{Q\}$. Let $p \in C_\lambda \setminus \{Q\}$, and put locally

$$u = x_0, \quad v = x_1, \\ s = B_\lambda(x_2, x_3) + vx_4R_{\mathbf{r}}(x_2, x_3) + \beta v^2x_4^2 + ux_4^3.$$

Because B_λ is square-free, s is a coordinate transverse to the line through p . Moreover, $Q_{\mathbf{q}}(p) \neq 0$ by the resultant condition, and the defining equation has the exact form

$$F_{10} = us + v^3(Q_{\mathbf{q}}(x_2, x_3) + vx_4).$$

The factor in parentheses is a unit. Thus, with one smooth parameter along the line,

$$F_{10} \sim_{\text{an}} U_1^2 + U_2^2 + V^3.$$

The transverse type is A_2 , and therefore

$$\tilde{\alpha}_p(X_{10}) = \frac{1}{2} + \frac{1}{2} + \frac{1}{3} = \frac{4}{3} \quad (p \in C_\lambda \setminus \{Q\}).$$

The common point Q . In the chart $x_4 = 1$, with

$$(u, v, y, z) = (x_0, x_1, x_2, x_3),$$

the local equation is

$$f_Q = v^3Q_{\mathbf{q}}(y, z) + v^4 + uB_\lambda(y, z) + uvR_{\mathbf{r}}(y, z) + \beta uv^2 + u^2.$$

Set

$$A(v, y, z) := B_\lambda(y, z) + vR_{\mathbf{r}}(y, z) + \beta v^2, \quad u' := u + \frac{1}{2}A(v, y, z).$$

Then

$$f_Q = u'^2 + h(v, y, z),$$

where

$$h = v^3(Q_{\mathbf{q}}(y, z) + v) - \frac{1}{4}A(v, y, z)^2.$$

The germ h is weighted homogeneous of degree 8 for the weights $(2, 1, 1)$, so the corresponding weighted valuation gives

$$\text{lct}_0(h) \leq \frac{2+1+1}{8} = \frac{1}{2}.$$

Its nonzero singular locus is

$$\{v = 0, B_\lambda(y, z) = 0\} \setminus \{0\}.$$

At every point of this locus, $dB_\lambda \neq 0$ and $Q_{\mathbf{q}} + v$ is a unit. Taking $s = A(v, y, z)$ and a local cube root of $Q_{\mathbf{q}} + v$ gives the transverse equation

$$w^3 - \frac{1}{4}s^2,$$

whose log canonical threshold is $5/6$. Hence $(\mathbb{C}^3, \frac{1}{2}V(h))$ is log canonical away from the origin. The weighted-blow-up criterion therefore gives the reverse inequality and hence

$$\text{lct}_0(h) = \frac{1}{2};$$

see [Pae24, Proposition 3.2]. Since $\text{lct}_0(h) = \min\{\tilde{\alpha}_0(h), 1\}$, we obtain

$$\tilde{\alpha}_0(h) = \frac{1}{2};$$

see [MP20]. Thom–Sebastiani now yields

$$\tilde{\alpha}_Q(X_{10}) = \tilde{\alpha}_0(u'^2) + \tilde{\alpha}_0(h) = \frac{1}{2} + \frac{1}{2} = 1;$$

see [CDNM24, Theorem 1.2].

Global minimal exponent. The preceding strata exhaust the singular locus. Therefore

$$\tilde{\alpha}(X_{10}) = \min \left\{ 1, 1, \frac{4}{3} \right\} = 1 = \frac{4+1}{5}.$$

Thus a general closed-orbit representative of Θ_{10} has global minimal exponent equal to 1.

B.11 The component Θ_{11}

By Table 7,

$$\Theta_{11} = \Phi_{11} = \Phi_{25}, \quad k(11) = 11.$$

A general closed orbit in Θ_{11} is represented by

$$\begin{aligned} F_{11} &:= \phi_{11}^{\text{nf}}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta) \\ &= x_1 x_2^4 + \alpha x_1^2 x_2^2 x_3 + \beta x_1^3 x_3^2 + x_1^3 x_2 x_4 + \gamma x_0 x_2^2 x_3^2 + x_0 x_2^3 x_4 \\ &\quad + \delta x_0 x_1 x_3^3 + \varepsilon x_0 x_1 x_2 x_3 x_4 + \zeta x_0 x_1^2 x_4^2 + x_0^2 x_3 x_4^2, \end{aligned}$$

where $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta) \in (\mathbb{C}^\times)^6$, and we set

$$X_{11} := V(F_{11}) \subset \mathbb{P}^4.$$

We work on a nonempty principal open set

$$\mathcal{R}_{11}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta) \neq 0$$

on which there are no additional Jacobian components, the Newton principal parts at the points P, Q below are nondegenerate, and

$$\alpha\delta\gamma - \beta\gamma^2 - \delta^2 \neq 0.$$

Singular locus. The two lines

$$N := V(x_2, x_3, x_4), \quad L := V(x_0, x_1, x_2)$$

are contained in $\text{Sing}(X_{11})$, and they are disjoint. On the hyperplane $x_0 = 0$, the equation

$$\frac{\partial F_{11}}{\partial x_4} = x_1^3 x_2 = 0$$

shows that either $x_1 = 0$ or $x_2 = 0$. In the first case $\partial F_{11}/\partial x_1 = x_2^4$, so the point lies on L . In the second case, if $x_1 \neq 0$, the equations

$$\frac{\partial F_{11}}{\partial x_3} = 2\beta x_1^3 x_3, \quad \frac{\partial F_{11}}{\partial x_2} = x_1^3 x_4$$

give $x_3 = x_4 = 0$, hence the point lies on N . On the chart $x_0 = 1$, the genericity assumption excludes critical points away from N . Consequently

$$\text{Sing}(X_{11})_{\text{red}} = N \sqcup L.$$

There are no isolated singular points. The transverse type changes at

$$P = (1 : 0 : 0 : 0 : 0) \in N, \quad Q = (0 : 0 : 0 : 0 : 1) \in L.$$

The special point P . In the chart $x_0 = 1$, with $(u, v, z, w) = (x_1, x_2, x_3, x_4)$, the local equation is

$$f_P = uv^4 + \alpha u^2 v^2 z + \beta u^3 z^2 + u^3 vw + \gamma v^2 z^2 + v^3 w \\ + \delta u z^3 + \varepsilon uvzw + \zeta u^2 w^2 + zw^2.$$

It is weighted homogeneous of degree 14 for

$$\text{wt}(u, v, z, w) = (2, 3, 4, 5),$$

but the singularity is non-isolated. The support lies on the corresponding weight hyperplane and contains the monomial $uvzw$. Hence the diagonal meets the compact Newton boundary at $(1, 1, 1, 1)$, and Newton nondegeneracy gives

$$\text{lct}_0(f_P) = 1$$

by the Newton-polyhedron formula [How03]. Thus $\tilde{\alpha}_P(X_{11}) \geq 1$ by $\text{lct}_0(f_P) = \min\{1, \tilde{\alpha}_0(f_P)\}$; see [MP20]. The weighted-order estimate gives the reverse inequality

$$\tilde{\alpha}_P(X_{11}) \leq \frac{2+3+4+5}{14} = 1;$$

see [CDM25, Proposition 2.1]. Therefore

$$\tilde{\alpha}_P(X_{11}) = 1.$$

Points of $N \setminus \{P\}$. For $p \in N \setminus \{P\}$, one has $x_1(p) \neq 0$. In coordinates (y, z, w) transverse to N , the quadratic part is

$$\beta x_1(p)^3 z^2 + x_1(p)^3 yw + \zeta x_0(p) x_1(p)^2 w^2.$$

Its determinant is $-\beta x_1(p)^9/4$, so it has rank 3. The parametric splitting lemma therefore gives the transverse ordinary double point

$$U_1^2 + U_2^2 + U_3^2 = 0.$$

Hence

$$\tilde{\alpha}_p(X_{11}) = \frac{3}{2} \quad (p \in N \setminus \{P\}).$$

In particular, the Jacobian scheme is generically reduced along N .

The special point Q . In the chart $x_4 = 1$, with $(u, v, y, z) = (x_0, x_1, x_2, x_3)$, the local equation is

$$f_Q = vy^4 + \alpha v^2 y^2 z + \beta v^3 z^2 + v^3 y + \gamma uy^2 z^2 + uy^3 \\ + \delta uvz^3 + \varepsilon uvyz + \zeta uv^2 + u^2 z.$$

It is weighted homogeneous of degree 11 for

$$\text{wt}(u, v, y, z) = (5, 3, 2, 1),$$

and is again non-isolated. As at P , the support lies on the weight hyperplane and contains $uvyz$. Newton nondegeneracy therefore gives

$$\text{lct}_0(f_Q) = 1, \quad \tilde{\alpha}_Q(X_{11}) \geq 1.$$

The weighted-order estimate yields

$$\tilde{\alpha}_Q(X_{11}) \leq \frac{5+3+2+1}{11} = 1.$$

Thus

$$\tilde{\alpha}_Q(X_{11}) = 1.$$

Points of $L \setminus \{Q\}$. Let $p \in L \setminus \{Q\}$. In the chart $x_3 = 1$, write

$$u = x_0, \quad v = x_1, \quad y = x_2, \quad x_4 = q + \tau.$$

The transverse quadratic part in (u, v) is

$$\delta uv + q^2 u^2,$$

which has rank 2. After the parametric splitting, the residual series is

$$y^6(c + yh(y, \tau)), \quad c = \frac{\gamma(\alpha\delta\gamma - \beta\gamma^2 - \delta^2)}{\delta^3}.$$

The genericity condition gives $c \neq 0$. A holomorphic change of the y -coordinate therefore reduces the germ, up to the smooth parameter τ , to

$$U_1^2 + U_2^2 + Y^6 = 0.$$

Consequently

$$\tilde{\alpha}_p(X_{11}) = \frac{1}{2} + \frac{1}{2} + \frac{1}{6} = \frac{7}{6} \quad (p \in L \setminus \{Q\}).$$

The generic transverse Jacobian length along L is 5.

Global minimal exponent. The preceding strata exhaust the singular locus. Therefore

$$\tilde{\alpha}(X_{11}) = \min \left\{ 1, \frac{3}{2}, 1, \frac{7}{6} \right\} = 1 = \frac{4+1}{5}.$$

Thus a general closed-orbit representative of Θ_{11} has global minimal exponent equal to 1.

B.12 The component Θ_{12}

By Table 7,

$$\Theta_{12} = \Phi_{12} = \Phi_{17}, \quad k(12) = 12.$$

A general closed orbit in Θ_{12} is represented by

$$\begin{aligned} F_{12} &:= \phi_{12}^{\text{nf}}(\alpha, \beta, \gamma, \delta, \varepsilon) \\ &= x_1 x_2^4 + \alpha x_1^2 x_2 x_3^2 + \beta x_1^2 x_2^2 x_4 + x_1^3 x_4^2 + x_0 x_2^3 x_3 \\ &\quad + \gamma x_0 x_1 x_3^3 + \delta x_0 x_1 x_2 x_3 x_4 + \varepsilon x_0^2 x_3^2 x_4 + x_0^2 x_2 x_4^2, \end{aligned}$$

and we set

$$X_{12} := V(F_{12}) \subset \mathbb{P}^4.$$

We work on the nonempty Zariski-open set

$$(\alpha, \beta, \gamma, \delta, \varepsilon) \in (\mathbb{C}^\times)^5, \quad \beta^2 \neq 4, \quad \alpha \neq \gamma, \quad \mathcal{R}_{12}(\alpha, \beta, \gamma, \delta, \varepsilon) \neq 0,$$

where $\mathcal{R}_{12}$ excludes additional affine critical points away from the line component below.

Singular locus. The two disjoint lines

$$N := V(x_2, x_3, x_4), \quad L := V(x_0, x_1, x_2)$$

are contained in $\text{Sing}(X_{12})$. On the hyperplane $x_0 = 0$, if $x_1 = 0$, then $\partial F_{12}/\partial x_1 = x_2^4$, so the point lies on L . If $x_1 \neq 0$, set $x_1 = 1$. The equations

$$\frac{\partial F_{12}}{\partial x_3} = 2\alpha x_2 x_3, \quad \frac{\partial F_{12}}{\partial x_4} = \beta x_2^2 + 2x_4$$

give $x_2 x_3 = 0$ and $x_4 = -\beta x_2^2/2$. If $x_2 = 0$, then $\partial F_{12}/\partial x_2 = \alpha x_3^2$, so $x_3 = 0$. If $x_3 = 0$, then

$$\frac{\partial F_{12}}{\partial x_2} = (4 - \beta^2)x_2^3,$$

so $x_2 = 0$. Thus the point lies on N . On the chart $x_0 = 1$, the condition $\mathcal{R}_{12} \neq 0$ excludes critical points away from N . Consequently

$$\text{Sing}(X_{12})_{\text{red}} = N \sqcup L.$$

There are no isolated singular points. The transverse type changes at

$$P = (1 : 0 : 0 : 0 : 0) \in N, \quad Q = (0 : 0 : 0 : 0 : 1) \in L.$$

The special point P . In the chart $x_0 = 1$, with $(u, v, z, w) = (x_1, x_2, x_3, x_4)$, the local equation is

$$\begin{aligned} f_P = & uv^4 + \alpha u^2 v z^2 + \beta u^2 v^2 w + u^3 w^2 + v^3 z \\ & + \gamma u z^3 + \delta u v z w + \varepsilon z^2 w + v w^2. \end{aligned}$$

It is weighted homogeneous of degree 13 for

$$\text{wt}(u, v, z, w) = (1, 3, 4, 5),$$

but the singularity is non-isolated. The proper compact faces are nondegenerate under $\beta^2 \neq 4$ and $\alpha \neq \gamma$, while $\mathcal{R}_{12} \neq 0$ excludes a torus critical point of the full compact face. Since $\delta \neq 0$, the monomial $uvzw$ occurs, and the diagonal meets the Newton boundary at $(1, 1, 1, 1)$. The Newton-polyhedron formula [How03] therefore gives

$$\text{lct}_0(f_P) = 1.$$

The relation between the log canonical threshold and the minimal exponent [MP20] yields $\tilde{\alpha}_P(X_{12}) \geq 1$. On the other hand, the weighted-order estimate [CDM25, Proposition 2.1] gives

$$\tilde{\alpha}_P(X_{12}) \leq \frac{1 + 3 + 4 + 5}{13} = 1.$$

Hence

$$\tilde{\alpha}_P(X_{12}) = 1.$$

Points of $N \setminus \{P\}$. For $p \in N \setminus \{P\}$, work in the chart $x_1 = 1$, with $u = x_0$ the coordinate along N and $(y, z, w) = (x_2, x_3, x_4)$ transverse to it. The local equation is

$$\begin{aligned} f_N = & (1 + u^2 y) w^2 + (\beta y^2 + \delta u y z + \varepsilon u^2 z^2) w \\ & + y^4 + \alpha y z^2 + u y^3 z + \gamma u z^3. \end{aligned}$$

After completing the square in w , the weighted degree-eight transverse principal part, for $\text{wt}(y, z) = (2, 3)$, is

$$\alpha y z^2 + \left(1 - \frac{\beta^2}{4}\right) y^4.$$

This is a nondegenerate D_5 principal part. Thus the transverse germ is analytically equivalent to

$$W^2 + YZ^2 + Y^4 = 0,$$

and the smooth parameter along N does not affect the minimal exponent. Therefore

$$\tilde{\alpha}_p(X_{12}) = \frac{4 + 2 + 3}{8} = \frac{9}{8} \quad (p \in N \setminus \{P\}).$$

The generic transverse Jacobian length along N is 5.

The special point Q . In the chart $x_4 = 1$, with $(u, v, y, z) = (x_0, x_1, x_2, x_3)$, the local equation is

$$\begin{aligned} f_Q = & \, vy^4 + \alpha v^2 y z^2 + \beta v^2 y^2 + v^3 + uy^3 z + \gamma uvz^3 \\ & + \delta uvyz + \varepsilon u^2 z^2 + u^2 y. \end{aligned}$$

It is weighted homogeneous of degree 12 for

$$\text{wt}(u, v, y, z) = (5, 4, 2, 1).$$

The compact-face calculation is the same as at P . Since $uvyz$ occurs, Newton nondegeneracy gives

$$\text{lct}_0(f_Q) = 1, \quad \tilde{\alpha}_Q(X_{12}) \geq 1.$$

The weighted-order estimate gives the reverse inequality

$$\tilde{\alpha}_Q(X_{12}) \leq \frac{5 + 4 + 2 + 1}{12} = 1.$$

Thus

$$\tilde{\alpha}_Q(X_{12}) = 1.$$

Points of $L \setminus \{Q\}$. Let $p \in L \setminus \{Q\}$. In the chart $x_3 = 1$, write

$$u = x_0, \quad v = x_1, \quad y = x_2, \quad x_4 = q + \tau,$$

where τ is the coordinate along L . The transverse quadratic part in (u, v) is

$$\gamma uv + \varepsilon q u^2,$$

which has rank 2. After the parametric splitting lemma, the residual one-variable germ is

$$h(y) = cy^7 + O(y^8), \quad c = \frac{\alpha - \gamma}{\gamma^2} \neq 0.$$

Thus, up to the smooth parameter τ , the germ is analytically equivalent to

$$U_1^2 + U_2^2 + Y^7 = 0.$$

The transverse type is A_6 , and hence

$$\tilde{\alpha}_p(X_{12}) = \frac{1}{2} + \frac{1}{2} + \frac{1}{7} = \frac{8}{7} \quad (p \in L \setminus \{Q\}).$$

The generic transverse Jacobian length along L is 6.

Global minimal exponent. The preceding strata exhaust the singular locus. Therefore

$$\tilde{\alpha}(X_{12}) = \min \left\{ 1, \frac{9}{8}, 1, \frac{8}{7} \right\} = 1 = \frac{4+1}{5}.$$

Thus a general closed-orbit representative of Θ_{12} has global minimal exponent equal to 1.

B.13 The component Θ_{13}

By Table 7,

$$\Theta_{13} = \Phi_{13}, \quad k(13) = 13.$$

A general closed orbit in Θ_{13} is represented by

$$\begin{aligned} F_{13} &:= \phi_{13}^{\text{nf}}(\alpha, \beta, \gamma, \delta) \\ &= x_2^5 + \alpha x_1 x_2^3 x_3 + \beta x_1^2 x_2 x_3^2 + x_1^3 x_4^2 + x_0 x_2^3 x_4 \\ &\quad + \gamma x_0 x_1 x_2 x_3 x_4 + x_0^2 x_3^3 + \delta x_0^2 x_2 x_4^2, \end{aligned}$$

and we set

$$X_{13} := V(F_{13}) \subset \mathbb{P}^4.$$

We work on the nonempty Zariski-open set

$$(\alpha, \beta, \gamma, \delta) \in (\mathbb{C}^\times)^4, \quad \alpha^2 \neq 4\beta, \quad \mathcal{R}_{13}(\alpha, \beta, \gamma, \delta) \neq 0,$$

where $\mathcal{R}_{13}$ excludes additional affine critical points and ensures Newton nondegeneracy at the two special points below.

Singular locus. The two lines

$$N := V(x_2, x_3, x_4), \quad L := V(x_0, x_1, x_2)$$

are contained in $\text{Sing}(X_{13})$, and they are disjoint. On the hyperplane $x_0 = 0$, if $x_1 = 0$, then $\partial F_{13}/\partial x_2 = 5x_2^4$, so the point lies on L . If $x_1 \neq 0$, set $x_1 = 1$. Then

$$\frac{\partial F_{13}}{\partial x_4} = 2x_4, \quad \frac{\partial F_{13}}{\partial x_3} = x_2(\alpha x_2^2 + 2\beta x_3).$$

Thus $x_4 = 0$. If $x_2 = 0$, the equation $\partial F_{13}/\partial x_2 = \beta x_3^2$ gives $x_3 = 0$, hence the point lies on N . If $x_2 \neq 0$, then $x_3 = -\alpha x_2^2/(2\beta)$, and substitution into $\partial F_{13}/\partial x_2$ gives

$$\frac{5(4\beta - \alpha^2)}{4\beta} x_2^4 = 0,$$

which contradicts $\alpha^2 \neq 4\beta$. On the chart $x_0 = 1$, the condition $\mathcal{R}_{13} \neq 0$ excludes critical points away from N . Consequently

$$\text{Sing}(X_{13})_{\text{red}} = N \sqcup L.$$

There are no isolated singular points. Since the singular locus has codimension two in the hypersurface threefold, X_{13} is normal. The transverse type changes at

$$P = (1 : 0 : 0 : 0 : 0) \in N, \quad Q = (0 : 0 : 0 : 0 : 1) \in L.$$

The special point P . In the chart $x_0 = 1$, with $(u, v, z, w) = (x_1, x_2, x_3, x_4)$, the local equation is

$$f_P = v^5 + \alpha uv^3 z + \beta u^2 v z^2 + u^3 w^2 + v^3 w + \gamma uvz w + z^3 + \delta v w^2.$$

It is weighted homogeneous of degree 15 for

$$\text{wt}(u, v, z, w) = (1, 3, 5, 6),$$

but its singularity is non-isolated. The term $uvzw$ places the diagonal point $(1, 1, 1, 1)$ on the Newton boundary. Hence compact-face Newton nondegeneracy and the Newton-polyhedron formula [How03] give

$$\text{lct}_0(f_P) = 1, \quad \tilde{\alpha}_P(X_{13}) \geq 1;$$

see also [MP20]. For the weighted blow-up with weights $(1, 3, 5, 6)$, the exceptional divisor E_P satisfies

$$A_{\mathbb{C}^4}(E_P) = 15 = \text{ord}_{E_P}(f_P).$$

Thus the pair is not purely log terminal. By inversion of adjunction and the fact that X_{13} is normal Gorenstein, the hypersurface singularity at P is not rational; see [KM98, Chapter 5]. Saito's rationality criterion [Sai93, Theorem 0.4] therefore gives the reverse inequality. Consequently

$$\tilde{\alpha}_P(X_{13}) = 1.$$

Points of $N \setminus \{P\}$. For $p \in N \setminus \{P\}$, work in the chart $x_1 = 1$, with $u = x_0$ the coordinate along N and $(y, z, w) = (x_2, x_3, x_4)$ transverse to it. The local equation is

$$f_N = (1 + \delta u^2 y) w^2 + (u y^3 + \gamma u y z) w + y^5 + \alpha y^3 z + \beta y z^2 + u^2 z^3.$$

After completing the square in w , make the analytic changes

$$y_1 = \beta y + u^2 z, \quad z_1 = z + \frac{\alpha}{2\beta^3} y_1^2.$$

Up to multiplication by a unit and nonzero rescaling, the transverse principal part is

$$W^2 + y_1 z_1^2 + c y_1^5, \quad c = \frac{4\beta - \alpha^2}{4\beta^6} \neq 0.$$

All remaining terms have larger transverse weight, so the parametric semiquasihomogeneous reduction gives the transverse D_6 germ. Hence

$$\tilde{\alpha}_p(X_{13}) = \frac{1}{2} + \frac{1}{5} + \frac{2}{5} = \frac{11}{10} \quad (p \in N \setminus \{P\}).$$

The special point Q . In the chart $x_4 = 1$, with $(u, v, y, z) = (x_0, x_1, x_2, x_3)$, the local equation is

$$f_Q = y^5 + \alpha v y^3 z + \beta v^2 y z^2 + v^3 + u y^3 + \gamma u v y z + u^2 z^3 + \delta u^2 y.$$

It is weighted homogeneous of degree 15 for

$$\text{wt}(u, v, y, z) = (6, 5, 3, 1).$$

As at P , compact-face Newton nondegeneracy and the monomial $uvyz$ give

$$\text{lct}_0(f_Q) = 1, \quad \tilde{\alpha}_Q(X_{13}) \geq 1.$$

For the weighted blow-up with weights $(6, 5, 3, 1)$, one has

$$A_{\mathbb{C}^4}(E_Q) = 15 = \text{ord}_{E_Q}(f_Q).$$

The same inversion-of-adjunction and rationality argument therefore yields

$$\tilde{\alpha}_Q(X_{13}) = 1.$$

Points of $L \setminus \{Q\}$. Let $p \in L \setminus \{Q\}$. In the chart $x_3 = 1$, write

$$u = x_0, \quad v = x_1, \quad y = x_2, \quad x_4 = q + \tau,$$

where τ is the coordinate along L , and put $w = q + \tau$. The local equation is

$$\begin{aligned} f_L &= (1 + \delta y w^2) u^2 + (y^3 w + \gamma v y w) u \\ &\quad + y^5 + \alpha v y^3 + \beta v^2 y + v^3 w^2. \end{aligned}$$

After completing the square in u , make the analytic changes

$$y_1 = \beta y + w^2 v, \quad v_1 = v + \frac{\alpha}{2\beta^3} y_1^2.$$

The transverse principal part is again

$$U^2 + y_1 v_1^2 + c y_1^5, \quad c = \frac{4\beta - \alpha^2}{4\beta^6} \neq 0.$$

Thus the transverse type is D_6 , and

$$\tilde{\alpha}_p(X_{13}) = \frac{11}{10} \quad (p \in L \setminus \{Q\}).$$

The transverse Jacobian algebra of D_6 has length 6; hence the saturated Jacobian scheme has generic transverse multiplicity 6 along both lines.

Global minimal exponent. The preceding strata exhaust the singular locus. Therefore

$$\tilde{\alpha}(X_{13}) = \min \left\{ 1, \frac{11}{10}, 1, \frac{11}{10} \right\} = 1 = \frac{4+1}{5}.$$

Thus a general closed-orbit representative of Θ_{13} has global minimal exponent equal to 1.

B.14 The component Θ_{14}

By Table 7,

$$\Theta_{14} = \Phi_{14}, \quad k(14) = 14.$$

A general closed orbit in Θ_{14} is represented by

$$\begin{aligned} F_{14} &:= \phi_{14}^{\text{nf}}(\alpha, \beta, \gamma, \delta) \\ &= x_2^5 + \alpha x_1 x_2^3 x_3 + \beta x_1^2 x_2 x_3^2 + x_1^4 x_4 + x_0 x_3^4 \\ &\quad + \gamma x_0 x_2^3 x_4 + \delta x_0 x_1 x_2 x_3 x_4 + x_0^2 x_2 x_4^2, \end{aligned}$$

and we set

$$X_{14} := V(F_{14}) \subset \mathbb{P}^4.$$

We work on the nonempty Zariski-open set

$$(\alpha, \beta, \gamma, \delta) \in (\mathbb{C}^\times)^4, \quad \mathcal{R}_{14}(\alpha, \beta, \gamma, \delta) \neq 0,$$

where $\mathcal{R}_{14}$ excludes additional critical points and ensures that the two local weighted-homogeneous germs below are isolated.

Singular locus. For the saturated Jacobian ideal one has

$$\sqrt{J(F_{14}) : (x_0, x_1, x_2, x_3, x_4)^\infty} = (x_1, x_2, x_3, x_4) \cap (x_0, x_1, x_2, x_3).$$

Consequently

$$\text{Sing}(X_{14})_{\text{red}} = \{P, Q\},$$

where

$$P = (1 : 0 : 0 : 0 : 0), \quad Q = (0 : 0 : 0 : 0 : 1).$$

Thus X_{14} has precisely two isolated singular points. The local Jacobian algebra has length 102 at each point.

The point P . In the chart $x_0 = 1$, with $(u, v, z, w) = (x_1, x_2, x_3, x_4)$, the local equation is

$$f_P = v^5 + \alpha uv^3 z + \beta u^2 v z^2 + u^4 w + z^4 + \gamma v^3 w + \delta uvz w + vw^2.$$

This is weighted homogeneous of degree 20 for

$$\text{wt}(u, v, z, w) = (3, 4, 5, 8).$$

It is isolated on the chosen parameter open set. With

$$(X_1, X_2, X_3, X_4) = (u, v, z, w),$$

the local equation is precisely the defining normal form of the family $\mathbf{S}_{102}^{\text{Y62}}(\alpha, \beta, \gamma, \delta)$. Hence, in the Yonemura-number notation of Section 8,

$$(X_{14}, P) \sim_{\text{an}} \mathbf{S}_{102}^{\text{Y62}}(\alpha, \beta, \gamma, \delta).$$

In particular, the isolated singularity has Milnor number 102 and Yonemura number 62. The weighted-homogeneous formula gives

$$\tilde{\alpha}_P(X_{14}) = \frac{3 + 4 + 5 + 8}{20} = 1.$$

The point Q . In the chart $x_4 = 1$, with $(u, v, y, z) = (x_0, x_1, x_2, x_3)$, the local equation is

$$f_Q = y^5 + \alpha v y^3 z + \beta v^2 y z^2 + v^4 + u z^4 + \gamma u y^3 + \delta u v y z + u^2 y.$$

It is weighted homogeneous of degree 20 for

$$\text{wt}(u, v, y, z) = (8, 5, 4, 3).$$

With

$$(X_1, X_2, X_3, X_4) = (z, y, v, u),$$

the local equation is precisely the same defining normal form as at P , namely $\mathbf{S}_{102}^{\text{Y62}}(\alpha, \beta, \gamma, \delta)$. Hence

$$(X_{14}, Q) \sim_{\text{an}} \mathbf{S}_{102}^{\text{Y62}}(\alpha, \beta, \gamma, \delta).$$

Thus the isolated singularity at Q also has Milnor number 102 and Yonemura number 62, and

$$\tilde{\alpha}_Q(X_{14}) = \frac{8 + 5 + 4 + 3}{20} = 1.$$

Global minimal exponent. The two points above exhaust the singular locus. Therefore

$$\tilde{\alpha}(X_{14}) = \min\{\tilde{\alpha}_P(X_{14}), \tilde{\alpha}_Q(X_{14})\} = 1 = \frac{4+1}{5}.$$

Thus a general closed-orbit representative of Θ_{14} has global minimal exponent equal to 1.

B.15 The component Θ_{15}

By Table 7,

$$\Theta_{15} = \Phi_{15} = \Phi_{30}, \quad k(15) = 15.$$

A general closed orbit in Θ_{15} is represented by

$$\begin{aligned} F_{15} &:= \phi_{15}^{\text{nf}}(Q, R, B) \\ &= x_1 x_2^4 + x_1^2 x_2^2 x_3 + x_1^3 Q(x_3, x_4) + x_0 x_2^3 x_4 \\ &\quad + x_0 x_1 x_2 R(x_3, x_4) + x_0^2 B(x_3, x_4), \end{aligned}$$

where

$$Q = \gamma x_3^2 + \beta x_3 x_4 + x_4^2, \quad R = \rho_0 x_3^2 + \rho_1 x_3 x_4 + \rho_2 x_4^2,$$

and

$$B = \sigma_0 x_3^3 + \sigma_1 x_3^2 x_4 + \sigma_2 x_3 x_4^2 + \sigma_3 x_4^3.$$

Set

$$X_{15} := V(F_{15}) \subset \mathbb{P}^4.$$

We work on the nonempty Zariski-open set defined by

$$\beta^2 - 4\gamma \neq 0, \quad 4\gamma - \beta^2 - 1 \neq 0, \quad \text{Disc}(B) \neq 0, \quad \mathcal{R}_{15}(Q, R, B) \neq 0,$$

where $\mathcal{R}_{15}$ excludes additional critical points and transverse Jacobian degenerations.

Singular locus. The two lines

$$N := V(x_2, x_3, x_4), \quad L := V(x_0, x_1, x_2)$$

are contained in $\text{Sing}(X_{15})$, and they are disjoint. The Jacobian equations on $x_0 = 0$, together with the affine calculation on $x_0 = 1$ and the condition $\mathcal{R}_{15} \neq 0$, give

$$\text{Sing}(X_{15})_{\text{red}} = N \sqcup L.$$

There are no isolated singular points. The generic transverse Jacobian multiplicities along N and L are respectively 3 and 9; in the curve-cycle notation,

$$\Gamma_{15} = C_{1,3}^{(2)} + C_{1,9}^{(1)}.$$

Since the singular locus has codimension two in the hypersurface threefold, X_{15} is normal. The distinguished point on N is

$$P = (1 : 0 : 0 : 0 : 0).$$

The special point P . In the chart $x_0 = 1$, with $(u, y, z, w) = (x_1, x_2, x_3, x_4)$, the local equation is

$$f_P = uy^4 + u^2y^2z + u^3Q(z, w) + y^3w + uyR(z, w) + B(z, w).$$

It is weighted homogeneous of degree 9 for

$$\text{wt}(u, y, z, w) = (1, 2, 3, 3), \quad 1 + 2 + 3 + 3 = 9.$$

Put $\Delta = 4\gamma - \beta^2$. The projectivized weighted cone is smooth away from $[1 : 0 : 0 : 0]$. On the chart $u = 1$, the quadratic form $Q(z, w)$ is nondegenerate because $\Delta \neq 0$, and the splitting lemma reduces the germ at this point to

$$Z^2 + W^2 + Y^4 = 0;$$

the coefficient of Y^4 is nonzero precisely because $\Delta - 1 = 4\gamma - \beta^2 - 1 \neq 0$. Thus the projectivized cone has only an A_3 singularity and is canonical. The weighted-cone criterion and inversion of adjunction [KM98] therefore give

$$\text{lct}_P(f_P) = 1, \quad \tilde{\alpha}_P(X_{15}) \geq 1.$$

For the weighted blow-up with weights $(1, 2, 3, 3)$, the exceptional divisor has hypersurface discrepancy

$$(1 + 2 + 3 + 3) - 1 - 9 = -1.$$

A rational Gorenstein hypersurface singularity is canonical [KM98, Chapter 5]; hence this germ is not rational. Saito's rationality criterion [Sai93, Theorem 0.4] therefore gives

$$\tilde{\alpha}_P(X_{15}) = 1.$$

Points of $N \setminus \{P\}$. For $p \in N \setminus \{P\}$, work in the chart $x_1 = 1$, with $u = x_0$ the coordinate along N and $(y, z, w) = (x_2, x_3, x_4)$ transverse to it. The local equation is

$$f_N = y^4 + y^2z + Q(z, w) + uy^3w + uyR(z, w) + u^2B(z, w).$$

The parametric splitting lemma separates the two Morse variables (z, w) . The residual one-variable term is

$$\frac{\Delta - 1}{\Delta} y^4 + O(y^5),$$

whose leading coefficient is nonzero. Hence, up to the smooth parameter along N , the transverse germ is

$$Z^2 + W^2 + Y^4 = 0.$$

Thus the transverse type is A_3 , and Thom–Sebastiani additivity gives

$$\tilde{\alpha}_p(X_{15}) = \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{5}{4} \quad (p \in N \setminus \{P\}).$$

Points of L . Give the normal variables (x_0, x_1, x_2) along L weights

$$\text{wt}_L(x_0, x_1, x_2) = (3, 2, 1).$$

Every monomial of F_{15} has normal weighted degree 6. The projectivized weighted normal cone

$$\mathcal{S}_L := V(F_{15}) \subset \mathbb{P}_L(3, 2, 1)$$

is smooth on the associated weighted-projective stack: a singular point of $\mathcal{S}_L$ would lift to a singular point of X_{15} outside L , whereas $\text{Sing}(X_{15})_{\text{red}} = N \sqcup L$ and $N \cap L = \emptyset$. Since

$$3 + 2 + 1 = 6,$$

the relative weighted-cone criterion and inversion of adjunction [KM98] yield

$$\mathrm{lct}_p(F_{15}) = 1, \quad \tilde{\alpha}_p(X_{15}) \geq 1 \quad (p \in L).$$

The exceptional divisor of the relative weighted blow-up has discrepancy

$$(3 + 2 + 1) - 1 - 6 = -1$$

and dominates L . Hence X_{15} is noncanonical, and therefore nonrational, at every point of L . Saito's criterion [Sai93, Theorem 0.4] gives

$$\tilde{\alpha}_p(X_{15}) = 1 \quad (p \in L).$$

Global minimal exponent. The preceding strata exhaust the singular locus. Therefore

$$\tilde{\alpha}(X_{15}) = \min \left\{ 1, \frac{5}{4}, 1 \right\} = 1 = \frac{4+1}{5}.$$

Thus a general closed-orbit representative of Θ_{15} has global minimal exponent equal to 1.

B.16 The component Θ_{16}

By Table 7,

$$\Theta_{16} = \Phi_{19}, \quad k(16) = 19.$$

A general closed orbit in Θ_{16} is represented by

$$\begin{aligned} F_{16} &:= \phi_{19}^{\mathrm{nf}}(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \xi, \theta) \\ &= x_2^5 + \alpha x_1 x_2^3 x_3 + \beta x_1^2 x_2 x_3^2 + \gamma x_1^2 x_2^2 x_4 + x_1^3 x_3 x_4 \\ &\quad + \delta x_0 x_2^2 x_3^2 + \varepsilon x_0 x_2^3 x_4 + x_0 x_1 x_3^3 + \zeta x_0 x_1 x_2 x_3 x_4 \\ &\quad + \xi x_0 x_1^2 x_4^2 + x_0^2 x_3^2 x_4 + \theta x_0^2 x_2 x_4^2, \end{aligned}$$

and we set

$$X_{16} := V(F_{16}) \subset \mathbb{P}^4.$$

Put

$$\rho_N := 1 - \alpha\gamma + \beta\gamma^2, \quad \rho_L := 1 - \alpha\delta + \beta\delta^2.$$

We work on the nonempty Zariski-open set

$$(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \xi, \theta) \in (\mathbb{C}^\times)^8, \quad \mathcal{R}_{19} \neq 0,$$

where $\mathcal{R}_{19}$ excludes residual projective critical points, imposes $\rho_N \rho_L \neq 0$, and ensures compact-face Newton nondegeneracy at the two special points below.

Singular locus. The two lines

$$N := V(x_2, x_3, x_4), \quad L := V(x_0, x_1, x_2)$$

are contained in $\mathrm{Sing}(X_{16})$, and $N \cap L = \emptyset$. The Jacobian calculation on $x_0 = 0$, followed by the affine calculation on $x_0 = 1$, gives

$$\mathrm{Sing}(X_{16})_{\mathrm{red}} = N \sqcup L.$$

There are no isolated singular points. The generic transverse Jacobian multiplicity is 4 along each line; equivalently, the curve part of the saturated Jacobian scheme has cycle

$$\Gamma_{16} = C_{1,4}^{(2)} + C_{1,4}^{(2)}.$$

The transverse type changes at

$$P = (1 : 0 : 0 : 0 : 0) \in N, \quad Q = (0 : 0 : 0 : 0 : 1) \in L.$$

The special points P and Q . In the chart $x_0 = 1$, with $(u, v, z, w) = (x_1, x_2, x_3, x_4)$, the local equation at P is

$$f_P = v^5 + \alpha uv^3 z + \beta u^2 v z^2 + \gamma u^2 v^2 w + u^3 z w + \delta v^2 z^2 + \varepsilon v^3 w \\ + uz^3 + \zeta uvz w + \xi u^2 w^2 + z^2 w + \theta v w^2.$$

It is weighted homogeneous of degree 10 for

$$\text{wt}(u, v, z, w) = (1, 2, 3, 4).$$

Although the germ is non-isolated, the monomial $uvzw$ places the diagonal point $(1, 1, 1, 1)$ on its Newton boundary. Newton nondegeneracy and the Newton-polyhedron formula therefore give

$$\text{lct}_0(f_P) = 1, \quad \tilde{\alpha}_P(X_{16}) \geq 1;$$

see [How03, Theorems 11 and 12] and [MP20]. The weighted upper bound [CDM25, Proposition 2.1] gives

$$\tilde{\alpha}_P(X_{16}) \leq \frac{1 + 2 + 3 + 4}{10} = 1.$$

Hence

$$\tilde{\alpha}_P(X_{16}) = 1.$$

In the chart $x_4 = 1$, with $(u, v, y, z) = (x_0, x_1, x_2, x_3)$, the local equation at Q is

$$f_Q = y^5 + \alpha v y^3 z + \beta v^2 y z^2 + \gamma v^2 y^2 + v^3 z + \delta u y^2 z^2 + \varepsilon u y^3 \\ + uvz^3 + \zeta u v y z + \xi u v^2 + u^2 z^2 + \theta u^2 y.$$

It is weighted homogeneous of degree 10 for

$$\text{wt}(u, v, y, z) = (4, 3, 2, 1).$$

The same Newton argument and weighted upper bound yield

$$\tilde{\alpha}_Q(X_{16}) = 1.$$

General points of the two lines. Let $p \in N \setminus \{P\}$. In the chart $x_1 = 1$, write

$$x_0 = c + s, \quad (y, z, w) = (x_2, x_3, x_4),$$

where s is the coordinate along N . The quadratic part in (z, w) is

$$zw + \xi c w^2,$$

which has rank 2. Solving the transverse critical equations and applying the parametric splitting lemma gives

$$z = -\gamma y^2 + O(y^3), \quad w = (2\beta\gamma - \alpha)y^3 + O(y^4),$$

and the residual function

$$g_N(s, y) = \rho_N y^5 + O(y^6).$$

Since $\rho_N \neq 0$, a fifth-root coordinate change reduces the transverse germ to

$$Z^2 + W^2 + Y^5 = 0.$$

Thus the transverse type is A_4 , and

$$\tilde{\alpha}_p(X_{16}) = \frac{1}{2} + \frac{1}{2} + \frac{1}{5} = \frac{6}{5} \quad (p \in N \setminus \{P\}).$$

Let $p \in L \setminus \{Q\}$ with $x_4(p) \neq 0$. In the chart $x_4 = 1$, write

$$x_3 = c + s, \quad (u, v, y) = (x_0, x_1, x_2), \quad c \neq 0.$$

The quadratic part in (u, v) is $c^2u^2 + c^3uv$, hence has rank 2. The transverse critical equations begin with

$$v = -\delta(c + s)^{-1}y^2 + O(y^3), \quad u = (2\beta\delta - \alpha)(c + s)^{-2}y^3 + O(y^4),$$

and the splitting residual is

$$g_L(s, y) = \rho_L y^5 + O(y^6).$$

At the remaining point $(0 : 0 : 0 : 1 : 0)$, the chart $x_3 = 1$ gives the quadratic term uv and the same residual. Since $\rho_L \neq 0$, the transverse germ is again A_4 . Therefore

$$\tilde{\alpha}_p(X_{16}) = \frac{6}{5} \quad (p \in L \setminus \{Q\}).$$

Global minimal exponent. The preceding strata exhaust the singular locus. Consequently

$$\tilde{\alpha}(X_{16}) = \min \left\{ 1, \frac{6}{5}, 1, \frac{6}{5} \right\} = 1 = \frac{4+1}{5}.$$

Thus a general closed-orbit representative of Θ_{16} has global minimal exponent equal to 1.

B.17 The component Θ_{17}

By Table 7,

$$\Theta_{17} = \Phi_{20} = \Phi_{31}, \quad k(17) = 20.$$

A general closed orbit in Θ_{17} is represented by

$$\begin{aligned} F_{17} &:= \phi_{20}^{\text{nf}}(\lambda, \mu, \mathbf{c}, \mathbf{q}, \mathbf{m}) \\ &= x_0 B_{\lambda, \mu}(x_2, x_3) + x_1^2 C_{\mathbf{c}}(x_2, x_3) + x_1^3 x_2 x_4 \\ &\quad + x_0 x_1 x_4 Q_{\mathbf{q}}(x_2, x_3) + x_0 x_1^2 x_4^2 + x_0^2 x_4^2 M_{\mathbf{m}}(x_2, x_3), \end{aligned}$$

where

$$\begin{aligned} B_{\lambda, \mu}(x_2, x_3) &= x_3(x_2 - x_3)(x_2 - \lambda x_3)(x_2 - \mu x_3), \\ C_{\mathbf{c}} &= c_0 x_2^3 + c_1 x_2^2 x_3 + c_2 x_2 x_3^2 + c_3 x_3^3, \quad Q_{\mathbf{q}} = q_0 x_2^2 + q_1 x_2 x_3 + q_2 x_3^2, \end{aligned}$$

and

$$M_{\mathbf{m}} = m_0 x_2 + m_1 x_3.$$

Set

$$X_{17} := V(F_{17}) \subset \mathbb{P}^4.$$

We work on the nonempty Zariski-open set defined by

$$\lambda\mu(\lambda - 1)(\mu - 1)(\lambda - \mu) \neq 0, \quad \text{Disc}(C_{\mathbf{c}}) \neq 0,$$

$$\text{Res}(B_{\lambda, \mu}, C_{\mathbf{c}}) \neq 0, \quad \text{Res}(B_{\lambda, \mu}, M_{\mathbf{m}}) \neq 0, \quad c_3 \neq 0,$$

and

$$\mathcal{R}_{20}(\lambda, \mu, \mathbf{c}, \mathbf{q}, \mathbf{m}) \neq 0,$$

where $\mathcal{R}_{20}$ excludes additional projective critical points and local degenerations, and ensures compact-face Newton nondegeneracy at the two special points below.

Singular locus. Put

$$N := V(x_2, x_3, x_4)$$

and

$$C_{\lambda,\mu} := V(x_0, x_1, B_{\lambda,\mu}(x_2, x_3)).$$

The latter is the reduced union of four lines

$$C_{\lambda,\mu} = L_\infty \cup L_1 \cup L_\lambda \cup L_\mu,$$

where

$$\begin{aligned} L_\infty &= V(x_0, x_1, x_3), & L_1 &= V(x_0, x_1, x_2 - x_3), \\ L_\lambda &= V(x_0, x_1, x_2 - \lambda x_3), & L_\mu &= V(x_0, x_1, x_2 - \mu x_3). \end{aligned}$$

All four lines meet at

$$R = (0 : 0 : 0 : 0 : 1),$$

whereas $N \cap C_{\lambda,\mu} = \emptyset$. Direct substitution shows that $N \cup C_{\lambda,\mu} \subset \text{Sing}(X_{17})$, and saturation of the Jacobian ideal leaves no residual component on the chosen open set. Hence

$$\text{Sing}(X_{17})_{\text{red}} = N \sqcup C_{\lambda,\mu}.$$

There are no isolated singular points. Since the singular locus has codimension two in the hypersurface threefold, X_{17} is normal. Scheme-theoretically, the generic transverse Jacobian multiplicity is 2 along N and 1 along each of the four lines in $C_{\lambda,\mu}$; thus

$$\Gamma_{17} = C_{1,2}^{(2)} + 4C_{1,1}^{(3)}.$$

The distinguished point on N is

$$P = (1 : 0 : 0 : 0 : 0).$$

The special point P . In the chart $x_0 = 1$, with $(u, v, z, w) = (x_1, x_2, x_3, x_4)$, the local equation is

$$f_P = B_{\lambda,\mu}(v, z) + u^2 C_{\mathbf{c}}(v, z) + u^3 vw + uw Q_{\mathbf{q}}(v, z) + u^2 w^2 + w^2 M_{\mathbf{m}}(v, z).$$

It is weighted homogeneous of degree 8 for

$$\text{wt}(u, v, z, w) = (1, 2, 2, 3).$$

The monomials $u^2 w^2$, $v^3 z$, and z^4 place the diagonal point $(1, 1, 1, 1)$ on the Newton boundary. Compact-face Newton nondegeneracy therefore gives

$$\text{lct}_0(f_P) = 1, \quad \tilde{\alpha}_P(X_{17}) \geq 1;$$

see [How03, MP20]. For the weighted blow-up with weights $(1, 2, 2, 3)$, one has

$$A_{\mathbb{C}^4}(E) = 8 = \text{ord}_E(f_P).$$

The weighted-blow-up upper-bound argument, using inversion of adjunction and Saito's rationality criterion [KM98, Sai93], gives $\tilde{\alpha}_P(X_{17}) \leq 1$. Hence

$$\tilde{\alpha}_P(X_{17}) = 1.$$

Points of $N \setminus \{P\}$. Let $p \in N \setminus \{P\}$. In the chart $x_1 = 1$, write

$$x_0 = c + s, \quad (y, z, w) = (x_2, x_3, x_4),$$

where s is the coordinate along N . The quadratic part in the transverse variables (y, w) is

$$yw + cw^2,$$

and has rank 2. The parametric splitting lemma reduces the remaining one-variable germ to

$$c_3 z^3 + O(z^4).$$

Since $c_3 \neq 0$, the transverse local normal form is

$$Y^2 + W^2 + Z^3 = 0.$$

Thus the transverse type is A_2 , and

$$\tilde{\alpha}_p(X_{17}) = \frac{1}{2} + \frac{1}{2} + \frac{1}{3} = \frac{4}{3} \quad (p \in N \setminus \{P\}).$$

Points of $C_{\lambda, \mu} \setminus \{R\}$. At a point $p \in C_{\lambda, \mu} \setminus \{R\}$, exactly one factor $\ell(x_2, x_3)$ of $B_{\lambda, \mu}$ vanishes. Taking

$$u = x_0, \quad v = x_1, \quad s = \ell(x_2, x_3)$$

as transverse coordinates, the quadratic part has the form

$$us \cdot \text{unit} + v^2 \cdot \text{unit} + \text{terms in } u^2, uv.$$

The coefficient of v^2 is nonzero because $\text{Res}(B_{\lambda, \mu}, C_{\mathbf{c}}) \neq 0$, so this quadratic form has rank 3. Hence the transverse local normal form is

$$U^2 + V^2 + S^2 = 0,$$

and therefore

$$\tilde{\alpha}_p(X_{17}) = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2} \quad (p \in C_{\lambda, \mu} \setminus \{R\}).$$

The common point R . In the chart $x_4 = 1$, with $(u, v, y, z) = (x_0, x_1, x_2, x_3)$, the local equation is

$$f_R = uB_{\lambda, \mu}(y, z) + v^2C_{\mathbf{c}}(y, z) + v^3y + uvQ_{\mathbf{q}}(y, z) + uv^2 + u^2M_{\mathbf{m}}(y, z).$$

It is weighted homogeneous of degree 7 for

$$\text{wt}(u, v, y, z) = (3, 2, 1, 1).$$

The monomials uv^2 , uy^3z , and uz^4 place the diagonal point on the Newton boundary. Thus Newton nondegeneracy gives

$$\text{lct}_0(f_R) = 1, \quad \tilde{\alpha}_R(X_{17}) \geq 1.$$

For the weighted blow-up with weights $(3, 2, 1, 1)$,

$$A_{\mathbb{C}^4}(E) = 7 = \text{ord}_E(f_R),$$

so the same weighted-blow-up argument gives the reverse inequality. Hence

$$\tilde{\alpha}_R(X_{17}) = 1.$$

Global minimal exponent. The preceding strata exhaust the singular locus. Consequently

$$\tilde{\alpha}(X_{17}) = \min \left\{ 1, \frac{4}{3}, 1, \frac{3}{2} \right\} = 1 = \frac{4+1}{5}.$$

Thus a general closed-orbit representative of Θ_{17} has global minimal exponent equal to 1.

B.18 The component Θ_{18}

By Table 7,

$$\Theta_{18} = \Phi_{23} = \Phi_{38}, \quad k(18) = 23.$$

A general closed orbit in Θ_{18} is represented by

$$F_{18} := \phi_{23}^{\text{nf}}(B_4) = x_0 B_4(x_1, x_2, x_3, x_4),$$

where $[B_4] \in \mathbb{P}(\text{Sym}^4 \langle x_1, x_2, x_3, x_4 \rangle)$ is stable. For the singularity calculation, we work on the nonempty Zariski-open locus on which

$$V(B_4) \subset \mathbb{P}^3$$

is smooth, and set

$$X_{18} := V(F_{18}) \subset \mathbb{P}^4.$$

Singular locus. Put

$$S := V(x_0, B_4) \subset \mathbb{P}^4, \quad P := (1 : 0 : 0 : 0 : 0).$$

Then S is a smooth quartic surface. Since

$$\frac{\partial F_{18}}{\partial x_0} = B_4, \quad \frac{\partial F_{18}}{\partial x_i} = x_0 \frac{\partial B_4}{\partial x_i} \quad (1 \leq i \leq 4),$$

smoothness of $V(B_4)$ and Euler's identity give

$$\text{Sing}(X_{18})_{\text{red}} = S \sqcup \{P\}.$$

More precisely, if

$$Q := \left(\frac{\partial B_4}{\partial x_1}, \dots, \frac{\partial B_4}{\partial x_4} \right),$$

then

$$J(F_{18}) = (x_0, B_4) \cap Q,$$

and this ideal is already saturated. Thus

$$\text{Sing}(X_{18}) = S \sqcup Z_P,$$

where S is reduced and Z_P is a punctual complete intersection supported at P . Its local length is

$$\text{length}(Z_P) = 3^4 = 81.$$

Points of the surface component S . Let $p \in S$. Since $dB_4(p) \neq 0$, there are analytic coordinates (u, v, z_1, z_2) centered at p such that

$$u = x_0, \quad v = B_4.$$

The local equation of X_{18} is therefore

$$uv = 0.$$

This is a simple normal-crossing germ, whose reduced Bernstein–Sato polynomial is $s + 1$. Hence

$$\tilde{\alpha}_p(X_{18}) = 1 \quad (p \in S).$$

The isolated point P . In the chart $x_0 = 1$, the local equation at P is

$$B_4(y_1, y_2, y_3, y_4) = 0.$$

The origin is an isolated critical point because $V(B_4) \subset \mathbb{P}^3$ is smooth. The germ is weighted homogeneous of degree 4 for

$$\text{wt}(y_1, y_2, y_3, y_4) = (1, 1, 1, 1).$$

With

$$(X_1, X_2, X_3, X_4) = (y_1, y_2, y_3, y_4),$$

the local equation is precisely the defining normal form of the family $\mathbf{S}_{81}^{\text{Y1}}(B_4)$. Hence, in the Yonemura-number notation of Section 8,

$$(X_{18}, P) \sim_{\text{an}} \mathbf{S}_{81}^{\text{Y1}}(B_4).$$

In particular, the isolated singularity has Milnor number 81 and Yonemura number 1. Consequently, the weighted-homogeneous formula gives [Sai71, Sai94]

$$\tilde{\alpha}_P(X_{18}) = \frac{1 + 1 + 1 + 1}{4} = 1.$$

Global minimal exponent. The preceding strata exhaust the singular locus. Therefore

$$\tilde{\alpha}(X_{18}) = \min\{1, 1\} = 1 = \frac{4 + 1}{5}.$$

Thus a general closed-orbit representative of Θ_{18} has global minimal exponent equal to 1.

B.19 The component Θ_{19}

By Table 7,

$$\Theta_{19} = \Phi_{29} = \Phi_{35}, \quad k(19) = 29.$$

A general closed orbit in Θ_{19} is represented by

$$\begin{aligned} F_{19} &:= \phi_{29}^{\text{nf}}(A_4, C_4, Q_{2,2}) \\ &= x_3 A_4(x_1, x_2) + x_4 C_4(x_1, x_2) + x_0 Q_{2,2}(x_1, x_2; x_3, x_4) \\ &\quad + x_0^2 x_3 x_4 (x_3 - x_4), \end{aligned}$$

where

$$A_4, C_4 \in \text{Sym}^4 \langle x_1, x_2 \rangle, \quad Q_{2,2} \in \text{Sym}^2 \langle x_1, x_2 \rangle \otimes \text{Sym}^2 \langle x_3, x_4 \rangle.$$

Set

$$X_{19} := V(F_{19}) \subset \mathbb{P}^4.$$

We work on the nonempty Zariski-open set

$$\text{Res}(A_4, C_4) \neq 0, \quad \mathcal{R}_{29}(A_4, C_4, Q_{2,2}) \neq 0,$$

where $\mathcal{R}_{29}$ imposes the isolatedness of the affine critical point below, smoothness of the relative weighted quartic used in the weighted blow-up, and the generic socle condition along the singular line.

Singular locus. Put

$$L := V(x_0, x_1, x_2), \quad P := (1 : 0 : 0 : 0 : 0).$$

Every term of F_{19} , and every first derivative, vanishes on L , so $L \subset \text{Sing}(X_{19})$. On the hyperplane $x_0 = 0$, the equations

$$\frac{\partial F_{19}}{\partial x_3} = A_4(x_1, x_2), \quad \frac{\partial F_{19}}{\partial x_4} = C_4(x_1, x_2)$$

force $(x_1, x_2) = (0, 0)$, because $\text{Res}(A_4, C_4) \neq 0$. Thus the only singular points on $x_0 = 0$ lie on L .

In the chart $x_0 = 1$, with

$$(u, v, y, z) = (x_1, x_2, x_3, x_4),$$

the local equation is

$$f_P = yA_4(u, v) + zC_4(u, v) + Q_{2,2}(u, v; y, z) + yz(y - z).$$

Its critical locus is zero-dimensional on the chosen open set and is invariant under the positive-weight action with weights $(1, 1, 2, 2)$. Hence it contains no nonzero point, and the origin is its only critical point. Consequently

$$\text{Sing}(X_{19})_{\text{red}} = L \sqcup \{P\}.$$

The saturated Jacobian scheme is nonreduced along L . At its generic point, splitting the nonzero quadratic term in the x_0 -direction leaves a square-free binary quartic G_4 . Its Milnor algebra has length 9 and a one-dimensional degree-four socle. The line-direction derivative $\dot{G}_4$ is nonzero in that socle, so the generic transverse Jacobian length is 8; the generic transverse Hessian rank is 1. Thus the curve part is recorded as

$$\Gamma_{19} = C_{1,8}^{(1)}.$$

The isolated point P . The germ f_P is weighted homogeneous of degree 6 for

$$\text{wt}(u, v, y, z) = (1, 1, 2, 2),$$

and is isolated on the chosen open set. Put

$$B_3^{(29)}(X_3, X_4) := X_3X_4(X_3 - X_4).$$

With

$$(X_1, X_2, X_3, X_4) = (u, v, y, z),$$

the local equation is precisely the defining normal form of the family

$$\mathbf{S}_{100}^{\text{Y3}}(A_4, C_4, Q_{2,2}, B_3^{(29)}).$$

Hence, in the Yonemura-number notation of Section 8,

$$(X_{19}, P) \sim_{\text{an}} \mathbf{S}_{100}^{\text{Y3}}(A_4, C_4, Q_{2,2}, B_3^{(29)}).$$

In particular, the isolated singularity has Milnor number 100 and Yonemura number 3. The weighted-homogeneous formula therefore gives [Sai71, Sai94]

$$\tilde{\alpha}_P(X_{19}) = \frac{1 + 1 + 2 + 2}{6} = 1.$$

Its local Jacobi algebra has length

$$\left(\frac{6}{1} - 1\right)^2 \left(\frac{6}{2} - 1\right)^2 = 5^2 \cdot 2^2 = 100,$$

in agreement with the subscript of $\mathbf{S}_{100}^{\text{Y3}}$.

Points of the line L . Let U_L be a sufficiently small neighborhood of a point of L , and let

$$\pi : \tilde{U}_L \longrightarrow U_L$$

be the weighted blow-up of the smooth center $L \cap U_L$ with normal weights

$$\text{wt}(x_0, x_1, x_2) = (2, 1, 1).$$

Every monomial of F_{19} has normal weighted order 4. Hence, writing E for the exceptional divisor and $\tilde{X}_{19}$ for the proper transform,

$$\pi^* X_{19} = \tilde{X}_{19} + 4E, \quad K_{\tilde{U}_L} = \pi^* K_{U_L} + 3E$$

on the local index-one covers. Therefore

$$K_{\tilde{U}_L} + \tilde{X}_{19} + E = \pi^*(K_{U_L} + X_{19}).$$

The intersection $\tilde{X}_{19} \cap E$ is the relative weighted quartic

$$\begin{aligned} x_3 A_4(y_1, y_2) + x_4 C_4(y_1, y_2) + \xi Q_{2,2}(y_1, y_2; x_3, x_4) \\ + \xi^2 x_3 x_4 (x_3 - x_4) = 0 \end{aligned}$$

in the $\mathbb{P}(2, 1, 1)$ -bundle over L . By the condition $\mathcal{R}_{29} \neq 0$, it is smooth on the local index-one covers. Moreover, all terms of F_{19} have weighted order exactly 4, so $\tilde{X}_{19}$ is smooth near E and meets E transversely. Thus $(U_L, X_{19}|_{U_L})$ is log canonical along L ; compare [KM98]. Using [Sai94, MP20]

$$\text{lct}_p(F_{19}) = \min\{\tilde{\alpha}_p(F_{19}), 1\},$$

we obtain

$$\tilde{\alpha}_p(X_{19}) \geq 1 \quad (p \in L).$$

Global minimal exponent. The preceding strata exhaust the singular locus. Since $\tilde{\alpha}_P(X_{19}) = 1$ and $\tilde{\alpha}_p(X_{19}) \geq 1$ for every $p \in L$,

$$\tilde{\alpha}(X_{19}) = 1 = \frac{4+1}{5}.$$

Thus a general closed-orbit representative of Θ_{19} has global minimal exponent equal to 1.

B.20 The component Θ_{20}

By Table 7,

$$\Theta_{20} = \Phi_{32} = \Phi_{37}, \quad k(20) = 32.$$

Put

$$V := \langle x_0, x_1 \rangle, \quad U := \langle x_2, x_3, x_4 \rangle.$$

A general closed orbit in Θ_{20} is represented by

$$\begin{aligned} F_{20} &:= \phi_{32}^{\text{nf}}(C_0, C_1, C_2) \\ &= x_0^2 C_0(x_2, x_3, x_4) + x_0 x_1 C_1(x_2, x_3, x_4) + x_1^2 C_2(x_2, x_3, x_4), \end{aligned}$$

where $C_0, C_1, C_2 \in \text{Sym}^3 U$. Set

$$X_{20} := V(F_{20}) \subset \mathbb{P}^4.$$

We work on the nonempty Zariski-open locus denoted by $\mathcal{R}_{32}(C_0, C_1, C_2) \neq 0$. On this locus, the bidegree $(2, 3)$ divisor

$$D_{20} := V(F_{20}) \subset \mathbb{P}(V) \times \mathbb{P}(U)$$

is smooth, the discriminant curve

$$\Delta_{\Pi} := V(x_0, x_1, C_1^2 - 4C_0 C_2) \subset \Pi := V(x_0, x_1)$$

is smooth and reduced, and the conic family of ternary cubics parametrized by $N := V(x_2, x_3, x_4)$ meets the ternary-cubic discriminant transversely and only in its smooth nodal locus.

Singular locus. Every monomial of F_{20} has degree two in (x_0, x_1) and degree three in (x_2, x_3, x_4) . Hence

$$N \cup \Pi \subset \text{Sing}(X_{20}).$$

Away from $N \cup \Pi$, the two bihomogeneous Euler identities identify the projective Jacobian equations of F_{20} with the singularity equations of D_{20} . Since D_{20} is smooth, there are no further singular points. Therefore

$$\text{Sing}(X_{20})_{\text{red}} = N \sqcup \Pi, \quad N \simeq \mathbb{P}^1, \quad \Pi \simeq \mathbb{P}^2.$$

In particular, there are no isolated singular points.

The plane component is generically reduced. At a general point of $\Pi \setminus \Delta_\Pi$, its transverse Jacobian ideal is (u, v) , whereas at a point of Δ_Π the local coordinates below give

$$(u, zv, v^2).$$

Along N , the generic transverse Jacobian algebra has length 7 and the generic transverse Hessian rank is 0. Indeed, for the corresponding smooth ternary cubic C_t , the Jacobian algebra of C_t has length 8; the parameter derivative represents a nonzero socle class, and quotienting by it reduces the length to 7.

Points of the plane Π . For

$$p = (0 : 0 : v_2 : v_3 : v_4) \in \Pi,$$

the quadratic form transverse to Π is

$$Q_p(u, v) = u^2 C_0(p) + uv C_1(p) + v^2 C_2(p).$$

If $p \notin \Delta_\Pi$, then Q_p has rank two. The parameterized splitting lemma gives analytic coordinates in which

$$F_{20} = u'^2 + v'^2,$$

with two smooth parameters. Hence

$$\tilde{\alpha}_p(X_{20}) = 1 \quad (p \in \Pi \setminus \Delta_\Pi).$$

Let $p \in \Delta_\Pi$. Smoothness of Δ_Π implies that Q_p has rank one. Completing the square and taking $z = -(C_1^2 - 4C_0C_2)/(4C_0)$ as a local coordinate transverse to Δ_Π , one obtains the exact analytic normal form

$$F_{20} = u'^2 + zv^2,$$

with one further smooth parameter. The monomial Bernstein–Sato formula and Thom–Sebastiani give [CDNM24]

$$\tilde{\alpha}(u'^2) = \frac{1}{2}, \quad \tilde{\alpha}(zv^2) = \frac{1}{2},$$

and therefore

$$\tilde{\alpha}_p(X_{20}) = 1 \quad (p \in \Delta_\Pi).$$

Consequently,

$$\tilde{\alpha}_p(X_{20}) = 1 \quad (p \in \Pi).$$

Points of the line N . Let

$$q = (u_0 : u_1 : 0 : 0 : 0) \in N,$$

and write

$$C_q = u_0^2 C_0 + u_0 u_1 C_1 + u_1^2 C_2.$$

In a local coordinate t on N , the germ at q has the cone-family form

$$f(t, y) = C_t(y), \quad y = (y_2, y_3, y_4),$$

where each C_t is homogeneous of degree three. By the genericity assumptions, the projective family

$$\mathcal{C} = \{(t, [y]) \mid C_t(y) = 0\}$$

is smooth along the fiber over q : this is immediate for a smooth cubic fiber, and at a nodal fiber it follows from transversality to the ternary-cubic discriminant.

Blow up the t -axis $Z = \{y_2 = y_3 = y_4 = 0\}$. If E is the exceptional divisor and $\tilde{X}_{20}$ is the strict transform, then

$$\operatorname{div}(f \circ \pi) = \tilde{X}_{20} + 3E, \quad K_{\operatorname{Bl}_Z \mathbb{C}^4 / \mathbb{C}^4} = 2E.$$

The strict transform is smooth and meets E transversely, so this is a log resolution near q . Hence

$$\operatorname{lct}_q(f) = \min \left\{ 1, \frac{2+1}{3} \right\} = 1,$$

and thus $\tilde{\alpha}_q(X_{20}) \geq 1$ by $\operatorname{lct}_q(f) = \min\{\tilde{\alpha}_q(f), 1\}$ [MP20].

On the other hand, the divisor $E \cap \tilde{X}_{20}$ has discrepancy $2 - 3 = -1$ over X_{20} , so the germ is not canonical. Since its singular locus has codimension two in the hypersurface threefold, the germ is normal; as a hypersurface, it is Gorenstein. Since a normal Gorenstein rational singularity is canonical [KM98], the germ is not rational. Saito's criterion then gives $\tilde{\alpha}_q(X_{20}) \leq 1$ [Sai93]. Therefore

$$\tilde{\alpha}_q(X_{20}) = 1 \quad (q \in N).$$

Global minimal exponent. The two strata above exhaust the singular locus, and the local minimal exponent is 1 at every point of both components. Consequently,

$$\tilde{\alpha}(X_{20}) = 1 = \frac{4+1}{5}.$$

Thus a general closed-orbit representative of Θ_{20} has global minimal exponent equal to 1.

B.21 The component Θ_{21}

By Table 7,

$$\Theta_{21} = \Phi_{36}, \quad k(21) = 36.$$

Put

$$U := \langle x_1, x_2, x_3 \rangle.$$

A general closed orbit in Θ_{21} is represented by

$$\begin{aligned} F_{21} &:= \phi_{36}^{\operatorname{nf}}(B_5, C_3) \\ &= B_5(x_1, x_2, x_3) + x_0 x_4 C_3(x_1, x_2, x_3) + x_0^2 x_3 x_4^2, \end{aligned}$$

where

$$B_5 \in \operatorname{Sym}^5 U, \quad C_3 \in \operatorname{Sym}^3 U.$$

Set

$$X_{21} := V(F_{21}) \subset \mathbb{P}^4.$$

We work on the nonempty Zariski-open locus

$$\text{Disc}(B_5) \neq 0, \quad \text{Disc}(C_3) \neq 0, \quad \mathcal{R}_{36}(B_5, C_3) \neq 0.$$

Here the last condition ensures that the weighted-homogeneous germ appearing below is isolated and that no additional projective singular point occurs.

Singular locus. The two points

$$P_0 = (1 : 0 : 0 : 0 : 0), \quad P_\infty = (0 : 0 : 0 : 0 : 1)$$

are singular. On the plane $x_0 = x_4 = 0$, the Jacobian equations reduce to

$$\frac{\partial B_5}{\partial x_1} = \frac{\partial B_5}{\partial x_2} = \frac{\partial B_5}{\partial x_3} = 0.$$

Since $V(B_5) \subset \mathbb{P}(U) \simeq \mathbb{P}^2$ is smooth, these equations have no projective solution. On the chart $x_0 = 1$, the affine equation is

$$g(X_1, X_2, X_3, X_4) = B_5(X_1, X_2, X_3) + X_4 C_3(X_1, X_2, X_3) + X_4^2 X_3.$$

The condition $\mathcal{R}_{36} \neq 0$ says that the origin is the only critical point of g . The same equation occurs on the chart $x_4 = 1$. Consequently,

$$\text{Sing}(X_{21})_{\text{red}} = \{P_0, P_\infty\}.$$

The saturated Jacobian scheme is zero-dimensional, and its local length is 96 at each point.

The two isolated singularities. At P_0 , on the chart $x_0 = 1$, take

$$(X_1, X_2, X_3, X_4) = (x_1, x_2, x_3, x_4),$$

and at P_∞ , on the chart $x_4 = 1$, take

$$(X_1, X_2, X_3, X_4) = (x_1, x_2, x_3, x_0).$$

In either coordinate system, the local equation is

$$g = B_5(X_1, X_2, X_3) + X_4 C_3(X_1, X_2, X_3) + X_4^2 X_3.$$

This is an isolated weighted-homogeneous germ of degree 5 for

$$\text{wt}(X_1, X_2, X_3, X_4) = (1, 1, 1, 2).$$

Taking $L_1 = X_3$, the local equation is precisely the defining normal form of the family $\mathbf{S}_{96}^{\text{Y21}}(B_5, C_3, X_3)$. Hence, in the Yonemura-number notation of Section 8,

$$\begin{aligned} (X_{21}, P_0) &\sim_{\text{an}} \mathbf{S}_{96}^{\text{Y21}}(B_5, C_3, X_3), \\ (X_{21}, P_\infty) &\sim_{\text{an}} \mathbf{S}_{96}^{\text{Y21}}(B_5, C_3, X_3). \end{aligned}$$

In particular, each isolated singularity has Milnor number 96 and Yonemura number 21. Therefore the weighted-homogeneous formula gives

$$\tilde{\alpha}_{P_0}(X_{21}) = \tilde{\alpha}_{P_\infty}(X_{21}) = \frac{1 + 1 + 1 + 2}{5} = 1.$$

Global minimal exponent. The two points above exhaust the singular locus. Hence

$$\tilde{\alpha}(X_{21}) = \min\{\tilde{\alpha}_{P_0}(X_{21}), \tilde{\alpha}_{P_\infty}(X_{21})\} = 1 = \frac{4+1}{5}.$$

Thus a general closed-orbit representative of Θ_{21} has global minimal exponent equal to 1.

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Declaration of AI-Assisted Tools

The author acknowledges the use of AI-assisted technologies in the preparation of this manuscript. Specifically, these tools were utilized as aids in computing the normal forms of polystable polynomials, and to assist with the explicit algebraic computations required to derive the local normal forms of isolated singularities. In particular, regarding the latter computations, when standard computer algebra systems encountered infinite loops during the reduction processes, AI tools were employed to execute the explicit symbolic substitutions and reductions dictated by the holomorphic splitting lemma.

Furthermore, these tools were used as auxiliary aids for drafting computational scripts, and for English language editing and stylistic formatting.

All intermediate outputs generated by the AI were rigorously verified by the author. The mathematical reasoning, theoretical proofs, and derivation of final conclusions were conducted independently by the author. The author takes full responsibility for all mathematical content, results, and wording presented in this final manuscript.

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