Boundary of the Moduli Space of Stable Cubic Fivefolds

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Abstract

We give a description of the strictly semistable boundary of the GIT moduli space of cubic fivefolds. Specifically, we classify all 21 boundary components and determine the *codimension-one wall adjacency* relation among them in the sense of Kirwan wall-crossing. The resulting adjacency graph is computed explicitly: it has 21 vertices and 56 edges (in particular, it has no isolated vertices). For each boundary component, we compute the minimal exponent of a general member and prove that it equals 7/3, which is the critical value appearing in the stability criterion for cubic fivefolds.

Our approach is explicit: we exhibit 21 normal forms representing the general polystable member in each boundary component and analyze the corresponding singularities. In particular, we identify exactly six analytic types of isolated singularities occurring on the boundary—*extremal cubic fivefold singularities*—each given by a quasi-homogeneous normal form (in some cases with moduli) and each having minimal exponent 7/3. We also treat the non-isolated cases: we classify the possible positive-dimensional singular loci—lines, conics, complete intersections of two quadrics, and an elliptic quartic curve—and compute their minimal exponents.

Introduction

This paper studies the Geometric Invariant Theory (GIT) compactification of the moduli space of cubic fivefolds, i.e. cubic hypersurfaces $X \subset \mathbb{P}^6$, via the natural action of SL(7) on

$$\mathbb{P}\big(\mathrm{Sym}^3\mathbb{C}^7\big) \quad \text{and the quotient} \quad \mathbb{P}\big(\mathrm{Sym}^3\mathbb{C}^7\big)//\mathrm{SL}(7).$$

Our goal is to describe not only the stable locus but also the boundary contributed by strictly semistable cubics, i.e. the image of the strictly semistable locus in the GIT quotient. While the case of cubic fourfolds has been intensively studied from both classical and modern perspectives, a comparably explicit description of the GIT boundary for cubic fivefolds has not been fully developed: in particular, one would like concrete normal forms for closed orbits, an explicit description of the boundary families, and a systematic analysis of the singularities that govern the geometry of degenerations.

From the GIT viewpoint, the boundary arises from the strictly semistable locus. Understanding it naturally breaks into three tasks: (i) identify polystable points (closed orbits) and construct explicit normal forms representing them; (ii)

understand the boundary families they generate, together with the corresponding parameter spaces and dimensions; and (iii) extract geometrically meaningful information by analyzing the singular loci $\mathrm{Sing}(X)$ of these boundary representatives. The main contribution of this paper is to carry out these steps as explicitly as possible for cubic fivefolds.

Irreducible components and normal forms. We first construct explicit polystable (closed-orbit) normal forms and the corresponding boundary families Φ_k $(1 \le k \le 21)$, together with their parameter spaces and dimensions. We then prove the non-inclusion among these 21 families (Theorem 3.5); in particular, the families Φ_k are pairwise distinct and give exactly the 21 irreducible components of the strictly semistable locus; see Theorem A. In any case, the construction provides a concrete coordinate-level model for the boundary representatives and their degenerations.

Singular loci on the boundary. Next, we determine the singular locus of a general closed-orbit representative X_k in each boundary family. We show that $\mathrm{Sing}(X_k)$ consists of a one-dimensional part (when present) together with finitely many isolated singular points. Moreover, the positive-dimensional part is a union of low-degree curves—namely lines, smooth conics, complete intersections of two quadrics, or an elliptic quartic—whereas isolated singularities occur only in finitely many families. Crucially, on these general representatives, all isolated boundary singularities are quasi-homogeneous and fall into precisely six analytic types. These statements are summarized in Theorem C.

The critical minimal exponent threshold and extremal singularities. A second theme of the paper is that these six isolated singularity types are not accidental by-products of the classification: they capture the *exact boundary* of stability in a sharp sense coming from singularity theory. Building on Park's framework relating (local) minimal exponents to GIT stability for hypersurfaces, we show that every isolated boundary singularity appearing here has local minimal exponent equal to

$$\frac{7}{3} = \frac{n+1}{d} \qquad \text{for } (n,d) = (6,3),$$

that is, it realizes the equality case at the critical threshold. In Park's terminology, this is precisely the borderline regime where semistability can still occur while stability is no longer forced. Motivated by this, we package these six analytic types as a distinguished class and refer to them as *extremal cubic fivefold singularities*; see Definition 6.3 and Theorem C.

Global minimal exponents on the boundary. The extremal value $\frac{7}{3}$ is not confined to isolated points. We compute minimal exponents along all positive-dimensional singular loci that occur on strictly semistable closed orbits (lines, conics, complete intersections of two quadrics, and an elliptic quartic), and show that possible drops occur only at finitely many explicitly described special points. As a consequence, the global minimal exponent of a general strictly semistable closed-orbit cubic fivefold in each boundary component is equal to the same critical threshold 7/3 (see Theorem D).

Adjacency of boundary families. Finally, we analyze codimension-one wall adjacency relations among the boundary families arising from the closed-orbit

normal forms. In a setting with many boundary families, their *connectivity* and *proximity* control the local structure of the moduli space and are closely tied to Kirwan-type wall-crossing. Using Kirwan's stratification together with an explicit slice-maximality computation on the weight set $I = \mathbb{Z}^7_{\geq 0}(3)$, we determine the full codimension-one wall-adjacency graph among the closed strata $\{\Phi_k\}_{k=1}^{21} \subset \Sigma^{\mathrm{ss}}$. Contrary to a sparse picture, the resulting adjacency graph is nontrivial: it has 21 vertices and 56 edges (Theorem E and Theorem 7.2), and in particular it has no isolated vertices. For clarity, we distinguish genuine wall adjacencies (mutual slice matches, drawn as solid edges in Figure 2) from *one-sided* slice matches (drawn as dotted arrows), which do not represent codimension-one adjacency but rather reflect higher-codimension incidence among faces. Altogether, this provides an explicit combinatorial skeleton for the boundary, and serves as a foundation for further investigations of local deformation models and related compactifications.

Idea of the proofs. Our approach combines standard GIT techniques—1-parameter-subgroup limits, computations of stabilizers and centralizers, and normalizations via residual group actions—with explicit calculations tailored to cubic fivefolds. For each boundary family we construct concrete normal forms and read off the structure of the singular locus directly from these models. The minimal exponent statement is obtained by applying Park's criterion to our explicit list of isolated singularities and verifying the critical equality $\widetilde{\alpha}_x(X) = \frac{7}{3}$. The adjacency results follow from comparing degenerations along Kirwan strata and tracking how the boundary normal forms specialize. Finally, the non-inclusion among the boundary families is established through a successive sieve of geometric and algebraic filters, utilizing the semicontinuity of dimensions, apolar Betti numbers, singular-locus Hilbert functions, minimal exponents, and Steenbrink spectra.

Organization. The paper provides a concrete "dictionary" for the GIT boundary of cubic fivefolds: the complete list of components, explicit closed-orbit normal forms, the associated singularity types, the critical minimal-exponent characterization, and the full adjacency graph. We hope that this dictionary will serve as a basis for subsequent work on the geometry of degenerations of cubic fivefolds, local deformation theory near boundary points, and comparisons with other notions of stability.

Theorem A (Boundary families and components). We construct 21 explicitly described SL(7)-invariant families

$$\Phi_1, \Phi_2, \ldots, \Phi_{21} \ \subset \ \mathbb{P}(W)^{ss} \setminus \mathbb{P}(W)^{s} \qquad (W = \mathrm{Sym}^3\mathbb{C}^7),$$

together with explicit closed-orbit (polystable) representatives in normal form; see Table 2 (Section 4). Moreover, we prove the non-inclusion among these 21 families (Theorem 3.5). In particular, the families Φ_k are pairwise distinct and give exactly the 21 irreducible components of $\mathbb{P}(W)^{ss} \setminus \mathbb{P}(W)^{s}$.

Theorem B (Closed-orbit test and normal-form construction). Let $G = \mathrm{SL}(7)$ act on $W = \mathrm{Sym}^3\mathbb{C}^7$. For each boundary family Φ_k $(k = 1, \ldots, 21)$, choose a 1-PS limit with stabilizer $H \subset G$ and consider the induced effective action of the centralizer $C_G(H)$ on W^H. Then closed orbits occurring in Φ_k can be represented by elements $\phi_k \in W^H$, and ϕ_k can be normalized to a normal

form by the residual $C_G(H)$-action. Moreover, the closedness of $C_G(H) \cdot \phi_k$ (hence of $G \cdot \phi_k$) is decided uniformly by a dichotomy: in the toric case by the convex-hull criterion for weights, and otherwise by the Casimiro–Florentino criterion via a positive linear identity. In particular, Table 2 lists explicit closed-orbit normal forms associated with all Φ_k.

Theorem C (Singularities on the boundary). Let ϕ_k^{nf} be a general closed–orbit representative associated with Φ_k (Table 2) and set $X_k := V(\phi_k^{\mathrm{nf}}) \subset \mathbb{P}^6$. Then $\mathrm{Sing}(X_k)$ is explicitly determined (Table 3): it is a union of low–degree curves (lines, smooth conics, complete intersections of two quadrics, or an elliptic quartic), possibly together with isolated points. Isolated points occur exactly for $k \in \{1, 5, 7, 8, 9, 11, 16, 19, 21\}$; they are quasi–homogeneous and fall into six analytic types (Definitions 5.1–5.6), including wild examples, and satisfy the extremal equality

$$\widetilde{\alpha}_x(X_k) = \frac{7}{3} = \frac{n+1}{d} \qquad \text{for } (n, d) = (6, 3).$$

Theorem D (Global minimal exponents on the boundary). For each $k \in \{1, \ldots, 21\}$ let $X_k := V(\phi_k^{\mathrm{nf}}) \subset \mathbb{P}^6$ be a general closed–orbit cubic fivefold. Then the *global minimal exponent* of X_k,

$$\widetilde{\alpha}(X_k) \;:=\; \min_{x \in \mathrm{Sing}(X_k)} \widetilde{\alpha}_x(X_k),$$

is equal to the extremal threshold value

$$\widetilde{\alpha}(X_k) = \frac{7}{3} \qquad \text{for all } k = 1, \ldots, 21.$$

More precisely, if $\mathrm{Sing}(X_k)$ contains isolated points, then by Theorem C each such point x satisfies $\widetilde{\alpha}_x(X_k) = 7/3$, hence $\widetilde{\alpha}(X_k) = \frac{7}{3}$. If $\mathrm{Sing}(X_k)$ contains positive–dimensional components (lines, conics, $(2, 2)$-complete intersections, or an elliptic quartic), then $\widetilde{\alpha}_x(X_k)$ is constant at a general point of each irreducible component, while it may drop at finitely many special points; in every case the minimum along each component is attained at an explicitly listed special point and equals $7/3$. Consequently, across the entire singular locus one has

$$\min_{x \in \mathrm{Sing}(X_k)} \widetilde{\alpha}_x(X_k) = \frac{7}{3}.$$

In particular, a general strictly semistable closed-orbit cubic fivefold in each of the 21 irreducible boundary components realizes the equality case

$$\widetilde{\alpha}(X_k) = \frac{n+1}{d} \qquad \text{for } (n, d) = (6, 3).$$

Theorem E (Codimension-one wall adjacency via wall-crossing). Write $\Sigma^{\mathrm{ss}} := \mathbb{P}(W)^{\mathrm{ss}} \setminus \mathbb{P}(W)^{\mathrm{s}}$ for the strictly semistable locus. In the sense of Kirwan wall-crossing (Definition 7.1), the codimension-one wall-adjacency graph among the closed strata $\{\Phi_k\}_{k=1}^{21} \subset \Sigma^{\mathrm{ss}}$ has 21 vertices and 56 edges. Equivalently, for each k the neighbor set

$$N(k) := \{\, j \neq k \mid \Phi_k \text{ is adjacent to } \Phi_j \,\}$$

is given explicitly in Table 6 (and visualized in Figure 2). All other unordered pairs $\{\Phi_i, \Phi_j\}$ are not adjacent. In particular, the adjacency graph has *no isolated vertices*. See Theorem 7.2 and [Kir84, DH98, Tha96].

Ideas and methods

Our starting point is a convex-geometric analysis of Hilbert–Mumford weights for the natural $\mathrm{SL}(7)$-action on $W = \mathrm{Sym}^3(\mathbb{C}^7)$ [MFK94, Dol03]. Fix a maximal torus $T \subset \mathrm{SL}(7)$, and for a cubic $f \in W$ write $\mathrm{Supp}(f) \subset I = \mathbb{Z}^7_{(3)} \cap \mathbb{Z}^7_{\geq 0}$ for its set of exponent vectors. We enumerate the *maximal T-strictly semistable supports* $I(r)_{\geq 0}$ whose convex hull contains the barycenter $\eta = (3/7, \ldots, 3/7)$. An algorithmic search over the faces of $\mathrm{Conv}(I)$ containing η produces 23 candidates up to permutation; exactly one is T-unstable and is discarded, leaving 22 families with respect to T. Passing to the full $\mathrm{SL}(7)$-action, there is a single residual identification $f_{21} \sim f_{22}$. We prove that no further identifications occur: namely, we establish the non-inclusion among the 21 boundary families (Theorem 3.5; see Section 8). This leads to the desired list of 21 $\mathrm{SL}(7)$-inequivalent boundary families (Algorithm 2.2, Proposition 3.4, and Theorem 3.6).

For each family, we obtain closed $\mathrm{SL}(7)$-orbits by taking appropriate one-parameter subgroup limits and applying Luna's centralizer reduction [Lun75]. If the centralizer is a torus, we apply the convex-hull criterion; if it is non-toric, we use the Casimiro–Florentino criterion [CF12]. This uniform dichotomy produces explicit closed-orbit normal forms, from which the family dimensions follow (Section 4; see also Table 2).

Organization of the paper

Section 1 recalls the numerical criterion for (semi)stability and reformulates it in convex–geometric terms via supports and weight half-spaces. Section 2 enumerates the maximal T-strictly semistable supports for a fixed maximal torus $T \subset \mathrm{SL}(7)$ (Algorithm 2.2), yielding 22 candidates up to permutation. Section 3 passes from the T-data to the full $\mathrm{SL}(7)$-action, removes the single residual identification, and obtains 21 $\mathrm{SL}(7)$-inequivalent boundary families $\Phi_1, \ldots, \Phi_{21}$ (with generic forms $f_1, \ldots, f_{21}$), together with the non-inclusion statement deferred to Section 8. Section 4 constructs explicit closed-orbit (polystable) representatives ϕ_k^{nf} for each family via 1-PS limits and Luna's centralizer reduction, certifies polystability by either the convex-hull criterion (toric centralizers) or the Casimiro–Florentino criterion (non-toric centralizers), and computes the dimensions of the corresponding components (summarized in Table 2). Section 5 determines the singular scheme $\mathrm{Sing}(X_k)$ of each closed-orbit cubic $X_k := V(\phi_k^{\mathrm{nf}}) \subset \mathbb{P}^6$, describing its positive-dimensional part (low-degree curves) and listing the isolated singularities; in particular, the isolated boundary singularities fall into exactly six quasi-homogeneous analytic types (summarized in Tables 3–4). Section 6 studies minimal exponents: it first proves that each of the six isolated boundary singularity types has local minimal exponent $7/3$, and then computes the global minimal exponent $\tilde{\alpha}(X_k)$ for every $k = 1, \ldots, 21$ by a case-by-case analysis along $\mathrm{Sing}(X_k)$. Section 7 determines the adjacency relations among the closed strata $\{\Phi_k\}$ via wall-crossing in Kirwan's stratification, showing that the adjacency graph has exactly 56 edges. Section 8 proves the non-inclusion among the 21 boundary families (Theorem 3.5) by a sieve of monotone necessary conditions ("filters"), culminating in fine geometric and algebraic obstructions based on limits of 1-cycles, Steenbrink spectra of isolated singularities, and generic stabilizer tori that remove the final candidates. Section 9 collects concluding remarks and outlines several future directions,

including deformation/K-moduli questions for the extremal singularities and Hodge-theoretic and integrable-systems perspectives.

We present below a table comparing cubic threefolds and fourfolds.

Table 1: Cubic threefolds and fourfolds: moduli dimension, GIT boundary components, and stability-singularity profile.

Cubic threefolds in $\mathbb{P}^4$ (See [Yok02].)
Moduli dimension. Number of monomials $\binom{5+3-1}{3} = 35$, projective dimension 34; $\dim \mathrm{PGL}_5 = 24$. Thus $\dim \mathcal{M}^{\mathrm{GIT}} = 34 - 24 = \mathbf{10}$.

Components of the GIT boundary (polystable closed orbits). Two irreducible components:

(i) a $\mathbb{P}^1$-family $\{\phi_{\alpha,\beta}\}$ with parameter $[\alpha : \beta]$, and

(ii) the isolated point represented by $\phi = vwz + x^3 + y^3$.

Singularity profile (stability). Stable $\Longleftrightarrow$ only double points of type A_n with $n \leq 4$. Semistable $\Longleftrightarrow$ only A_n $(n \leq 5)$, D_4, or A_∞ double points. Inside the $\mathbb{P}^1$-component, the special member with $\alpha^2 = 4\beta$ is the secant threefold (the singular locus is the rational normal curve).

Adjacency/closures. Both boundary components consist of closed orbits; the $\mathbb{P}^1$-component and the isolated point are distinct boundary components (there is no specialization of one to the other).

Cubic fourfolds in $\mathbb{P}^5$ (See [Laz09, Yok08, Huy23].)
Moduli dimension. Number of monomials $\binom{6+3-1}{3} = 56$, projective dimension 55; $\dim \mathrm{PGL}_6 = 35$. Thus $\dim \mathcal{M}^{\mathrm{GIT}} = \mathbf{20}$.

Boundary (closed orbit) families and their dimensions. There are six types, denoted $[C.1]$-$[C.6]$, of respective dimensions $1, 2, 3, 1, 1, 0$. A convenient set of normal forms is:

[C.1] $u\,q_1(w,x,y,z) + v\,q_2(w,x,y,z)$ with $V(u,v,q_1,q_2)$ smooth.

[C.2] $u(xy + xz + yz + \alpha z^2) + v^2x + w^2y + 2vwz$ (generic α).

[C.3] $uy^2 + v^2z + l_1(w,x)\,uz + 2\,l_2(w,x)\,vy + c(w,x)$ with $l_2^2 \nmid c$, $l_1 \nmid c$.

[C.4] $uvw + c(x,y,z)$ with $V(u,v,w,c)$ smooth.

[C.5] $\alpha\,uy^2 + v^2z + w^2x - uxz + 2vwy$ $(\alpha \neq 0)$.

[C.6] $uvw + xyz$.

Stability via singularities. A cubic fourfold with only isolated simple (ADE) singularities is GIT stable. Conversely, non-stability occurs if any of the following conditions holds: (1) $\mathrm{Sing}(X)$ contains a conic; (2) $\mathrm{Sing}(X)$ contains a line; (3) $\mathrm{Sing}(X)$ contains the intersection of two quadrics; (4) X has a double point of rank ≤ 2; (5) a rank 3 double point with a hyperplane section whose singular locus is a line with ranks ≤ 2 along it; (6) a rank 3 double point whose tangent-cone singular locus is a 2-plane in X.

Adjacency (specialization) among boundary strata. If we denote the families $[C.1]$ to $[C.6]$ by $C_1, S_2, V_3, C_4, C_5, P_6$, then

$$P_6 \subset \overline{S_2} \cap \overline{V_3}, \qquad P_6 \in \overline{C_1} \cap \overline{C_4} \cap \overline{C_5}.$$

Cubic fivefolds in $\mathbb{P}^6$

Moduli dimension. Number of monomials $\binom{7+3-1}{3} = 84$, projective dimension 83; $\dim \mathrm{PGL}_7 = 48$. Thus $\dim \mathcal{M}^{\mathrm{GIT}} = \mathbf{35}$.

Components of the GIT boundary (strictly semistable locus).

Remark 0.1 (Cubic threefolds and fourfolds rephrased via minimal exponents)**.** *The classical GIT classifications of cubic threefolds and cubic fourfolds admit a succinct reinterpretation in terms of the (global) minimal exponent $\widetilde{\alpha}$ in the sense of Park.*

Cubic threefolds. *Let $X \subset \mathbb{P}^4$ be a cubic threefold. The critical value in Park's criterion is $\frac{n+1}{d} = \frac{5}{3}$. Park's theorem gives the sufficient conditions $\widetilde{\alpha}(X) > \frac{5}{3} \Rightarrow X$ is GIT stable and $\widetilde{\alpha}(X) \geq \frac{5}{3} \Rightarrow X$ is GIT semistable [Par25, Theorem A]. In fact, the explicit GIT analysis of cubic threefolds (see [Yok02]) shows that*

$$X \text{ is stable } \iff \widetilde{\alpha}(X) > \tfrac{5}{3}$$

$$X \text{ is semistable and not GIT-equivalent to the chordal cubic } \mathcal{T} \iff \widetilde{\alpha}(X) \geq \tfrac{5}{3}.$$

The remaining strictly semistable closed orbit is the chordal cubic $\mathcal{T}$, which is GIT polystable and satisfies $\widetilde{\alpha}(\mathcal{T}) = \frac{3}{2}$ [Par25, Remark 7.6(2)].

Cubic fourfolds. *Let $X \subset \mathbb{P}^5$ be a cubic fourfold. Here $\frac{n+1}{d} = 2$. The classical results of Yokoyama–Laza (see [Yok08, Laz09]) imply that a cubic fourfold with only isolated simple (ADE) singularities is GIT stable, hence in particular $\widetilde{\alpha}(X) > 2$. Moreover, away from the one-parameter family χ of GIT polystable cubics, semistability is characterized by the threshold $\widetilde{\alpha}(X) \geq 2$:*

$$X \text{ semistable and not GIT-equivalent to a member of } \chi$$

$$\Rightarrow \widetilde{\alpha}(X) \geq 2 \Rightarrow X \text{ semistable.}$$

[Par25, Remark 7.6(3)].
The exceptional family χ forms the polystable locus with $\widetilde{\alpha}(X) < 2$: the secant to the Veronese surface $\omega \in \chi$ has $\widetilde{\alpha}(\omega) = \frac{3}{2}$, while for all other $X \in \chi \setminus \{\omega\}$ one has $\widetilde{\alpha}(X) = \frac{11}{6}$ [Par25, Remark 7.6(3)].

These lower-dimensional cases provide useful prototypes for cubic fivefolds, where the critical value is $\frac{n+1}{d} = \frac{7}{3}$ and our boundary isolated singularities realize the equality case at this threshold.

Scripts used in this paper

The scripts used in this paper are publicly available at [Shi26]. In particular, the archive includes the scripts for Sections 2, 5, 7, 8.

Use of AI Tools

The author used ChatGPT and Gemini as auxiliary tools in several parts of the project. In Section 4, they were used to help explore and organize computations leading to candidate normal forms. In Section 5, they were used in the same way to help derive and organize candidate normal forms for isolated singularities. In Section 6, they were used to support the organization and exploration of the minimal exponent computations (e.g., structuring case-by-case checks and assisting with the interpretation and presentation of intermediate computational outputs). These tools were also used to support coding related to these computations (for example, drafting and debugging scripts). Finally, under the author's supervision, they were used for English language editing and for improving exposition, organization, and internal consistency of the manuscript through iterative discussions. All mathematical statements and results were ultimately checked and verified by explicit calculations carried out by the author, who takes full responsibility for the content.

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1 Numerical criterion for cubic fivefolds

In this section, we review the numerical criterion for stability or semistability of cubic fivefolds. We use the following notations.

- Let $\mathbb{C}[x_0, \cdots, x_6]_3$ be the set of homogeneous polynomials of degree 3.
- For a vector $\mathbf{x} \in \mathbb{Q}^7$, $\mathrm{wt}(\mathbf{x}) = \sum_{k=0}^{6} x_k$ is called the weight of $\mathbf{x}$.
- We define $\mathbb{Z}^7_{\geq 0} = \{\mathbf{x} = (x_0, x_1, \cdots, x_6) \in \mathbb{Z}^7 | x_k \geq 0 (k = 0, 1, \cdots, 6)\}$,

  $\mathbb{Z}^7_{(d)} = \{\mathbf{x} \in \mathbb{Z}^7 | \mathrm{wt}(\mathbf{x}) = d\}$,

  $I = \mathbb{Z}^7_{(3)} \cap \mathbb{Z}^7_{\geq 0}$ and it is simply called the simplex.
- For $\mathbf{r} \in \mathbb{Q}^7$, we define $I(\mathbf{r})_{\geq 0} = \{\mathbf{i} \in I | \mathbf{r} \cdot \mathbf{i} \geq 0\}$, $I(\mathbf{r})_{>0} = \{\mathbf{i} \in I | \mathbf{r} \cdot \mathbf{i} > 0\}$ and $I(\mathbf{r})_{=0} = \{\mathbf{i} \in I | \mathbf{r} \cdot \mathbf{i} = 0\}$, here $\cdot$ denotes the standard inner product of vectors.

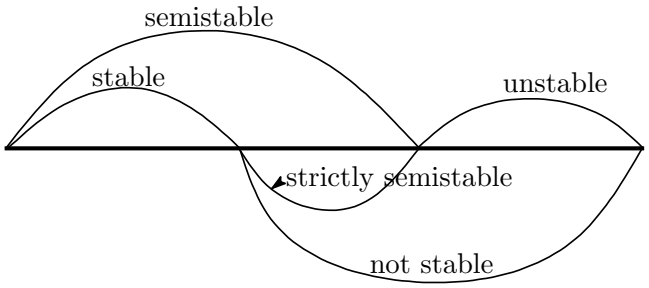

Figure 1: various concepts of stability

- For a polynomial $f = \sum_{\mathrm{wt}(\mathbf{i})=3} a_{\mathbf{i}} x^{\mathbf{i}} \in \mathbb{C}[x_0, \cdots, x_6]_3$, we define the support of f by $\mathrm{Supp}(f) = \{\mathbf{i} \in I | a_{\mathbf{i}} \neq 0\}$
- We set $\eta = (3/7, 3/7, 3/7, 3/7, 3/7, 3/7, 3/7) \in \mathbb{Q}^7$ and it is called the barycenter of the simplex I.
- A vector $\mathbf{r} \in \mathbb{Z}^7$ is said to be reduced when there is no integer α such that $|\alpha| \geq 2$ and $\frac{1}{\alpha}\mathbf{r} \in \mathbb{Z}^7$

We fix a maximal torus T of SL(7). Consider a one-parameter subgroup (1-PS for short) $\lambda : \mathbb{G}_m \to \mathrm{SL}(7)$ whose image is contained in T. For a suitable basis of $\mathbb{C}^7$, λ can be expressed as a diagonal matrix $\mathrm{diag}(t^{r_0}, t^{r_1}, \cdots, t^{r_6})$ where $t \neq 0$ is a parameter of $\mathbb{G}_m$. Let us choose and fix such basis. Then λ corresponds to an element $\mathbf{r} = (r_0, r_1, \cdots, r_6)$ in $\mathbb{Z}^7_{(0)}$. We can regard an element of $\mathbb{Z}^7_{(0)}$ as a 1-PS of T.

Definition 1.1. *Let s be a subset of I. We say that s is not stable (resp. unstable) with respect to T when $s \subseteq I(\mathbf{r})_{\geq 0}$ (resp. $s \subseteq I(\mathbf{r})_{>0}$) for some 1-PS $\mathbf{r}$. For $0 \neq f \in \mathbb{C}[x_0, \cdots, x_6]_3$, we say that f is not stable (resp. unstable) with respect to T when $\mathrm{Supp}(f) \subseteq I$ is not stable (resp. unstable) with respect to T.*

The following theorem is the numerical criterion for stability via the language of convex geometry.

Theorem 1.2. *The cubic fivefold defined by $f \in \mathbb{C}[x_0, \cdots, x_6]_3$ is not stable (resp. unstable) if and only if there exists an element $\sigma \in \mathrm{SL}(7)$ such that f^σ is not stable (resp. unstable) with respect to T.*

In particular, f is strictly semistable if and only if

(1) *There exist $\sigma \in \mathrm{SL}(7)$ such that f^σ is not stable with respect to T, and*

(2) *For any $\sigma \in \mathrm{SL}(7)$, f^σ is semistable with respect to T.*

Proof. See Theorem 9.1 of [Dol03]. □

2 Maximal strictly semistable cubic fivefolds with respect to the maximal torus T

In this section, we list the irreducible components corresponding to strictly semistable cubic fivefolds. For this purpose, we list all strictly semistable cubic

fivefolds with respect to the maximal torus T. To solve this problem, we will consider the set of maximal strictly semistable subsets of I. The order in the set of subsets of I is given by inclusion. For this purpose, we list the set of all maximal elements of $\mathcal{S} = \{I(\mathbf{r})_{\geq 0} | \mathbf{r} \in \mathbb{Z}^7_{(0)}\}$.

We solve this problem computationally. We need an algorithm which enables us to obtain them in finitely many steps. Before giving such an algorithm, we remark that $I(\mathbf{r})_{\geq 0}$ and $I(\mathbf{r}')_{\geq 0}$ might be the same for two different vectors $\mathbf{r}, \mathbf{r}' \in \mathbb{Z}^7_{(0)}$.

Lemma 2.1. *Let $I(\mathbf{r})_{\geq 0}$ be a maximal element of $\mathcal{S}$, where $\mathbf{r} \in \mathbb{Z}^7_{(0)}$. Then there exist 5 elements $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_5 \in I$ and a vector $\mathbf{r}' \in \mathbb{Z}^7_{(0)}$ such that they satisfy the following three conditions:*

(1) *The vector subspace W of $\mathbb{Q}^7$ spanned by $\mathbf{x}_1, \cdots, \mathbf{x}_5, \eta$ over $\mathbb{Q}$ has dimension 6*

(2) *The vector $\mathbf{r}'$ is orthogonal to the subspace W of $\mathbb{Q}^7$.*

(3) $I(\mathbf{r})_{\geq 0} = I(\mathbf{r}')_{\geq 0}$

Proof. Let us put $C = I(r) \cup \{\eta\}$. We consider the convex-hull $\check{C}$ of C in $\mathbb{Q}^7$. Let F be a face of $\check{C}$ containing the point η. There is a normal vector $\mathbf{r}'$ of F in $\mathbb{Z}^7_{(0)}$ such that $\check{C} \subseteq \{\mathbf{x} \in \mathbb{Q}^7 | \mathbf{r}' \cdot \mathbf{x} \geq 0\}$. We have $\mathrm{wt}(\mathbf{r}') = 0$ because the hyperplane defined by $\{\mathbf{x} \in \mathbb{Q}^7 | \mathbf{r}' \cdot \mathbf{x} = 0\}$ passes through the point η. By the definition of the faces of a convex set in $\mathbb{Q}^7$, we can take 5 points $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_5$ from the set $I \cap F$ such that $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_5, \eta$ are linearly independent over $\mathbb{Q}$. In general we have $I(\mathbf{r})_{\geq 0} \subseteq I(\mathbf{r}')_{\geq 0}$, and by the assumption that $I(\mathbf{r})_{\geq 0}$ is maximal in $\mathcal{S}$, we conclude that $I(\mathbf{r})_{\geq 0} = I(\mathbf{r}')_{\geq 0}$. □

By this lemma, we can determine the set of maximal elements of $\mathcal{S}$ up to permutations of coordinates in finite steps using the following algorithm.

Algorithm 2.2. *Let $\mathcal{F}$ be the set of all ordered 5-tuples $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_5)$ of* pairwise distinct *points of I. Fix a total order on $\mathcal{F}$. As initial data, set $\mathcal{S}' = \varnothing$ and let $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_5)$ be the minimum element of $\mathcal{F}$. We modify $\mathcal{S}'$ by the following algorithm.*

- *Step 1. Let $W \subset \mathbb{Q}^7$ be the $\mathbb{Q}$-subspace spanned by $\mathbf{x}_1, \ldots, \mathbf{x}_5, \eta$. If $\dim_{\mathbb{Q}}(W) = 6$, then take a* reduced *normal vector $\mathbf{r} = (r_0, \ldots, r_6) \in \mathbb{Z}^7_{(0)} \setminus \{0\}$ of W (i.e. $\mathbf{r} \cdot \mathbf{w} = 0$ for all $\mathbf{w} \in W$) and go to Step 2; otherwise, go to Step 5.*
- *Step 2. If $\mathbf{r}$ satisfies $r_0 \geq \cdots \geq r_6$ or $r_0 \leq \cdots \leq r_6$, then go to Step 3; otherwise, go to Step 5.*
- *Step 3. If $r_0 \geq \cdots \geq r_6$ (resp. $r_0 \leq \cdots \leq r_6$), add $I(\mathbf{r})_{\geq 0}$ (resp. $I(-\mathbf{r})_{\geq 0}$) to $\mathcal{S}'$ and go to Step 4.*
- *Step 4. Delete all elements of $\mathcal{S}'$ that are not maximal in $\mathcal{S}'$ (with respect to inclusion), and go to Step 5.*
- *Step 5. Replace the element $\mathbf{x}$ with the next element if $\mathbf{x}$ is not the maximum element, and go to Step 1. Otherwise, stop the algorithm.*

We note that Step 2 removes the S_7 symmetry on the variables $x_0, \cdots, x_6$. We also note that Step 4 is not essential but serves as technical measure to save memory. After running this algorithm with the aid of a computer, we find 23 elements $I(\mathbf{r}_1)_{\geq 0}, \cdots, I(\mathbf{r}_{23})_{\geq 0}$ in $\mathcal{S}'$, where $\mathbf{r}_k = (r_0, \cdots, r_6) \in \mathbb{Z}^7_{(0)}$ is a reduced vector with $r_0 \geq \cdots \geq r_6$. When we compute the convex-hulls of $I(\mathbf{r}_1)_{\geq 0}, \cdots, I(\mathbf{r}_{23})_{\geq 0}$ in $\mathbb{Q}^7$, only one of the convex-hulls of $I(\mathbf{r}_k)_{\geq 0}$ does not contain η. We denote it as $I(\mathbf{r}_{23})_{\geq 0}$. As $I(\mathbf{r}_{23})_{\geq 0}$ is unstable with respect to T, we remove it from the list. Thus, we can conclude that there are 22 maximal strictly semistable cubic fivefolds for the fixed maximal torus T. Because of this algorithm, we have the following proposition.

Proposition 2.3. *The set $\mathcal{M} = \{I(\mathbf{r}_1)_{\geq 0}, \cdots, I(\mathbf{r}_{22})_{\geq 0}\}$ is given as follows.*

$\mathbf{r}_1 = (8,3,2,-1,-2,-4,-6)$	$\mathbf{r}_2 = (6,4,1,-1,-2,-3,-5)$
$\mathbf{r}_3 = (4,2,1,-1,-1,-2,-3)$	$\mathbf{r}_4 = (3,2,1,0,-1,-2,-3)$
$\mathbf{r}_5 = (4,2,1,0,-1,-2,-4)$	$\mathbf{r}_6 = (5,3,2,1,-1,-4,-6)$
$\mathbf{r}_7 = (6,4,2,1,-2,-3,-8)$	$\mathbf{r}_8 = (4,1,1,0,-2,-2,-2)$
$\mathbf{r}_9 = (2,2,0,0,-1,-1,-2)$	$\mathbf{r}_{10} = (2,1,0,0,-1,-1,-1)$
$\mathbf{r}_{11} = (2,0,0,0,0,-1,-1)$	$\mathbf{r}_{12} = (3,2,1,1,-1,-2,-4)$
$\mathbf{r}_{13} = (2,1,1,0,-1,-1,-2)$	$\mathbf{r}_{14} = (2,2,0,-1,-1,-1,-1)$
$\mathbf{r}_{15} = (2,1,1,0,0,-2,-2)$	$\mathbf{r}_{16} = (2,1,0,0,0,-1,-2)$
$\mathbf{r}_{17} = (1,1,1,0,0,-1,-2)$	$\mathbf{r}_{18} = (1,1,0,0,0,-1,-1)$
$\mathbf{r}_{19} = (2,2,2,0,-1,-1,-4)$	$\mathbf{r}_{20} = (1,1,1,1,0,-2,-2)$
$\mathbf{r}_{21} = (1,1,0,0,0,0,-2)$	$\mathbf{r}_{22} = (1,0,0,0,0,0,-1)$

For example, $I(\mathbf{r}_1)_{\geq 0}$ is
$I(\mathbf{r}_1)_{\geq 0} = \{x_0^3, x_0^2x_1, x_0^2x_2, x_0^2x_3, x_0^2x_4, x_0^2x_5, x_0^2x_6, x_0x_1^2, x_0x_1x_2, x_0x_1x_3, x_0x_1x_4,$
$x_0x_1x_5, x_0x_1x_6, x_0x_2^2, x_0x_2x_3, x_0x_2x_4, x_0x_2x_5, x_0x_2x_6, x_0x_3^2, x_0x_3x_4, x_0x_3x_5, x_0x_3x_6,$
$x_0x_4^2, x_0x_4x_5, x_0x_4x_6, x_0x_5^2, x_1^3, x_1^2x_2, x_1^2x_3, x_1^2x_4, x_1^2x_5, x_1^2x_6, x_1x_2^2, x_1x_2x_3, x_1x_2x_4,$
$x_1x_2x_5, x_1x_3^2, x_1x_3x_4, x_2^3, x_2^2x_3, x_2^2x_4, x_2^2x_5, x_2x_3^2\}$.
Here we use the notation $x_0^{i_0}x_1^{i_1}\cdots x_6^{i_6}$ for an element $(i_0, i_1, \cdots, i_6) \in \mathbb{Z}^7_{(3)}$ in order to save space.

Remark 2.4. *The following vectors can serve as r_{23}, i.e., there are several vectors that yield the set $I(\mathbf{r}_{23})_{\geq 0}$. For example, we can take $\mathbf{r}_{23} = (8,5,3,2,-4,-4,-10)$.*

Remark 2.5. *The algorithm in this section has been comprehensively generalized by* [GMMS23].

3 21 maximal strictly semistable cubic fivefolds under action of SL(7)

An element $I(\mathbf{r}_k)_{\geq 0}$ of $\mathcal{M}$ represents a family of cubic fivefolds whose defining polynomial's support is contained in $I(\mathbf{r}_k)_{\geq 0}$. In this section, we analyze the inclusion relations among $I(\mathbf{r}_k)_{\geq 0}$ under the action of SL(7). Let f_k be a generic polynomial whose support is $I(\mathbf{r}_k)_{\geq 0}$. $(k = 1, 2, \cdots, 22)$. If we express f_k directly, it becomes too long, so we introduce notation.

Definition 3.1. *The symbols c, q, l, α stand for a cubic form, a quadratic form, a linear form, and a constant term, respectively. Similarly, the symbols q_i, l_i, α_i denote the i-th quadratic form, a linear form, and a constant term, respectively.*

The following theorem is a direct consequence of the list in Proposition 2.3.

Theorem 3.2. *Using the above notations, the generic polynomials of $f_1, \cdots, f_{22}$ are the following forms.*

- $f_1 = c(x_0,x_1,x_2) + q_1(x_0,x_1,x_2)x_3 + l_1(x_0,x_1,x_2)x_3^2 + \{q_2(x_0,x_1,x_2) + l_2(x_0,x_1)x_3\}x_4 + \alpha_1 x_0 x_4^2 + \{q_3(x_0,x_1,x_2) + x_0 l_3(x_3,x_4)\}x_5 + \alpha_2 x_0 x_5^2 + \{q_4(x_0,x_1) + x_0 l_4(x_2,x_3,x_4)\}x_6$

- $f_2 = c(x_0,x_1,x_2)+q_1(x_0,x_1,x_2)x_3+l_1(x_0,x_1)x_3^2+\{q_2(x_0,x_1,x_2)+l_2(x_0,x_1)x_3\}x_4+ l_3(x_0,x_1)x_4^2+\{q_3(x_0,x_1)+l_4(x_0,x_1)x_2+l_5(x_0,x_1)x_3+\alpha_1 x_0 x_4\}x_5+\alpha_2 x_0 x_5^2+ \{q_4(x_0,x_1) + l_6(x_0,x_1)x_2 + \alpha_3 x_0 x_3\}x_6$

- $f_3 = c(x_0,x_1,x_2)+q_1(x_0,x_1,x_2)x_3+l_1(x_0,x_1)x_3^2+\{q_2(x_0,x_1,x_2)+l_2(x_0,x_1)x_3\}x_4+ l_3(x_0,x_1)x_4^2 + \{q_3(x_0,x_1,x_2) + x_0 l_4(x_3,x_4)\}x_5 + \alpha_1 x_0 x_5^2 + \{q_4(x_0,x_1) + l_5(x_0,x_1)x_2 + x_0 l_6(x_3,x_4)\}x_6$

- $f_4 = c(x_0,x_1,x_2)+\{q_1(x_0,x_1,x_2)+l_1(x_0,x_1,x_2)x_3+\alpha_1 x_3^2\}x_3+\{q_2(x_0,x_1,x_2)+ l_2(x_0,x_1,x_2)x_3\}x_4+l_3(x_0,x_1)x_4^2+\{q_3(x_0,x_1,x_2)+l_4(x_0,x_1)x_3+\alpha_2 x_0 x_4\}x_5+ \{q_4(x_0,x_1) + l_5(x_0,x_1)x_2 + \alpha_3 x_0 x_3\}x_6$

- $f_5 = c(x_0,x_1,x_2,x_3)+\{q_1(x_0,x_1,x_2)+l_1(x_0,x_1,x_2)x_3\}x_4+l_2(x_0,x_1)x_4^2+ \{q_2(x_0,x_1,x_2)+l_3(x_0,x_1)x_3+\alpha_1 x_0 x_4\}x_5+\alpha_3 x_0 x_5^2+\{q_3(x_0,x_1)+\alpha_4 x_0 x_2+ \alpha_2 x_0 x_3\}x_6$

- $f_6 = c(x_0,x_1,x_2,x_3)+q_1(x_0,x_1,x_2,x_3)x_4+l_1(x_0,x_1,x_2)x_4^2+\{q_2(x_0,x_1,x_2)+ l_2(x_0,x_1)x_3 + \alpha_1 x_0 x_4\}x_5 + \{q_3(x_0,x_1) + x_0 l_3(x_2,x_3)\}x_6$

- $f_7 = c(x_0,x_1,x_2,x_3)+q_1(x_0,x_1,x_2,x_3)x_4+l_1(x_0,x_1)x_4^2+\{q_2(x_0,x_1,x_2)+ l_2(x_0,x_1,x_2)x_3 + \alpha_1 x_0 x_4\}x_5 + \alpha_2 x_0 x_5^2 + \{q_3(x_0,x_1) + \alpha_3 x_0 x_2\}x_6$

- $f_8 = c(x_0,x_1,x_2,x_3)+\{q_1(x_0,x_1,x_2)+\alpha_1 x_0 x_3\}x_4+\alpha_2 x_0 x_4^2+\{q_2(x_0,x_1,x_2)+ \alpha_3 x_0 x_3+\alpha_4 x_0 x_4\}x_5+\alpha_5 x_0 x_5^2+\{q_3(x_0,x_1,x_2)+\alpha_6 x_0 x_3+\alpha_7 x_0 x_4+\alpha_8 x_0 x_5\}x_6+ \alpha_9 x_0 x_6^2$

- $f_9 = c(x_0,x_1,x_2,x_3)+\{q_1(x_0,x_1)+l_1(x_0,x_1)x_2+l_2(x_0,x_1)x_3\}x_4+l_3(x_0,x_1)x_4^2+ \{q_2(x_0,x_1) + l_4(x_0,x_1)x_2 + l_5(x_0,x_1)x_3 + l_6(x_0,x_1)x_4\}x_5 + l_7(x_0,x_1)x_5^2 + \{q_3(x_0,x_1) + l_8(x_0,x_1)x_2 + l_9(x_0,x_1)x_3\}x_6$

- $f_{10} = c(x_0,x_1,x_2,x_3)+\{q_1(x_0,x_1)+l_1(x_0,x_1)x_2+l_2(x_0,x_1)x_3\}x_4+\alpha_1 x_0 x_4^2+ \{q_2(x_0,x_1)+l_3(x_0,x_1)x_2+l_4(x_0,x_1)x_3\}x_5+\alpha_2 x_0 x_5^2+\{q_3(x_0,x_1)+l_5(x_0,x_1)x_2+ l_6(x_0,x_1)x_3\}x_6 + \alpha_3 x_0 x_6^2 + \alpha_4 x_0 x_4 x_5 + \alpha_5 x_0 x_4 x_6 + \alpha_6 x_0 x_5 x_6$

- $f_{11} = c(x_0,x_1,x_2,x_3,x_4)+x_0 l_1(x_0,x_1,x_2,x_3,x_4)x_5+\alpha_1 x_0 x_5^2+x_0 l_2(x_0,x_1,x_2,x_3,x_4)x_6+ \alpha_2 x_0 x_6^2 + \alpha_3 x_0 x_5 x_6$

- $f_{12} = c(x_0,x_1,x_2,x_3)+q_1(x_0,x_1,x_2,x_3)x_4+l_1(x_0,x_1)x_4^2+\{q_2(x_0,x_1,x_2,x_3)+ \alpha_1 x_0 x_4\}x_5 + \{q_3(x_0,x_1) + x_0 l_2(x_2,x_3)\}x_6$

- $f_{13} = c(x_0,x_1,x_2,x_3) + \{q_1(x_0,x_1,x_2) + l_1(x_0,x_1,x_2)x_3\}x_4 + \alpha_1 x_0 x_4^2 + \{q_2(x_0,x_1,x_2) + l_2(x_0,x_1,x_2)x_3 + \alpha_2 x_0 x_4\}x_5 + \alpha_3 x_0 x_5^2 + \{q_3(x_0,x_1,x_2) + \alpha_4 x_0 x_3\}x_6$

- $f_{14} = \alpha x_2^3 + x_0 q_1(x_0,\ldots,x_6) + x_1 q_2(x_1,\ldots,x_6)$

- $f_{15} = c(x_0, x_1, x_2, x_3, x_4) + \{x_0 l_1(x_0, x_1, x_2, x_3, x_4) + q_1(x_1, x_2)\}x_5 + \{x_0 l_2(x_0, x_1, x_2, x_3, x_4) + q_2(x_1, x_2)\}x_6$
- $f_{16} = c(x_0, x_1, x_2, x_3, x_4) + \{q_1(x_0, x_1) + l_1(x_0, x_1)x_2 + l_2(x_0, x_1)x_3 + l_3(x_0, x_1)x_4\}x_5 + \alpha_1 x_0 x_5^2 + \{q_2(x_0, x_1) + x_0 l_4(x_2, x_3, x_4)\}x_6$
- $f_{17} = c(x_0, x_1, x_2, x_3, x_4) + \{q_1(x_0, x_1, x_2) + l_1(x_0, x_1, x_2) l_2(x_3, x_4)\}x_5 + q_2(x_0, x_1, x_2)x_6$
- $f_{18} = c(x_0, x_1, x_2, x_3, x_4) + \{x_0 l_1(x_0, x_1, x_2, x_3, x_4) + x_1 l_2(x_0, x_1, x_2, x_3, x_4)\}x_5 + \{x_0 l_3(x_0, x_1, x_2, x_3, x_4) + x_1 l_4(x_0, x_1, x_2, x_3, x_4)\}x_6$
- $f_{19} = c(x_0, x_1, x_2, x_3) + \{q_1(x_0, x_1, x_2) + l_1(x_0, x_1, x_2)x_3\}x_4 + l_2(x_0, x_1, x_2)x_4^2 + \{q_2(x_0, x_1, x_2) + l_3(x_0, x_1, x_2)x_3 + l_4(x_0, x_1, x_2)x_4\}x_5 + l_5(x_0, x_1, x_2)x_5^2 + q_3(x_0, x_1, x_2)x_6$
- $f_{20} = c(x_0, x_1, x_2, x_3, x_4) + q_1(x_0, x_1, x_2, x_3)x_5 + q_2(x_0, x_1, x_2, x_3)x_6$
- $f_{21} = c(x_0, x_1, x_2, x_3, x_4, x_5) + q(x_0, x_1)x_6$
- $f_{22} = c(x_0, x_1, x_2, x_3, x_4, x_5) + x_0 l(x_0, x_1, x_2, x_3, x_4, x_5)x_6$

For an element σ in SL(7) and $\mathbb{J} \subseteq I$, we set $\mathbb{J}^\sigma = \cup_f \mathrm{Supp}(f^\sigma)$, where f runs through all polynomials with $\mathrm{Supp}(f) \subseteq \mathbb{J}$.

Definition 3.3. *We denote*

$$I(\mathbf{r}_k)_{\geq 0} \subseteq I(\mathbf{r}_l)_{\geq 0} \bmod \mathrm{SL}(7)$$

when there exists $\sigma \in \mathrm{SL}(7)$ such that $I(\mathbf{r}_k)^\sigma_{\geq 0} \subseteq I(\mathbf{r}_l)_{\geq 0}$ and say that $I(\mathbf{r}_k)_{\geq 0}$ is included in $I(\mathbf{r}_l)_{\geq 0}$ modulo SL(7).

We construct a smaller subset $\mathcal{M}'$ of $\mathcal{M}$ such that (1) any element $I(\mathbf{r}_k)_{\geq 0}$ in $\mathcal{M}$ is included in some element $I(\mathbf{r}_l)_{\geq 0}$ in $\mathcal{M}'$ mod SL(7), (2) any element $I(\mathbf{r}_k)_{\geq 0}$ in $\mathcal{M}'$ is not included in any other $I(\mathbf{r}_l)_{\geq 0}$ in $\mathcal{M}'$ mod SL(7) $(1 \leq l \leq 22)$

Proposition 3.4. *There are two relations*

- $I(\mathbf{r}_{21})_{\geq 0} \subseteq I(\mathbf{r}_{22})_{\geq 0} \bmod \mathrm{SL}(7)$
- $I(\mathbf{r}_{22})_{\geq 0} \subseteq I(\mathbf{r}_{21})_{\geq 0} \bmod \mathrm{SL}(7)$.

Proof. $f_{22} = c(x_0, \cdots, x_5) + x_0 l(x_0, \cdots, x_5)x_6$

$\equiv c(x_0, \cdots, x_5) + x_0 l(x_0, x_1)x_6$

$\equiv c(x_0, \cdots, x_5) + q(x_0, x_1)x_6$

$= f_{21}$

Here $\equiv$ means equality after an SL(7) change of coordinates. □

From this proposition, we can remove f_{22} from the list. Thus, we obtain a list of 21 types of cubic fivefolds.

Theorem 3.5. *For any $1 \leq k, l \leq 21$ with $k \neq l$, there is no element $g \in \mathrm{SL}(7)$ such that*

$$g \cdot I(\mathbf{r}_k)_{\geq 0} \subseteq I(\mathbf{r}_l)_{\geq 0}.$$

Proof. We postpone the proof of this theorem to Section 8. □

Theorem 3.6. *The strictly semistable locus*

$$\mathbb{P}(W)^{ss} \setminus \mathbb{P}(W)^{s}$$

has 21 *irreducible components, represented by* $f_1, \ldots, f_{21}$.

4 Closed orbits of 21 families

In this section, we find the closed orbits in the 21 families of strictly semistable cubic fivefolds. We define $\lambda_k(t)\colon \mathbb{G}_m \to \mathrm{SL}(7)$ as 1-PSs corresponding to $\mathbf{r}_k$. *We shall use the following convex–geometric criterion repeatedly in this section, so we record it at the outset.*

For the following theorem, see [PV94].

Theorem 4.1 (Convex-hull criterion)**.** *Let T be an algebraic torus acting linearly on a finite-dimensional vector space, V, and let $v \in V$. Subsequently, the following conditions are equivalent:*

(1) the T-orbit $T \cdot v$ is closed in V;

(2) 0 is an interior point of the convex-hull of $\mathrm{Supp}(v)$ *in* $X(T)_{\mathbb{R}}$.

Here, $\mathrm{Supp}(v)$ *denotes the set of T-weights that occur in v, and $X(T)_{\mathbb{R}}$ is the real vector space spanned by the character lattice $X(T)$.*

The following series of definitions and theorems will be extremely useful for determining polystability in cases where the centralizer is not a torus and the convex-hull criterion cannot be applied. Because they will be used repeatedly from this point on, we state them here beforehand.

Notation 4.2. *Let G be a reductive algebraic group, and let X be an affine variety equipped with an algebraic G-action.*

(1) *We denote by $Y(G)$ the set of one-parameter subgroups of G.*

(2) *When $x \in X$, we put $\Lambda_x = \{\lambda(t) \in Y(G)\colon \lim_{t\to 0} \lambda(t)\cdot x \text{ exists}\}$.*

(3) *When $\lambda \in Y(G)$, we define $P(\lambda) = \{g \in G\colon \lim_{t\to 0} \lambda(t)\cdot g\cdot\lambda(t)^{-1} \text{ exists}\}$. It is a parabolic subgroup of G.*

Definition 4.3. *We say that a subset $\Lambda \subset Y(G)$ is symmetric if given any $\lambda \in \Lambda$, there is another 1-PS $\lambda' \in \Lambda$ such that $P(\lambda) \cap P(\lambda')$ is a Levi subgroup of both $P(\lambda)$ and $P(\lambda')$.*

We need the following theorem (Theorem 1.1 of [CF12]):

Theorem 4.4. *(Casimiro–Florentino criterion) Let G be a reductive algebraic group and X be an affine G-variety. Then, a point $x \in X$ is polystable if and only if Λ_x is symmetric.*

Lemma 4.5 (rank 1 symmetry for Casimiro–Florentino)**.** *Let G be a reductive algebraic group acting on an affine G-variety X, and let $x \in X$. Assume that there exists a one-parameter subgroup $\mu \in Y(G)$ such that*

$$\Lambda_x \;=\; \{\mu_k \mid k \in \mathbb{Z}\} \qquad \mu_k(t) := \mu(t^k).$$

Then Λ_x is symmetric in the sense of Definition 4.3. Consequently, by the Casimiro–Florentino criterion (Theorem 4.4), the point x is polystable.

Proof. Fix $k \neq 0$. It is standard that $P(\mu_k)$ and $P(\mu_{-k})$ are opposite parabolic subgroups and that

$$P(\mu_k) \cap P(\mu_{-k}) \;=\; C_G\big(\mu(\mathbb{G}_m)\big)$$

which is a Levi subgroup of both $P(\mu_k)$ and $P(\mu_{-k})$. The inclusion $C_G\big(\mu(\mathbb{G}_m)\big) \subset P(\mu_k) \cap P(\mu_{-k})$ is immediate: if g centralizes $\mu(\mathbb{G}_m)$, then $\mu_k(t)\, g\, \mu_k(t)^{-1} = g$ for all $t \in \mathbb{G}_m$, so both limits as $t \to 0$ and $t \to \infty$ exist. For the reverse inclusion, take $g \in P(\mu_k) \cap P(\mu_{-k})$ and consider

$$\phi : \mathbb{G}_m \longrightarrow G, \qquad \phi(t) := \mu_k(t)\, g\, \mu_k(t)^{-1}.$$

By assumption, the limits $\lim_{t\to 0} \phi(t)$ and $\lim_{t\to\infty} \phi(t)$ both exist (the latter because $g \in P(\mu_{-k})$ and $\phi(1/s) = \mu_{-k}(s)\, g\, \mu_{-k}(s)^{-1}$ for $s \to 0$). Hence, ϕ extends to a morphism $\tilde{\phi} : \mathbb{P}^1 \to G$. As G is affine, $\tilde{\phi}$ is constant; in particular,

$$\mu_k(t)\, g\, \mu_k(t)^{-1} = \phi(t) = \phi(1) = g \quad \text{for all } t \in \mathbb{G}_m.$$

Thus, g centralizes $\mu_k(\mathbb{G}_m) = \mu(\mathbb{G}_m)$, i.e. $g \in C_G\big(\mu(\mathbb{G}_m)\big)$. This proves $P(\mu_k) \cap P(\mu_{-k}) = C_G\big(\mu(\mathbb{G}_m)\big)$.

Based on the hypothesis $\mu_k, \mu_{-k} \in \Lambda_x$, so the requirement in Definition 4.3 is satisfied for every $\lambda = \mu_k$. Hence, Λ_x is symmetric. The last assertion follows from Theorem 4.4 (Casimiro–Florentino). □

Convention 4.6. *Throughout this section, we fix the following setup and notation.*

(1) ***Coordinates and the maximal torus.*** *We work on $W = \mathrm{Sym}^3\mathbb{C}^7$ with homogeneous coordinates $(x_0, \dots, x_6)$. We fix the diagonal maximal torus $T \subset \mathrm{SL}(7)$ acting by $\mathrm{diag}(\mu_0, \dots, \mu_6)$ with $\prod_i \mu_i = 1$. A one-parameter subgroup (1-PS) is written as*

$$\lambda(t) = \mathrm{diag}\big(t^{a_0}, \dots, t^{a_6}\big), \qquad \sum_{i=0}^{6} a_i = 0.$$

(2) ***1-PS limits and centralizer reduction.*** *For each $k \in \{1, \dots, 21\}$ let λ_k correspond to r_k in Proposition 2.3, and let f_k be the generic member from Theorem 3.2. We set the 1-PS limit*

$$\phi_k := \lim_{t\to 0} \lambda_k(t) \cdot f_k.$$

Put $H := \lambda_k(\mathbb{G}_m)$ and write $C_G(H)$ for the centralizer in $G = \mathrm{SL}(7)$. Let $W^H \subset W$ be the H-fixed subspace. By Luna's centralizer reduction [Lun75],

closedness of $\mathrm{SL}(7)\cdot\phi_k$ *in* $\mathbb{P}(W)^{ss}$ *is equivalent to closedness of* $C_G(H)\cdot\phi_k$ *in* W^H.

(3) ***Two criteria for polystability.*** *If* $C_G(H)$ *is a torus, we certify closedness by the* convex-hull criterion *(Theorem 4.1). If* $C_G(H)$ *is non-toric, we use the Casimiro–Florentino criterion (Theorem 4.4) as follows: for* $\lambda \in Y(C_G(H))$ *we write* $\mathrm{wt}_\lambda(x_i)$ *for the* λ*-weight on* x_i*, and* $w(m)$ *for the induced weight of a monomial* m*. We denote by* S *the linear constraint coming from* $\det = 1$ *on* $C_G(H)$ *(the "trace" of block weights), so* $S = 0$ *for every* λ*. We then exhibit a positive linear identity*

$$\sum_j c_j\, w(m_j) \;=\; C\cdot S \qquad (c_j > 0,\ C > 0).$$

If $\lambda \in \Lambda_{\phi_k}$*, then every* $w(m_j) \geq 0$ *and* $S = 0$*; hence, all* $w(m_j) = 0$*; solving yields a symmetric* 1*-PS* μ_k*, so* Λ_{ϕ_k} *is symmetric and* ϕ_k *is polystable.*

(4) ***Normal forms and coefficient normalizations.*** *Passing to "normal form," we are allowed to: (i) multiply by a nonzero scalar; (ii) act by* $C_G(H)/H$ *(e.g., block* $GL(2)$*,* $GL(3)$ *actions) to diagonalize blocks; and (iii) use diagonal elements of* T *(with* $\prod \mu_i = 1$*) to normalize nonzero coefficients to* 1*. Parameters* $(\alpha, \rho, \sigma, \dots)$ *record the residual moduli.*

(5) ***Dimension count.*** *Component dimensions are computed as*

$$\dim(W^H) - \dim_{\mathrm{eff}}\big(C_G(H)\big) - 1,$$

where $\dim_{\mathrm{eff}}$ *is the dimension of the effective* $C_G(H)$*-action on* W^H *(central tori acting trivially are subtracted). We state explicitly when a central factor acts trivially.*

(6) ***Weights and symbols.*** *We freely reuse symbols* a_i *for nonzero coefficients of* ϕ_k *prior to normalization. For 1-PS families obtained in the CF-check, we write* $\mu_k(t)$*. All such conventions are in force throughout §4.*

Let us now determine, for each $k = 1, 2, \dots, 21$, a polynomial whose $\mathrm{SL}(7)$-orbit is closed. The procedure is uniform across all cases. First, we take a 1-PS limit to produce a candidate ϕ_k for a polystable point. Next, to apply Luna's criterion, we take the stabilizer $H \subset G = \mathrm{SL}(7)$ of ϕ_k; we choose H as large as possible so that its centralizer $C_G(H)$ is as small as possible, which simplifies the closedness check. Write W^H for the H-fixed locus in the ambient representation $W = \mathrm{Sym}^3\mathbb{C}^7$. If $C_G(H)$ is a torus, we apply the convex-hull criterion (Theorem4.1) to show that $C_G(H)\cdot\phi_k$ is closed in W^H; if $C_G(H)$ is not a torus, we instead apply the Casimiro–Florentino criterion (Theorem 4.4) to obtain the same conclusion. In either case, $C_G(H)\cdot\phi_k$ is closed in W^H; hence, by Luna's criterion, the orbit $\mathrm{SL}(7)\cdot\phi_k$ is closed in $\mathbb{P}(\mathrm{Sym}^3\mathbb{C}^7)^{ss}$. Lastly, by determining a normal form of ϕ_k under the action of $C_G(H)/H$, we fix the dimension of the corresponding component of the moduli space for that k.

4.1 Case $k = 1$

1-PS limit. Set

$$\lambda_1(t) = \mathrm{diag}\big(t^8,\, t^3,\, t^2,\, t^{-1},\, t^{-2},\, t^{-4},\, t^{-6}\big), \quad t \in \mathbb{G}_m.$$

For a generic f_1 as in Section 3, the 1-PS limit is

$$\phi_1 := \lim_{t\to 0} \lambda_1(t)\cdot f_1 = a_1\, x_2x_3^2 + a_2\, x_1x_3x_4 + a_3\, x_2^2x_5 + a_4\, x_0x_5^2 + a_5\, x_1^2x_6 + a_6\, x_0x_4x_6.$$

H **and** $C_G(H)$**.** Let $H = \lambda_1(\mathbb{G}_m)$. The diagonal weights on $\langle x_0, \dots, x_6 \rangle$ are pairwise distinct; hence, $C_G(H) = T$ (the maximal diagonal torus). Each monomial of ϕ_1 has H-weight 0, so $\phi_1 \in W^H$.

Polystability (Luna + convex-hull). By Luna's criterion (see [Lun75]), closedness of the SL(7)-orbit of ϕ_1 is equivalent to closedness of the T-orbit in W^H. By the convex-hull criterion (Theorem 4.1), it suffices to check that 0 is an interior point of $\mathrm{Conv}(\mathrm{Supp}(\phi_1)) \subset X(T)_{\mathbb{R}} \simeq \mathbb{R}^7/\mathbb{R}(1,\dots,1)$. This holds because the exponent vectors of the six monomials satisfy

$$\begin{aligned} &2(0,0,1,2,0,0,0) + 2(0,1,0,1,1,0,0) + 2(0,0,2,0,0,1,0) \\ &\quad + 2(1,0,0,0,0,2,0) + 2(0,2,0,0,0,0,1) + 4(1,0,0,0,1,0,1) = 6(1,1,1,1,1,1,1), \end{aligned} \tag{1}$$

Hence, ϕ_1 is polystable and $\mathrm{SL}(7) \cdot \phi_1$ is closed.

Normal form and component dimension. Let $\mathrm{diag}(\mu_0, \dots, \mu_6) \in T$ with $\prod_{i=0}^{6} \mu_i = 1$. Along with the overall projective scaling, this acts on the six coefficients of ϕ_1 via the characters determined by the exponent vectors in (1); we may normalize all six coefficients to 1 simultaneously. Thus, a normal form is

$$\phi_1^{\mathrm{nf}} = x_2x_3^2 + x_1x_3x_4 + x_2^2x_5 + x_0x_5^2 + x_1^2x_6 + x_0x_4x_6.$$

Residual T**-stabilizer.** Recall that $H = \lambda_1(\mathbb{G}_m)$ and $C_G(H) = T$, and that every monomial of φ_1 has H-weight 0, so H acts trivially on W^H and the effective group on $\mathbb{P}(W^H)$ is T/H. Let $\tau = \mathrm{diag}(\mu_0, \dots, \mu_6) \in T$ with $\prod_{i=0}^{6} \mu_i = 1$. For the normal form

$$\phi_1^{\mathrm{nf}} = x_2x_3^2 + x_1x_3x_4 + x_2^2x_5 + x_0x_5^2 + x_1^2x_6 + x_0x_4x_6,$$

the induced T-characters on the six monomials are

$$\begin{aligned} &\chi_1 = \mu_2\mu_3^2 \quad (x_2x_3^2), && \chi_2 = \mu_1\mu_3\mu_4 \quad (x_1x_3x_4), \\ &\chi_3 = \mu_2^2\mu_5 \quad (x_2^2x_5), && \chi_4 = \mu_0\mu_5^2 \quad (x_0x_5^2), \\ &\chi_5 = \mu_1^2\mu_6 \quad (x_1^2x_6), && \chi_6 = \mu_0\mu_4\mu_6 \quad (x_0x_4x_6). \end{aligned}$$

Since we work projectively, τ stabilizes $[\phi_1^{\mathrm{nf}}] \in \mathbb{P}(W^H)$ if and only if $\tau \cdot \phi_1^{\mathrm{nf}} = c\,\phi_1^{\mathrm{nf}}$ for some $c \in \mathbb{C}^\times$, equivalently

$$\chi_1 = \chi_2 = \chi_3 = \chi_4 = \chi_5 = \chi_6.$$

Solving these equations (together with $\prod \mu_i = 1$) yields $\mu_4^7 = \mu_3^{14}$, hence $\mu_4 = \mu_3^2\xi$ with $\xi^7 = 1$. Writing $t := \mu_3^{-1}$, we can factor

$$\tau = \lambda_1(t) \cdot \delta_\xi, \qquad \delta_\xi := \mathrm{diag}(\xi^{-9}, \xi^{-4}, \xi^{-3}, 1, \xi, \xi^3, \xi^5).$$

Therefore

$$\mathrm{Stab}_T\big([\phi_1^{\mathrm{nf}}]\big) = H \cdot \{\delta_\xi : \ \xi^7 = 1\} \ \cong \ H \times \mu_7,$$

so the induced stabilizer in the effective group T/H is the finite group μ_7. In particular, after normalizing all coefficients to 1, no continuous torus symmetry remains; hence the corresponding boundary component is zero-dimensional.

4.2 Case $k = 2$

1-PS limit. Set

$$\lambda_2(t) = \operatorname{diag}\big(t^6,\, t^4,\, t,\, t^{-1},\, t^{-2},\, t^{-3},\, t^{-5}\big), \qquad t \in \mathbb{G}_m.$$

For a generic f_2 as in Section 3, the 1-PS limit is

$$\phi_2 := \lim_{t\to 0} \lambda_2(t)\cdot f_2 = a_1x_2^2x_4 + a_2x_1x_4^2 + a_3x_1x_3x_5 + a_4x_0x_5^2 + a_5x_1x_2x_6 + a_6x_0x_3x_6.$$

H and $C_G(H)$. Let $H = \lambda_2(\mathbb{G}_m)$. The diagonal weights on $\langle x_0, \ldots, x_6\rangle$ are pairwise distinct; hence, $C_G(H) = T$ (the maximal diagonal torus). Each monomial of ϕ_2 has H-weight 0, so $\phi_2 \in W^H$.

Polystability (Luna + convex-hull). By Luna's criterion, closedness of the SL(7)-orbit of ϕ_2 is equivalent to closedness of the T-orbit in the H-fixed subspace. By the convex-hull criterion (Theorem 4.1), it suffices to check that 0 lies in the interior of $\operatorname{Conv}(\operatorname{Supp}(\phi_2)) \subset X(T)_{\mathbb{R}} \simeq \mathbb{R}^7/\mathbb{R}(1,\ldots,1)$. This holds because

$$\begin{aligned} &2(0,0,2,0,1,0,0) + 2(0,1,0,0,2,0,0) + 2(0,1,0,1,0,1,0) \\ &\quad + 2(1,0,0,0,0,2,0) + 2(0,1,1,0,0,0,1) + 4(1,0,0,1,0,0,1) \qquad (2)\\ &= 6(1,1,1,1,1,1,1), \end{aligned}$$

written in terms of the exponent vectors of the six monomials of ϕ_2. Hence, ϕ_2 is polystable and the orbit $\mathrm{SL}(7)\cdot\phi_2$ is closed.

Normal form and component dimension. A diagonal scaling $\operatorname{diag}(\mu_0,\ldots,\mu_6) \in T$ with $\prod \mu_i = 1$, together with an overall scalar, acts on the six coefficients via the characters determined by the exponent vectors in (2). Thus we may normalize all six coefficients to 1 simultaneously. Therefore, a normal form is

$$\phi_2^{\mathrm{nf}} = x_2^2x_4 + x_1x_4^2 + x_1x_3x_5 + x_0x_5^2 + x_1x_2x_6 + x_0x_3x_6.$$

The residual T-stabilizer is finite; hence, the corresponding component Φ_2 of the moduli is zero-dimensional.

4.3 Case $k = 3$

1-PS limit. Set

$$\lambda_3(t) = \operatorname{diag}\big(t^4,\, t^2,\, t,\, t^{-1},\, t^{-1},\, t^{-2},\, t^{-3}\big), \qquad t \in \mathbb{G}_m.$$

For a generic f_3 as in Section 3, the 1-PS limit is

$$\begin{aligned} \phi_3 := \lim_{t\to 0} \lambda_3(t)\cdot f_3 = a_1\, x_1x_3^2 + a_2\, x_1x_3x_4 + a_3\, x_1x_4^2 \\ + a_4\, x_2^2x_5 + a_5\, x_0x_5^2 + a_6\, x_1x_2x_6 + a_7\, x_0x_3x_6 + a_8\, x_0x_4x_6. \end{aligned}$$

H and $C_G(H)$. Let $H = \lambda_3(\mathbb{G}_m)$. The multiplicities of the diagonal weights on $\langle x_0,\ldots,x_6\rangle$ are 1 on x_0, x_1, x_2, x_5, x_6 and 2 on $\langle x_3, x_4\rangle$; hence,

$$C_G(H) = \left\{ \begin{array}{l} \operatorname{diag}(\alpha_0,\alpha_1,\alpha_2) \oplus A \oplus \operatorname{diag}(\alpha_5,\alpha_6) : \\ \quad \alpha_i \in \mathbb{G}_m,\ A \in \mathrm{GL}(2),\ \alpha_0\alpha_1\alpha_2 \det(A)\, \alpha_5\alpha_6 = 1 \end{array} \right\} \cong \mathrm{SL}(2) \times \mathbb{G}_m^5.$$

Each monomial of ϕ_3 has H-weight 0, so ϕ_3 is H-fixed.

Polystability (Luna + Casimiro–Florentino). By Luna's slice/centralizer reduction, the closedness of the SL(7)-orbit of ϕ_3 is equivalent to polystability for the $C_G(H)$-action on the H-fixed subspace. After conjugating inside the SL(2)-block, any $\lambda \in Y(C_G(H))$ may be chosen with weights

$$\mathrm{wt}(x_0,\dots,x_6) = (a_0,\, a_1,\, a_2,\, c{+}n,\, c{-}n,\, a_5,\, a_6), \qquad S := a_0{+}a_1{+}a_2{+}2c{+}a_5{+}a_6 = 0$$

(as in Convention 4.6. Let w_i be the λ-weight of the i-th monomial of ϕ_3. Then

$$\begin{aligned} &w_1 = a_1 + 2(c+n), && w_2 = a_1 + 2c, && w_3 = a_1 + 2(c-n),\\ &w_4 = 2a_2 + a_5, && w_5 = a_0 + 2a_5, && w_6 = a_1 + a_2 + a_6,\\ &w_7 = a_0 + (c+n) + a_6, && w_8 = a_0 + (c-n) + a_6. \end{aligned}$$

A direct computation yields the positive linear identity

$$w_1 + w_2 + 2w_3 + 2w_4 + 2w_5 + 2w_6 + 3w_7 + w_8 \;=\; 6S. \tag{3}$$

If $\lambda \in \Lambda_{\phi_3}$, then all $w_i \geq 0$ and $S = 0$; by (3) they must all vanish. Solving gives

$$n = 0, \quad a_1 = -2c, \quad a_5 = -2a_2, \quad a_0 = 4a_2, \quad a_6 = 2c - a_2, \quad a_2 = -c.$$

Thus, with $k \in \mathbb{Z}$,

$$\mu_k(t) := \mathrm{diag}\big(t^{4k},\, t^{2k},\, t^{k},\, t^{-k},\, t^{-k},\, t^{-2k},\, t^{-3k}\big), \qquad \Lambda_{\phi_3} = \{\mu_k \mid k \in \mathbb{Z}\}.$$

Therefore Λ_{ϕ_3} is symmetric, and by the Casimiro–Florentino criterion ϕ_3 is polystable; in particular, $\mathrm{SL}(7)\cdot\phi_3$ is closed.

Normal form and component dimension. On the H-fixed slice, the eight weight-zero monomials are

$$x_1x_3^2,\; x_1x_3x_4,\; x_1x_4^2,\; x_2^2x_5,\; x_0x_5^2,\; x_1x_2x_6,\; x_0x_3x_6,\; x_0x_4x_6,$$

Therefore, W^H denotes their span. i.e.

$$W^H = \underbrace{\langle x_1\rangle \otimes \mathrm{Sym}^2\langle x_3,x_4\rangle}_{\text{(I)}} \oplus \underbrace{\mathrm{Sym}^2\langle x_2\rangle \otimes x_5}_{\text{(II)}} \oplus \underbrace{x_0 \otimes \mathrm{Sym}^2\langle x_5\rangle}_{\text{(III)}} \oplus \underbrace{x_0 \otimes \langle x_3,x_4\rangle \otimes x_6}_{\text{(IV)}} \oplus \underbrace{x_1 \otimes \langle x_2\rangle \otimes x_6}_{\text{(V)}}.$$

The centralizer is

$$C_G(H) \cong \mathrm{SL}(2) \times \mathbb{G}_m^5,$$

acting by SL(2) on $\langle x_3,x_4\rangle$ and by a diagonal torus on $\langle x_0,x_1,x_2,x_5,x_6\rangle$ (subject to the product-one condition).

(I) binary quadratic on $\langle x_3,x_4\rangle$. Write the x_1-part as a binary quadratic $Q = a_1x_3^2 + a_2x_3x_4 + a_3x_4^2 \in \mathrm{Sym}^2\langle x_3,x_4\rangle$. As $\Delta(Q)$ is SL(2)-invariant while rescaling x_1 scales Q (and hence Δ) homogeneously, for a generic (nondegenerate) Q, there exists $A \in \mathrm{SL}(2)$ and a rescaling of x_1 such that $Q \sim x_3^2{+}x_3x_4{+}x_4^2$. Thus, the block (I) is fixed to $x_1x_3^2 + x_1x_3x_4 + x_1x_4^2$.

(II)(III)(IV)(V) torus normalizations. Let $T' = \{\mathrm{diag}(\mu_0, \mu_1, \mu_2) \oplus I_2 \oplus \mathrm{diag}(\mu_5, \mu_6) : \ \mu_0\mu_1\mu_2\mu_5\mu_6 = 1\}$. Under T', the coefficients transform by the characters determined by exponent vectors: $x_2^2x_5$ by $\mu_2^2\mu_5$, $x_0x_5^2$ by $\mu_0\mu_5^2$, $x_1x_2x_6$ by $\mu_1\mu_2\mu_6$, while $x_0x_3x_6$ and $x_0x_4x_6$ both by $\mu_0\mu_6$. Using $\mu_0, \mu_1, \mu_2, \mu_5, \mu_6$ (together with an overall scalar), we set $x_2^2x_5$, $x_0x_5^2$, and $x_1x_2x_6$ to have coefficient 1. More precisely, we are imposing the conditions

$$\mu_2^2\mu_5 = 1, \qquad \mu_0\mu_5^2 = 1, \qquad \mu_1\mu_2\mu_6 = 1,$$

together with the product-one constraint

$$\mu_0\mu_1\mu_2\mu_5\mu_6 = 1.$$

We can solve these equations explicitly: choose $\mu_2, \mu_6 \in \mathbb{C}^\times$ freely, set

$$\mu_5 = \mu_2^{-2}, \qquad \mu_0 = \mu_5^{-2} = \mu_2^4, \qquad \mu_1 = (\mu_2\mu_6)^{-1}.$$

Then the product-one constraint $\mu_0\mu_1\mu_2\mu_5\mu_6 = 1$ is automatically satisfied. Consequently, we can normalize the coefficients of $x_2^2x_5$, $x_0x_5^2$, and $x_1x_2x_6$ to be all equal to 1.

(IV)The remaining pair and the residual SL(2)-symmetry. The pair $(x_0x_3x_6,\ x_0x_4x_6)$ is scaled by the same torus character $\mu_0\mu_6$, so the diagonal torus T' alone does not change their ratio. However, after fixing the nondegenerate binary quadratic

$$Q \sim x_3^2 + x_3x_4 + x_4^2,$$

the stabilizer $\mathrm{Stab}_{\mathrm{SL}(2)}(Q)$ is *not* finite: it is a one-dimensional torus (isomorphic to $\mathrm{SO}(2, \mathbb{C}) \cong \mathbb{G}_m$). Its induced action on $\mathbb{P}\langle x_3, x_4\rangle$ has two fixed points (the two roots of Q) and a single open orbit. Hence, for a generic element (i.e. when the linear form in block (IV) is not proportional to a root of Q), we may act by $\mathrm{Stab}_{\mathrm{SL}(2)}(Q)$ to normalize that linear form to $x_3 + x_4$. Using the remaining diagonal torus (and projective rescaling), we normalize its coefficient to 1.

Consequently, no continuous parameter remains, and a convenient normal form is

$$\phi_3^{\mathrm{nf}} = x_1x_3^2 + x_1x_3x_4 + x_1x_4^2 + x_2^2x_5 + x_0x_5^2 + x_1x_2x_6 + x_0x_3x_6 + x_0x_4x_6.$$

Finally, $\dim W^H = 8$ and $\dim C_G(H) = 8$, while $H \simeq \mathbb{G}_m$ acts trivially on W^H. Thus the effective group dimension is $8 - 1 = 7$, and after projectivizing we obtain

$$\dim(\Phi_3) = 8 - 7 - 1 = 0.$$

4.4 Case $k = 4$

1-PS limit.

$$\lambda_4(t) = \mathrm{diag}\big(t^3,\, t^2,\, t,\, 1,\, t^{-1},\, t^{-2},\, t^{-3}\big), \quad t \in \mathbb{G}_m.$$

For a generic f_4 as in Section 3, the 1-PS limit is

$$\begin{aligned}\phi_4 &:= \lim_{t\to 0} \lambda_4(t) \cdot f_4 \\ &= a_1\, x_3^3 + a_2\, x_2x_3x_4 + a_3\, x_1x_4^2 + a_4\, x_2^2x_5 + a_5\, x_1x_3x_5 + a_6\, x_0x_4x_5 + a_7\, x_1x_2x_6 + a_8\, x_0x_3x_6.\end{aligned}$$

H and $C_G(H)$. Let $H = \lambda_4(\mathbb{G}_m)$. The diagonal weights on $\langle x_0, \dots, x_6\rangle$ are pairwise distinct; hence $C_G(H) = T$ (the maximal diagonal torus). Each monomial of ϕ_4 has H-weight 0, so $\phi_4 \in W^H$.

Polystability (Luna + convex-hull). By Luna's criterion, closedness of the SL(7)-orbit of ϕ_4 is equivalent to closedness of the T-orbit in W^H. By the convex-hull criterion (Theorem 4.1), it suffices to check that 0 is an interior point of

$$\mathrm{Conv}(\mathrm{Supp}(\phi_4)) \subset X(T)_{\mathbb{R}} \cong \mathbb{R}^7/\mathbb{R}(1,\dots,1),$$

which holds because the exponent vectors satisfy

$$\begin{aligned}(0,0,0,3,0,0,0) + (0,0,1,1,1,0,0) + (0,1,0,0,2,0,0) + (0,0,2,0,0,1,0)\\ + 2(0,1,0,1,0,1,0) + 6(1,0,0,0,1,1,0) + 6(0,1,1,0,0,0,1)\\ + 3(1,0,0,1,0,0,1) = 9(1,1,1,1,1,1,1).\end{aligned} \tag{4}$$

Hence, ϕ_4 is polystable and $\mathrm{SL}(7)\cdot\phi_4$ is closed.

Normal form and component dimension. Since every monomial of ϕ_4 has H-weight 0, the subgroup $H \subset T$ acts *trivially* on W^H. Hence the T-action on $\mathbb{P}(W^H) \cong \mathbb{P}^7$ factors through the effective torus T/H, which has dimension 5.

For a point of $\mathbb{P}(W^H)$ with all eight coefficients nonzero, the stabilizer in T/H is finite (equivalently, the eight exponent vectors have affine rank 5), so a general (T/H)-orbit has dimension 5. Therefore the corresponding boundary component has dimension

$$\dim \Phi_4 = \dim \mathbb{P}(W^H) - \dim(T/H) = 7 - 5 = 2.$$

Using the 5 parameters of T/H together with the overall projective scaling, we can normalize *six* of the eight coefficients to 1, leaving two essential parameters. A convenient normal form is

$$\phi_4^{\mathrm{nf}}(\alpha,\beta) = x_3^3 + x_2x_3x_4 + x_1x_4^2 + x_2^2x_5 + x_1x_3x_5 + x_0x_4x_5 + \alpha\, x_1x_2x_6 + \beta\, x_0x_3x_6, \qquad (\alpha,\beta) \in (\mathbb{C}^\times)^2.$$

In particular, the corresponding component Φ_4 of the moduli is two-dimensional.

4.5 Case $k = 5$

1-PS limit. Set

$$\lambda_5(t) = \mathrm{diag}\big(t^4,\, t^2,\, t,\, 1,\, t^{-1},\, t^{-2},\, t^{-4}\big), \qquad t \in \mathbb{G}_m.$$

For a generic f_5 as in Section 3, the 1-PS limit is

$$\phi_5 := \lim_{t\to 0} \lambda_5(t)\cdot f_5 = a_1x_3^3 + a_2x_2x_3x_4 + a_3x_1x_4^2 + a_4x_2^2x_5 + a_5x_1x_3x_5 + a_6x_0x_5^2 + a_7x_1^2x_6 + a_8x_0x_3x_6.$$

H and $C_G(H)$. Let $H = \lambda_5(\mathbb{G}_m)$. The diagonal weights on $\langle x_0, \dots, x_6\rangle$ are pairwise distinct; hence, $C_G(H) = T$ (the maximal diagonal torus). Each monomial of ϕ_5 has H-weight 0, so $\phi_5 \in W^H$.

Polystability (Luna + convex-hull). By Luna's criterion, closedness of the SL(7)-orbit of ϕ_5 is equivalent to closedness of the T-orbit in the H-fixed subspace. By the convex-hull criterion (Theorem 4.1), it suffices to check that 0 lies in the interior of $\mathrm{Conv}(\mathrm{Supp}(\phi_5)) \subset X(T)_{\mathbb{R}} \cong \mathbb{R}^7/\mathbb{R}(1,\dots,1)$. This holds because the exponent vectors of the eight monomials satisfy

$$\begin{aligned}&(0,0,0,3,0,0,0) + (0,0,1,1,1,0,0)\\ &\quad + 10(0,1,0,0,2,0,0) + 10(0,0,2,0,0,1,0)\\ &\quad + (0,1,0,1,0,1,0) + 5(1,0,0,0,0,2,0)\\ &\quad + 5(0,2,0,0,0,0,1) + 16(1,0,0,1,0,0,1)\\ &= 21(1,1,1,1,1,1,1),\end{aligned} \tag{5}$$

Hence, ϕ_5 is polystable and $\mathrm{SL}(7)\cdot\phi_5$ is closed.

Normal form and component dimension. Since every monomial of ϕ_5 has H-weight 0, the subgroup $H \subset T$ acts *trivially* on W^H. Hence the T-action on $\mathbb{P}(W^H) \cong \mathbb{P}^7$ factors through the effective torus T/H, which has dimension 5.

For a point of $\mathbb{P}(W^H)$ with all eight coefficients nonzero, the stabilizer in T/H is finite (equivalently, the eight exponent vectors have affine rank 5), so a general (T/H)-orbit has dimension 5. Therefore the corresponding boundary component has dimension

$$\dim \Phi_5 = \dim \mathbb{P}(W^H) - \dim(T/H) = 7 - 5 = 2.$$

Using the 5 parameters of T/H together with the overall projective scaling, we can normalize *six* of the eight coefficients to 1, leaving two essential parameters. A convenient normal form is

$$\phi_5^{\mathrm{nf}}(\alpha,\beta) = x_3^3 + x_2x_3x_4 + x_1x_4^2 + x_2^2x_5 + x_1x_3x_5 + x_0x_5^2 + \alpha\, x_1^2x_6 + \beta\, x_0x_3x_6, \qquad (\alpha,\beta) \in (\mathbb{C}^\times)^2.$$

In particular, the corresponding component Φ_5 of the moduli is two-dimensional.

4.6 Case $k = 6$

1-PS limit. Set

$$\lambda_6(t) = \mathrm{diag}\big(t^5,\, t^3,\, t^2,\, t,\, t^{-1},\, t^{-4},\, t^{-6}\big), \quad t \in \mathbb{G}_m.$$

For a generic f_6 as in Section 3, the 1-PS limit is

$$\phi_6 := \lim_{t\to 0} \lambda_6(t)\cdot f_6 = a_1\, x_2x_4^2 + a_2\, x_2^2x_5 + a_3\, x_1x_3x_5 + a_4\, x_0x_4x_5 + a_5\, x_1^2x_6 + a_6\, x_0x_3x_6.$$

H and $C_G(H)$. Let $H = \lambda_6(\mathbb{G}_m)$. The diagonal weights on $\langle x_0,\dots,x_6\rangle$ are pairwise distinct; hence $C_G(H) = T$ (the maximal diagonal torus). Each monomial of ϕ_6 has H-weight 0, so $\phi_6 \in W^H$.

Polystability (Luna + convex-hull). Based on Luna's criterion, closedness of the SL(7)-orbit of ϕ_6 is equivalent to closedness of the T-orbit in the H-fixed subspace. By Theorem 4.1, it suffices to check that 0 is an interior point of

$$\mathrm{Conv}(\mathrm{Supp}(\phi_6)) \subset X(T)_{\mathbb{R}} \cong \mathbb{R}^7/\mathbb{R}(1,\dots,1),$$

which holds because the exponent vectors of the six monomials of ϕ_6 satisfy the positive relation

$$\begin{aligned}&(0,0,1,0,2,0,0)+(0,0,2,0,0,1,0)+(0,1,0,1,0,1,0)\\&\quad+(1,0,0,0,1,1,0)+(0,2,0,0,0,0,1)+2(1,0,0,1,0,0,1)\\&=3(1,1,1,1,1,1,1).\end{aligned}\tag{6}$$

Hence, ϕ_6 is polystable and $\mathrm{SL}(7)\cdot\phi_6$ is closed.

Normal form and component dimension. A diagonal scaling $\mathrm{diag}(\mu_0,\cdots,\mu_6)\in T$ with $\Pi\mu_i=1$, together with an overall scalar, acts on the six coefficients via the characters determined by the exponent vectors in (6). Hence we may normalize all six coefficients to 1 simultaneously. A normal form is therefore

$$\phi_6^{\mathrm{nf}}=x_2x_4^2+x_2^2x_5+x_1x_3x_5+x_0x_4x_5+x_1^2x_6+x_0x_3x_6.$$

The residual T-stabilizer is finite; hence, the corresponding component Φ_6 of the moduli is zero-dimensional.

4.7 Case $k=7$

1-PS limit. Set

$$\lambda_7(t)=\mathrm{diag}\big(t^6,\,t^4,\,t^2,\,t,\,t^{-2},\,t^{-3},\,t^{-8}\big)\,,\qquad t\in\mathbb{G}_m.$$

For a generic f_7 as in Section 3, the 1-PS limit is

$$\phi_7:=\lim_{t\to0}\lambda_7(t)\cdot f_7=a_1\,x_3^2x_4+a_2\,x_1x_4^2+a_3\,x_2x_3x_5+a_4\,x_0x_5^2+a_5\,x_1^2x_6+a_6\,x_0x_2x_6.$$

H and $C_G(H)$. Let $H=\lambda_7(\mathbb{G}_m)$. The diagonal weights on $\langle x_0,\ldots,x_6\rangle$ are pairwise distinct; hence $C_G(H)=T$ (the maximal diagonal torus). Each monomial of ϕ_7 has H-weight 0, so ϕ_7 lies in the H-fixed subspace.

Polystability (Luna + convex-hull). By Luna's criterion, closedness of the $\mathrm{SL}(7)$-orbit of ϕ_7 is equivalent to closedness of the T-orbit in the H-fixed subspace. By the convex-hull criterion (Theorem 4.1), it suffices to check that 0 is an interior point of

$$\mathrm{Conv}(\mathrm{Supp}(\phi_7))\subset X(T)_{\mathbb{R}}\cong\mathbb{R}^7/\mathbb{R}(1,\ldots,1),$$

which holds because

$$\begin{aligned}&(0,0,0,2,1,0,0)+(0,1,0,0,2,0,0)+(0,0,1,1,0,1,0)\\&\quad+(1,0,0,0,0,2,0)+(0,2,0,0,0,0,1)+2(1,0,1,0,0,0,1)\\&=3(1,1,1,1,1,1,1),\end{aligned}\tag{7}$$

written in terms of the exponent vectors of the six monomials of ϕ_7. Hence, ϕ_7 is polystable and $\mathrm{SL}(7)\cdot\phi_7$ is closed.

Normal form and component dimension. A diagonal scaling $\mathrm{diag}(\mu_0, \dots, \mu_6) \in T$ with $\prod \mu_i = 1$, together with an overall scalar, acts on the six coefficients via the characters determined by the exponent vectors in (7). Thus we may normalize all six coefficients to 1 simultaneously. Therefore, a normal form is

$$\phi_7^{\mathrm{nf}} = x_3^2x_4 + x_1x_4^2 + x_2x_3x_5 + x_0x_5^2 + x_1^2x_6 + x_0x_2x_6.$$

The residual T-stabilizer is finite; hence, the corresponding component Φ_7 of the moduli is zero-dimensional.

4.8 Case $k = 8$

1-PS limit. Set

$$\lambda_8(t) = \mathrm{diag}\big(t^4,\, t,\, t,\, 1,\, t^{-2},\, t^{-2},\, t^{-2}\big)\,, \qquad t \in \mathbb{G}_m.$$

For a generic f_8 as in Section 3, the 1-PS limit is

$$\begin{aligned}\phi_8 := \lim_{t\to 0} \lambda_8(t)\cdot f_8 = {} & a_1x_3^3 + a_2x_1^2x_4 + a_3x_1x_2x_4 + a_4x_2^2x_4 + a_6x_1^2x_5 + a_7x_1x_2x_5 + a_8x_2^2x_5 \\ & + a_{11}x_1^2x_6 + a_{12}x_1x_2x_6 + a_{13}x_2^2x_6 + a_5x_0x_4^2 + a_9x_0x_4x_5 + a_{14}x_0x_4x_6 \\ & + a_{10}x_0x_5^2 + a_{15}x_0x_5x_6 + a_{16}x_0x_6^2.\end{aligned}$$

H and $C_G(H)$. Let $H = \lambda_8(\mathbb{G}_m)$. The weights on $\langle x_0, \dots, x_6\rangle$ are $(4, 1, 1, 0, -2, -2, -2)$ with multiplicities $(1, 2, 1, 3)$; hence,

$$C_G(H) = \Big\{\mathrm{diag}(\alpha)\oplus A\oplus\mathrm{diag}(\beta)\oplus B :\ \alpha,\beta\in\mathbb{G}_m,\ A\in\mathrm{GL}(2),\ B\in\mathrm{GL}(3),\ \alpha\beta\det(A)\det(B) = 1\Big\}.$$

Thus $C_G(H) \cong \big(\mathbb{G}_m \times \mathrm{GL}(2) \times \mathbb{G}_m \times \mathrm{GL}(3)\big) \cap \mathrm{SL}(7)$ and $\dim C_G(H) = 14$. Every monomial of ϕ_8 has H-weight 0; hence, $\phi_8 \in W^H$.

Polystability (Luna + Casimiro–Florentino). Based on Luna's reduction, the closedness of the SL(7)-orbit of ϕ_8 is equivalent to polystability for the $C_G(H)$-action on W^H. After conjugating inside the GL(2)- and GL(3)-blocks, any $\lambda \in Y(C_G(H))$ may be taken with

$$\mathrm{wt}(x_0, \dots, x_6) = (\alpha,\, s+u,\, s-u,\, \beta,\, \gamma+v_1,\, \gamma+v_2,\, \gamma+v_3),$$

where $\alpha, \beta, s, u, \gamma, v_i \in \mathbb{Z}$, $v_1 + v_2 + v_3 = 0$, and

$$S := \alpha + \beta + 2s + 3\gamma = 0$$

is the SL-constraint fixed in Convention 4.6. Let $w(\cdot)$ be the weight of a monomial. A direct computation yields the positive linear identity

$$\begin{aligned}\sum_{j=4}^{6}\Big(w(x_1^2x_j) + w(x_2^2x_j) + 2\,w(x_1x_2x_j)\Big) + 3w(x_0x_4^2) + 3w(x_0x_5^2) + 3w(x_0x_6^2) \\ + w(x_0x_4x_5) + w(x_0x_4x_6) + w(x_0x_5x_6) + 4\,w(x_3^3) = 12S.\end{aligned} \tag{8}$$

If $\lambda \in \Lambda_{\phi_8}$, then all the above weights are ≥ 0 and $S = 0$; therefore, by (8) they must all vanish. Solving gives

$$\beta = 0, \quad v_1 = v_2 = v_3 = 0, \quad u = 0, \quad 2s + \gamma = 0, \quad \alpha + 2\gamma = 0.$$

Putting $s = k \in \mathbb{Z}$, we obtain

$$\mu_k(t) = \operatorname{diag}\bigl(t^{4k},\, t^k,\, t^k,\, 1,\, t^{-2k},\, t^{-2k},\, t^{-2k}\bigr), \qquad \Lambda_{\phi_8} = \{\mu_k \mid k \in \mathbb{Z}\},$$

which is symmetric. Hence, by the Casimiro–Florentino criterion, ϕ_8 is polystable.

Normal form and component dimension. On the H-fixed subspace

$$W^H = \underbrace{\operatorname{Sym}^2\langle x_1, x_2\rangle \otimes \langle x_4, x_5, x_6\rangle}_{\text{(I)}} \oplus \underbrace{x_0 \otimes \operatorname{Sym}^2\langle x_4, x_5, x_6\rangle}_{\text{(II)}} \oplus \underbrace{\langle x_3^3\rangle}_{\text{(III)}},$$

use $\mathrm{GL}(3)$ on $\langle x_4, x_5, x_6\rangle$ to diagonalize the quadratic, then the left/right actions on $\operatorname{Sym}^2\langle x_1, x_2\rangle \otimes \langle x_4, x_5, x_6\rangle$ to diagonalize the 3×3 block (SVD-type reduction under left $\operatorname{Sym}^2\mathrm{GL}(2)$ and right $\mathrm{O}(U, q)$, here $U = \langle x_4, x_5, x_6\rangle$ and $q = x_4^2 + x_5^2 + x_6^2$) and normalize one diagonal entry to 1; the remaining two appear as parameters ρ, σ. A normal form is

$$\phi_8^{\mathrm{nf}}(\rho, \sigma) = x_3^3 + x_0x_4^2 + x_0x_5^2 + x_0x_6^2 + x_1^2x_4 + \rho\, x_1x_2x_5 + \sigma\, x_2^2x_6, \qquad (\rho, \sigma) \in (\mathbb{C}^\times)^2.$$

As $\dim W^H = 16$ and the effective action has dimension 13 (after projectivizing), the closed component has dimension $16 - 13 - 1 = 2$.

Remark 4.7. *Here $\mathrm{O}(U, q)$ denotes the orthogonal group of the quadratic space (U, q), i.e. $\mathrm{O}(U, q) = \{\, g \in \mathrm{GL}(U) \mid q(gu) = q(u)\ \forall u \in U\}$. In the chosen basis $U = \langle x_4, x_5, x_6\rangle$ with $q = x_4^2 + x_5^2 + x_6^2$, this is $\{\, B \in \mathrm{GL}_3 \mid B^{\mathsf{T}}B = I_3\,\}$.*

The residual T-stabilizer is finite; hence, the corresponding component Φ_8 of the moduli is two-dimensional.

4.9 Case $k = 9$

1-PS limit. Set

$$\lambda_9(t) = \operatorname{diag}\bigl(t^2,\, t^2,\, 1,\, 1,\, t^{-1},\, t^{-1},\, t^{-2}\bigr), \qquad t \in \mathbb{G}_m.$$

For a generic f_9 as in Section 3, the 1-PS limit is

$$\begin{aligned}\phi_9 := \lim_{t\to 0} \lambda_9(t) \cdot f_9 = {} & a_1x_2^3 + a_2x_2^2x_3 + a_3x_2x_3^2 + a_4x_3^3 \\ & + a_5x_0x_4^2 + a_6x_1x_4^2 + a_7x_0x_4x_5 + a_8x_1x_4x_5 \\ & + a_9x_0x_5^2 + a_{10}x_1x_5^2 \\ & + a_{11}x_0x_2x_6 + a_{12}x_1x_2x_6 + a_{13}x_0x_3x_6 + a_{14}x_1x_3x_6.\end{aligned}$$

(All monomials have H-weight 0.)

H **and** $C_G(H)$**.** Let $H = \lambda_9(\mathbb{G}_m)$. The H-weights on $\langle x_0, \dots, x_6\rangle$ are $(2, 2, 0, 0, -1, -1, -2)$ with block decomposition $\langle x_0, x_1\rangle$, $\langle x_2, x_3\rangle$, $\langle x_4, x_5\rangle$, $\langle x_6\rangle$. Hence,

$$C_G(H) = \Big\{ A\oplus B\oplus C\oplus\gamma :\ A, B, C \in \mathrm{GL}(2),\ \gamma \in \mathbb{G}_m,\ \det(A)\det(B)\det(C)\gamma = 1 \Big\}$$
$$\cong\ \big(\mathrm{GL}(2)^3 \times \mathbb{G}_m\big) \cap \mathrm{SL}(7).$$

Polystability (Luna + Casimiro–Florentino). By Luna's reduction, the closedness of the SL(7)-orbit of ϕ_9 is equivalent to polystability for the $C_G(H)$-action on the H-fixed locus. After conjugating inside each GL(2)-block, any $\lambda \in Y\big(C_G(H)\big)$ may be taken with

$$\mathrm{wt}(x_0, \dots, x_6) = (a{+}u,\ a{-}u,\ b{+}v,\ b{-}v,\ c{+}w,\ c{-}w,\ d), \qquad S := 2a{+}2b{+}2c{+}d = 0.$$

A direct computation yields the positive linear identity

$$\begin{aligned}&\big[w(x_2^3) + w(x_2^2x_3) + w(x_2x_3^2) + w(x_3^3)\big]\\ &\quad + 2\big[w(x_0x_4^2) + w(x_0x_4x_5) + w(x_0x_5^2) + w(x_1x_4^2) + w(x_1x_4x_5) + w(x_1x_5^2)\big]\\ &\quad + 3\big[w(x_0x_2x_6) + w(x_1x_2x_6) + w(x_0x_3x_6) + w(x_1x_3x_6)\big] = 12\,S. \qquad (9)\end{aligned}$$

If $\lambda \in \Lambda_{\phi_9}$, then all the above weights are ≥ 0 and $S = 0$; hence, by (9) they all vanish. Solving gives

$$b = v = u = w = 0, \qquad a + 2c = 0, \qquad a + d = 0.$$

Writing $a = 2k$ with $k \in \mathbb{Z}$ we obtain

$$\mu_k(t) := \mathrm{diag}\big(t^{2k},\ t^{2k},\ 1,\ 1,\ t^{-k},\ t^{-k},\ t^{-2k}\big), \qquad \Lambda_{\phi_9} = \{\mu_k \mid k \in \mathbb{Z}\}.$$

Thus Λ_{ϕ_9} is symmetric, and by the Casimiro–Florentino criterion ϕ_9 is polystable; in particular $\mathrm{SL}(7)\cdot\phi_9$ is closed.

Normal form and component dimension. From now on, we normalize coefficients under the $C_G(H)$-action. Decompose

$$W^H = \underbrace{\mathrm{Sym}^3\langle x_2, x_3\rangle}_{\text{(I)}} \ \oplus\ \underbrace{\langle x_0, x_1\rangle \otimes \mathrm{Sym}^2\langle x_4, x_5\rangle}_{\text{(II)}} \ \oplus\ \underbrace{\langle x_0, x_1\rangle \otimes \langle x_2, x_3\rangle \otimes \langle x_6\rangle}_{\text{(III)}}.$$

(I) Binary cubic $\mathrm{Sym}^3\langle x_2, x_3\rangle$**.** A general element is $\mathrm{GL}(\langle x_2, x_3\rangle)$-equivalent (up to overall scaling) to

$$x_2^2x_3 + \tau\, x_2x_3^2, \qquad \tau \in \mathbb{C}^\times,$$

leaving one parameter τ.

Indeed, over $\mathbb{C}$ any binary cubic $F \in \mathrm{Sym}^3\langle x_2, x_3\rangle$ factors as

$$F = L_1L_2L_3, \qquad L_i \in \langle x_2, x_3\rangle,$$

and for a general F the three linear factors (equivalently, the three roots on $\mathbb{P}^1 = \mathbb{P}(\langle x_2, x_3\rangle)$) are distinct. Since $\mathrm{PGL}(\langle x_2, x_3\rangle)$ acts 3–transitively on $\mathbb{P}^1$,

after a change of basis we may send two of the roots to $[1:0]$ and $[0:1]$, i.e. arrange $L_1 = x_2$ and $L_2 = x_3$. Thus

$$F = x_2x_3(\alpha x_2 + \beta x_3), \qquad \alpha, \beta \in \mathbb{C}^\times.$$

Up to an overall scalar we may take $\alpha = 1$ and set $\tau := \beta/\alpha$, giving

$$F \sim x_2^2x_3 + \tau\, x_2x_3^2.$$

(II) $\langle x_0, x_1\rangle \otimes \mathrm{Sym}^2\langle x_4, x_5\rangle$ **(2×3 block).** Via the right $\mathrm{GL}(\langle x_4, x_5\rangle)$ (through Sym^2) diagonalize a reference quadratic to $q = x_4^2 + x_5^2$; the residual right group is $O(U, q)$ on $U = \langle x_4, x_5\rangle$. Using the left $\mathrm{GL}(\langle x_0, x_1\rangle)$ together with this right orthogonal action (an SVD-type reduction), eliminate the x_4x_5 cross term and equalize the x_4^2 entries. After central torus/projective scalings,

$$x_0(x_4^2 + \rho\, x_5^2) + x_1(x_4^2 + x_5^2), \qquad \rho \in \mathbb{C}^\times.$$

(III) $\langle x_0, x_1\rangle \otimes \langle x_2, x_3\rangle \otimes \langle x_6\rangle$ **(2×2 block).** With $\mathrm{GL}(\langle x_0, x_1\rangle)$, $\mathrm{GL}(\langle x_2, x_3\rangle)$ (respecting the choice in (I)), and scaling x_6, we diagonalize to

$$x_0x_2x_6 + x_1x_3x_6,$$

and normalize the coefficients to 1.

Combining the three steps yields the normal form

$$\phi_9^{\mathrm{nf}}(\tau, \rho) = x_2^2x_3 + \tau\, x_2x_3^2 + x_0x_4^2 + \rho\, x_0x_5^2 + x_1x_4^2 + x_1x_5^2 + x_0x_2x_6 + x_1x_3x_6, \qquad (\tau, \rho) \in (\mathbb{C}^\times)^2.$$

As $\dim W^H = 14$ and $\dim C_G(H) = 12$, the effective action has dimension 11; after projectivizing, we obtain $14 - 11 - 1 = 2$. The residual T-stabilizer is finite; hence, the corresponding component Φ_9 of the moduli is two-dimensional.

4.10 Case $k = 10$

1-PS limit. Set

$$\lambda_{10}(t) = \mathrm{diag}\big(t^2,\, t,\, 1,\, 1,\, t^{-1},\, t^{-1},\, t^{-1}\big), \qquad t \in \mathbb{G}_m.$$

For a generic f_{10} as in Section 3, the 1-PS limit is

$$\begin{aligned}\phi_{10} := \lim_{t\to 0} \lambda_{10}(t) \cdot f_{10} = {} & a_1x_2^3 + a_2x_2^2x_3 + a_3x_2x_3^2 + a_4x_3^3 \\ & + a_5x_1x_2x_4 + a_6x_1x_3x_4 + a_8x_1x_2x_5 + a_9x_1x_3x_5 \\ & + a_{12}x_1x_2x_6 + a_{13}x_1x_3x_6 \\ & + a_7x_0x_4^2 + a_{10}x_0x_4x_5 + a_{11}x_0x_5^2 + a_{14}x_0x_4x_6 + a_{15}x_0x_5x_6 + a_{16}x_0x_6^2.\end{aligned}$$

H **and** $C_G(H)$**.** Let $H = \lambda_{10}(\mathbb{G}_m)$. The H-weights on $\langle x_0, \dots, x_6\rangle$ are $(2, 1, 0, 0, -1, -1, -1)$ with block multiplicities $(1, 1, 2, 3)$; hence,

$$\begin{aligned}C_G(H) &= \Big\{\ \mathrm{diag}(\alpha) \oplus \mathrm{diag}(\beta) \oplus A \oplus B :\ \alpha, \beta \in \mathbb{G}_m,\ A \in \mathrm{GL}(2),\ B \in \mathrm{GL}(3),\ \alpha\beta \det(A) \det(B) = 1 \Big\} \\ &\cong \big(\mathbb{G}_m \times \mathbb{G}_m \times \mathrm{GL}(2) \times \mathrm{GL}(3)\big) \cap \mathrm{SL}(7). \qquad (10)\end{aligned}$$

Every monomial of ϕ_{10} has H-weight 0, so $\phi_{10} \in W^H$.

Polystability (Luna + Casimiro–Florentino). By Luna's reduction, the closedness of the SL(7)-orbit of ϕ_{10} is equivalent to polystability for the $C_G(H)$-action on W^H. After conjugating within the GL(2)- and GL(3)-blocks, any $\lambda \in Y\big(C_G(H)\big)$ may be taken as

$$\mathrm{wt}(x_0, \dots, x_6) = (\alpha,\, \beta,\, s+u,\, s-u,\, \gamma+v_1,\, \gamma+v_2,\, \gamma+v_3),$$

where $\alpha, \beta, s, u, \gamma, v_i \in \mathbb{Z}$ with $v_1 + v_2 + v_3 = 0$, and by Convention 4.6 the SL-constraint is

$$S := \alpha + \beta + 2s + 3\gamma = 0.$$

A direct computation gives the positive linear identity

$$\begin{aligned}
&\big[w(x_2^3) + w(x_2^2x_3) + w(x_2x_3^2) + w(x_3^3)\big] \\
&\quad + 2\big[w(x_0x_4^2) + w(x_0x_4x_5) + w(x_0x_5^2) + w(x_0x_4x_6) + w(x_0x_5x_6) + w(x_0x_6^2)\big] \\
&\quad + 2\big[w(x_1x_2x_4) + w(x_1x_3x_4) + w(x_1x_2x_5) + w(x_1x_3x_5) + w(x_1x_2x_6) + w(x_1x_3x_6)\big] \;=\; 12\,S.
\end{aligned} \tag{11}$$

If $\lambda \in \Lambda_{\phi_{10}}$, then all the weights above are ≥ 0 and $S = 0$; hence, by (11), they all vanish. Solving yields

$$\beta = -\gamma, \qquad s = 0, \qquad u = 0, \qquad v_1 = v_2 = v_3 = 0, \qquad \alpha = -2\gamma.$$

Writing $\gamma = -k$ with $k \in \mathbb{Z}$ we obtain

$$\mu_k(t) := \mathrm{diag}\big(t^{2k},\, t^k,\, 1,\, 1,\, t^{-k},\, t^{-k},\, t^{-k}\big), \qquad \Lambda_{\phi_{10}} = \{\mu_k \mid k \in \mathbb{Z}\}.$$

Thus $\Lambda_{\phi_{10}}$ is symmetric, and by the Casimiro–Florentino criterion, ϕ_{10} is polystable; in particular $\mathrm{SL}(7) \cdot \phi_{10}$ is closed.

Normal form and component dimension. Set

$$U := \langle x_2, x_3 \rangle, \qquad V := \langle x_4, x_5, x_6 \rangle.$$

Then the H-fixed subspace decomposes as

$$W^H = \underbrace{\mathrm{Sym}^3 U}_{\text{(I)}} \;\oplus\; \underbrace{x_1 \otimes U \otimes V}_{\text{(II)}} \;\oplus\; \underbrace{x_0 \otimes \mathrm{Sym}^2 V}_{\text{(III)}}.$$

We normalize a *general* element of W^H under the $C_G(H)$-action, taking care that the $\mathrm{GL}(U)$-freedom is shared by blocks (I) and (II).

(III) The quadratic block $x_0 \otimes \mathrm{Sym}^2 V$. For a general element, the quadratic form on V is nondegenerate. Acting by $\mathrm{GL}(V) \subset C_G(H)$ and absorbing an overall scalar into the x_0-factor (using the central torus/projective scaling), we may assume the quadratic block is

$$x_0\, q, \qquad q = x_4x_5 + x_6^2.$$

After this normalization, the residual right group on V is the orthogonal group

$$\mathrm{O}(V, q) = \{\, B \in \mathrm{GL}(V) \mid q(Bv) = q(v) \,\}.$$

(II) The mixed block $x_1 \otimes U \otimes V$. Write the mixed term as $x_1 m$ with $m \in U \otimes V$, and represent m by a 2×3 matrix after choosing bases of U and V. For a general element, $\operatorname{rank}(m) = 2$ and the restriction $q|_{\operatorname{Im}(m)}$ is nondegenerate. Equivalently, pulling back q along m yields a nondegenerate binary quadratic form on U:

$$q_m(u) := q(m(u)) \in \operatorname{Sym}^2 U^\vee.$$

Using $\mathrm{GL}(U)$ we may put q_m into split form. Then, by Witt's extension theorem, there exists $B \in \mathrm{O}(V, q)$ sending the resulting hyperbolic pair in $\operatorname{Im}(m)$ to $\langle x_4, x_5 \rangle$. Hence, after the $\mathrm{GL}(U) \times \mathrm{O}(V, q)$-action and scaling x_1, we may assume

$$x_1 m = x_1(x_2 x_5 + x_3 x_4).$$

(If $\operatorname{rank}(m) = 1$ or $q|_{\operatorname{Im}(m)}$ is degenerate, one lands in a proper closed subset; for the generic closed orbit in this component we work on the open set above.)

Residual symmetry after (II) and (III). After fixing (II) and (III) as above, the stabilizer contains the one-dimensional torus

$$T = \left\{ x_2 \mapsto t\, x_2,\ x_3 \mapsto t^{-1} x_3,\ x_4 \mapsto t\, x_4,\ x_5 \mapsto t^{-1} x_5,\ x_6 \mapsto x_6 \right\} \subset C_G(H), \qquad t \in \mathbb{G}_m,$$

which preserves both $x_2 x_5 + x_3 x_4$ and $x_4 x_5 + x_6^2$.

(I) The cubic block $\operatorname{Sym}^3 U$. A general element of $\operatorname{Sym}^3 U$ is

$$c_0 x_2^3 + c_1 x_2^2 x_3 + c_2 x_2 x_3^2 + c_3 x_3^3.$$

Under T, these monomials have weights t^3, t^1, t^{-1}, t^{-3}, respectively. On the dense open set where $c_0 c_3 \neq 0$, using the T-parameter and projective scaling we normalize $c_0 = c_3 = 1$. Writing $\tau := c_1$ and $\rho := c_2$, we obtain

$$x_2^3 + \tau\, x_2^2 x_3 + \rho\, x_2 x_3^2 + x_3^3.$$

Combining (I)–(III), we may take the closed-orbit representative in Case $k = 10$ to be

$$\phi_{10}^{\mathrm{nf}}(\tau, \rho) = x_2^3 + \tau\, x_2^2 x_3 + \rho\, x_2 x_3^2 + x_3^3 + x_1 x_2 x_5 + x_1 x_3 x_4 + x_0 x_4 x_5 + x_0 x_6^2, \qquad (\tau, \rho) \in (\mathbb{C}^\times)^2$$

As $\dim W^H = 16$ and $\dim C_G(H) = 14$, the effective action has dimension 13; after projectivizing, we obtain $16 - 13 - 1 = 2$. *The residual T-stabilizer of a general representative is finite; hence, the corresponding component Φ_{10} of the moduli is two-dimensional.*

4.11 Case $k = 11$

1-PS limit. Set

$$\lambda_{11}(t) = \operatorname{diag}\big(t^2,\ 1,\ 1,\ 1,\ 1,\ t^{-1},\ t^{-1}\big), \qquad t \in \mathbb{G}_m.$$

For a generic f_{11} as in Section 3, the 1-PS limit is

$$\phi_{11} := \lim_{t \to 0} \lambda_{11}(t) \cdot f_{11} = F(x_1, x_2, x_3, x_4) + x_0\, Q(x_5, x_6),$$

where $F \in \operatorname{Sym}^3 \langle x_1, x_2, x_3, x_4 \rangle$ is a generic cubic form and $Q \in \operatorname{Sym}^2 \langle x_5, x_6 \rangle$ is a generic quadratic form.

H **and** $C_G(H)$**.** Let $H = \lambda_{11}(\mathbb{G}_m)$. The H-weights on $\langle x_0, \dots, x_6\rangle$ are $(2, 0, 0, 0, 0, -1, -1)$ with block decomposition $\langle x_0\rangle \oplus \langle x_1, x_2, x_3, x_4\rangle \oplus \langle x_5, x_6\rangle$. Hence,

$$C_G(H) = \Big\{ \operatorname{diag}(\alpha) \oplus A \oplus B \ : \ \alpha \in \mathbb{G}_m,\ A \in \mathrm{GL}(4),\ B \in \mathrm{GL}(2),\ \alpha \det(A) \det(B) = 1 \Big\}$$
$$\cong \big(\mathbb{G}_m \times \mathrm{GL}(4) \times \mathrm{GL}(2)\big) \cap \mathrm{SL}(7). \tag{12}$$

Every monomial of ϕ_{11} has H-weight 0, so $\phi_{11} \in W^H$.

Polystability (Luna + Casimiro–Florentino). By Luna's reduction, the closedness of the SL(7)-orbit of ϕ_{11} is equivalent to polystability for the $C_G(H)$-action on W^H. After conjugating inside the GL(4)- and GL(2)-blocks, any $\lambda \in Y\big(C_G(H)\big)$ may be taken with

$$\mathrm{wt}(x_0, \dots, x_6) = (\alpha,\ s_1,\ s_2,\ s_3,\ s_4,\ t{+}u,\ t{-}u), \qquad S := \alpha{+}(s_1{+}s_2{+}s_3{+}s_4){+}2t = 0$$

(Convention 4.6). Summing the λ-weights of the 20 cubic monomials in F gives $15(s_1 + s_2 + s_3 + s_4)$, while the weights of the quadratic terms x_0Q sum to $3\alpha + 6t$. Using the relation $S = 0$, we obtain the positive linear identity

$$w(F) + 5\, w(x_0Q) \ = \ 15(s_1 + s_2 + s_3 + s_4) + 15(\alpha + 2t) \ = \ 15\, S. \tag{13}$$

If $\lambda \in \Lambda_{\phi_{11}}$, then all weights are ≥ 0 and $S = 0$; by (13) they all vanish. Solving this implies that $\Lambda_{\phi_{11}}$ is symmetric, and by the Casimiro–Florentino criterion, ϕ_{11} is polystable.

Normal form and component dimension. We normalize generic elements under the action of the centralizer $C_G(H)$. The H-fixed subspace decomposes as

$$W^H = \mathrm{Sym}^3\langle x_1, x_2, x_3, x_4\rangle \ \oplus \ x_0 \otimes \mathrm{Sym}^2\langle x_5, x_6\rangle.$$

(1) **Quadratic part.** Under the GL(2)-action on $\langle x_5, x_6\rangle$, a generic binary quadratic is normalized to $x_5^2 + x_6^2$. The stabilizer of this form contains the orthogonal group O(2), contributing 1 dimension to the stabilizer of the generic point.
(2) **Cubic part.** Under the GL(4)-action on $\langle x_1, x_2, x_3, x_4\rangle$, a generic cubic surface can be written as the sum of five cubes of linear forms (**Sylvester's Pentahedron Theorem**). Let $L_0, \dots, L_4$ be fixed general linear forms in $x_1, \dots, x_4$. Then the cubic part is $\sum_{i=0}^4 c_i L_i^3$.

Thus, a closed-orbit representative is:

$$\phi_{11}^{\mathrm{nf}}(c) = \sum_{i=0}^{4} c_i L_i(x_1, x_2, x_3, x_4)^3 + x_0(x_5^2 + x_6^2),$$

where the parameters are $[c_0 : \cdots : c_4] \in \mathbb{P}^4$.

The dimension of the component is computed as follows: $\dim W^H = 20(\text{cubic}){+}$ $3(\text{quadratic}) = 23$. The effective group dimension is $\dim C_G(H) - 1 = 19$ (since H acts trivially). Since the generic stabilizer has dimension 1 (due to O(2)), the orbit dimension is $19 - 1 = 18$.

$$\dim(\Phi_{11}) = 23 - 18 - 1 = 4.$$

(Note: This case is structurally isomorphic to Case $k = 21$.)

4.12 Case $k = 12$

1-PS limit. Set

$$\lambda_{12}(t) = \operatorname{diag}\bigl(t^3,\, t^2,\, t,\, t,\, t^{-1},\, t^{-2},\, t^{-4}\bigr), \qquad t \in \mathbb{G}_m.$$

For a generic f_{12} as in Section 3, the 1-PS limit is

$$\phi_{12} := \lim_{t\to 0} \lambda_{12}(t)\cdot f_{12} = a_1x_1x_4^2 + a_2x_2^2x_5 + a_3x_2x_3x_5 + a_4x_3^2x_5 + a_5x_0x_4x_5 + a_6x_1^2x_6 + a_7x_0x_2x_6 + a_8x_0x_3x_6.$$

H and $C_G(H)$. Let $H = \lambda_{12}(\mathbb{G}_m)$. The H-weights on $\langle x_0, \dots, x_6\rangle$ are $(3, 2, 1, 1, -1, -2, -4)$ with multiplicities $(1, 1, 2, 1, 1, 1)$; hence,

$$\begin{aligned} C_G(H) = \Bigl\{\ \operatorname{diag}(\alpha)\oplus\operatorname{diag}(\beta)\oplus A\oplus\operatorname{diag}(\gamma,\delta,\varepsilon)\ :\ \alpha,\beta,\gamma,\delta,\varepsilon\in\mathbb{G}_m,\ A\in\mathrm{GL}(2),\\ \alpha\beta\det(A)\gamma\delta\varepsilon = 1\ \Bigr\}\ \cong\ \bigl(\mathbb{G}_m^5\times\mathrm{GL}(2)\bigr)\cap\mathrm{SL}(7), \qquad \dim C_G(H) = 8. \end{aligned} \tag{14}$$

Every monomial of ϕ_{12} has H-weight 0.

Polystability (Luna + Casimiro–Florentino). By Luna's reduction, the closedness of the SL(7)-orbit of ϕ_{12} is equivalent to polystability for the $C_G(H)$-action on W^H. After conjugating inside the GL(2)-block on $\langle x_2, x_3\rangle$, any $\lambda \in Y(C_G(H))$ may be taken with

$$\mathrm{wt}(x_0,\dots,x_6) = (\alpha,\ \beta,\ s+u,\ s-u,\ \gamma,\ \delta,\ \varepsilon), \qquad S := \alpha+\beta+2s+\gamma+\delta+\varepsilon = 0.$$

A direct computation yields the positive identity

$$\begin{aligned} 2w(x_1x_4^2) + w(x_2^2x_5) + w(x_2x_3x_5) + 2w(x_3^2x_5) + 2w(x_0x_4x_5) + 2w(x_1^2x_6)\\ + 3w(x_0x_2x_6) + w(x_0x_3x_6) = 6S. \end{aligned} \tag{15}$$

If $\lambda \in \Lambda_{\phi_{12}}$, then all the weights on the left are ≥ 0 and $S = 0$; by (15) they all vanish. Solving gives

$$u = 0, \qquad \beta = -2\gamma, \qquad \delta = -2s, \qquad \varepsilon = -2\beta = 4\gamma, \qquad \alpha = -3\gamma, \qquad s = -\gamma.$$

Writing $k = -\gamma \in \mathbb{Z}$, we obtain

$$\mu_k(t) = \operatorname{diag}\bigl(t^{3k},\, t^{2k},\, t^k,\, t^k,\, t^{-k},\, t^{-2k},\, t^{-4k}\bigr), \qquad \Lambda_{\phi_{12}} = \{\mu_k \mid k\in\mathbb{Z}\}$$

Thus $\Lambda_{\phi_{12}}$ is symmetric, and by the Casimiro–Florentino criterion ϕ_{12} is polystable; in particular, $\mathrm{SL}(7)\cdot\phi_{12}$ is closed.

Normal form and component dimension. Work on the H-fixed slice

$$W^H = \underbrace{x_1\otimes\mathrm{Sym}^2\langle x_4\rangle}_{\text{(I)}} \oplus \underbrace{\mathrm{Sym}^2\langle x_2,x_3\rangle\otimes x_5}_{\text{(II)}} \oplus \underbrace{x_0\otimes\langle x_4\rangle\otimes x_5}_{\text{(III)}} \oplus \underbrace{\mathrm{Sym}^2\langle x_1\rangle\otimes x_6}_{\text{(IV)}}$$

$$\oplus\ \underbrace{x_0\otimes\langle x_2,x_3\rangle\otimes x_6}_{\text{(V)}}.$$

Diagonalize the binary quadratic in the (II)**-block.** Using $\mathrm{GL}(\langle x_2, x_3\rangle)$, bring $\mathrm{Sym}^2\langle x_2, x_3\rangle \otimes x_5$ to $x_2^2x_5 + x_3^2x_5$; the cross term $x_2x_3x_5$ is eliminated.

Align the (V)**-block.** Within $x_0 \otimes \langle x_2, x_3\rangle \otimes x_6$, use the same GL(2) to align this block to $x_0x_2x_6$ (so the $x_0x_3x_6$ entry vanishes).

Normalize coefficients by torus scalings and projective scaling in (I)**,** (III)**,** (V)**.** Use the 1-dimensional tori on the 1-dimensional weight spaces and the overall projective scaling to set the remaining nonzero coefficients to 1.

This yields the closed orbit normal form

$$\phi_{12}^{\mathrm{nf}} = x_1x_4^2 + x_2^2x_5 + x_3^2x_5 + x_0x_4x_5 + x_1^2x_6 + x_0x_2x_6.$$

Finally, $\dim W^H = 8$ and $\dim C_G(H) = 8$; the effective action has dimension 7 (with H acting trivially). After projectivizing, we obtain $8 - 7 - 1 = 0$; hence, the corresponding boundary component is zero-dimensional.

4.13 Case $k = 13$

1-PS limit. Set

$$\lambda_{13}(t) = \mathrm{diag}\big(t^2,\, t,\, t,\, 1,\, t^{-1},\, t^{-1},\, t^{-2}\big), \qquad t \in \mathbb{G}_m.$$

For a generic f_{13} as in Section 3, the 1-PS limit is

$$\begin{aligned}\phi_{13} := \lim_{t\to 0} \lambda_{13}(t)\cdot f_{13} = {} & a_1x_3^3 + a_2x_1x_3x_4 + a_3x_2x_3x_4 + a_4x_0x_4^2 + a_5x_1x_3x_5 + a_6x_2x_3x_5 \\ & + a_7x_0x_4x_5 + a_8x_0x_5^2 + a_9x_1^2x_6 + a_{10}x_1x_2x_6 + a_{11}x_2^2x_6 + a_{12}x_0x_3x_6.\end{aligned}$$

This is H-fixed for $H = \lambda_{13}(\mathbb{G}_m)$.

H and $C_G(H)$. The H-weights on $\langle x_0, \dots, x_6\rangle$ are $(2, 1, 1, 0, -1, -1, -2)$ with blocks $\langle x_0\rangle \oplus \langle x_1, x_2\rangle \oplus \langle x_3\rangle \oplus \langle x_4, x_5\rangle \oplus \langle x_6\rangle$. Hence,

$$\begin{aligned}C_G(H) = \Big\{ \mathrm{diag}(\alpha) \,\oplus\, A \,\oplus\, \mathrm{diag}(\beta) \,\oplus\, B \,\oplus\, \mathrm{diag}(\gamma) :\ \alpha,\beta,\gamma \in \mathbb{G}_m,\ A, B \in \mathrm{GL}(2), \\ \alpha\,\det(A)\,\beta\,\det(B)\,\gamma = 1 \Big\} \\ \cong \big(\mathbb{G}_m \times \mathrm{GL}(2) \times \mathbb{G}_m \times \mathrm{GL}(2) \times \mathbb{G}_m\big) \cap \mathrm{SL}(7), \qquad \dim C_G(H) = 10.\end{aligned} \tag{16}$$

Each monomial of ϕ_{13} has H-weight 0.

Polystability (Luna + Casimiro–Florentino). By Luna's reduction, the closedness of the SL(7)-orbit of ϕ_{13} is equivalent to polystability for the $C_G(H)$-action on W^H. After conjugating within the two GL(2)-blocks, any $\lambda \in Y(C_G(H))$ may be taken as

$$\mathrm{wt}(x_0, \dots, x_6) = (\alpha,\, s{+}u,\, s{-}u,\, \beta,\, t{+}v,\, t{-}v,\, \gamma), \qquad S := \alpha{+}2s{+}\beta{+}2t{+}\gamma = 0 \ \text{(Convention 4.6)}.$$

A direct computation yields the positive identity

$$\begin{aligned}&\big[w(x_1^2x_6) + w(x_2^2x_6) + 2\,w(x_1x_2x_6)\big] + \big[w(x_0x_4^2) + 2\,w(x_0x_4x_5) + w(x_0x_5^2)\big] \\ &\quad + \big[w(x_1x_3x_4) + w(x_2x_3x_4) + w(x_1x_3x_5) + w(x_2x_3x_5)\big] + 2\,w(x_0x_3x_6) = 6S.\end{aligned} \tag{17}$$

If $\lambda \in \Lambda_{\phi_{13}}$, then the twelve weights on the left are ≥ 0 and $S = 0$; hence, they all vanish. Solving gives

$$u = 0, \quad v = 0, \quad \alpha + 2t = 0, \quad 2s + \gamma = 0, \quad s + \beta + t = 0, \quad \alpha + \beta + \gamma = 0.$$

Writing $s = k \in \mathbb{Z}$ yields

$$(\alpha, s, \beta, t, \gamma) = (2k,\, k,\, 0,\, -k,\, -2k), \qquad \mu_k(t) := \operatorname{diag}\bigl(t^{2k},\, t^{k},\, t^{k},\, 1,\, t^{-k},\, t^{-k},\, t^{-2k}\bigr),$$

so $\Lambda_{\phi_{13}} = \{\mu_k \mid k \in \mathbb{Z}\}$ is symmetric; by the Casimiro–Florentino criterion ϕ_{13} is polystable; hence, $\mathrm{SL}(7) \cdot \phi_{13}$ is closed.

Normal form and component dimension. We display

$$W^H = \underbrace{\operatorname{Sym}^3\langle x_3\rangle}_{\text{(I)}} \oplus \underbrace{x_0 \otimes \operatorname{Sym}^2\langle x_4, x_5\rangle}_{\text{(II)}} \oplus \underbrace{\langle x_1, x_2\rangle \otimes \langle x_4, x_5\rangle \otimes \langle x_3\rangle}_{\text{(III)}}$$
$$\oplus \underbrace{\operatorname{Sym}^2\langle x_1, x_2\rangle \otimes \langle x_6\rangle}_{\text{(IV)}} \oplus \underbrace{x_0 \otimes \langle x_3\rangle \otimes x_6}_{\text{(V)}}.$$

Normalize block by block under the $C_G(H)$-action:

(II) $x_0 \otimes \operatorname{Sym}^2\langle x_4, x_5\rangle$**.** Diagonalize the binary quadratic to

$$x_0 x_4^2 + x_0 x_5^2.$$

(III) $\langle x_1, x_2\rangle \otimes \langle x_4, x_5\rangle \otimes \langle x_3\rangle$**.** View the four x_3-bilinear terms as a 2×2 matrix on $\langle x_1, x_2\rangle \otimes \langle x_4, x_5\rangle$ and bring it to diagonal form,

$$x_1 x_3 x_4 + \rho\, x_2 x_3 x_5, \qquad \rho \in \mathbb{C}^\times.$$

(IV) $\operatorname{Sym}^2\langle x_1, x_2\rangle \otimes \langle x_6\rangle$**.** Diagonalize the symmetric 2×2 form to

$$x_1^2 x_6 + \sigma\, x_2^2 x_6, \qquad \sigma \in \mathbb{C}^\times.$$

(I) and (V). Use the three torus factors on x_0, x_3, x_6 together with projective scaling to normalize the remaining nonzero coefficients to 1.

This yields the convenient normal form

$$\phi_{13}^{\mathrm{nf}}(\rho, \sigma) = x_3^3 + x_0 x_4^2 + x_0 x_5^2 + x_1 x_3 x_4 + \rho\, x_2 x_3 x_5 + x_1^2 x_6 + \sigma\, x_2^2 x_6 + x_0 x_3 x_6, \qquad (\rho, \sigma) \in (\mathbb{C}^\times)^2.$$

Here, $\dim(W^H) = 12$ and $\dim C_G(H) = 10$; as H acts trivially, the effective group dimension is 9. After projectivizing, we obtain $12 - 9 - 1 = 2$. *The residual T-stabilizer is finite; hence, the corresponding component Φ_{13} of the moduli is two-dimensional.*

4.14 Case $k = 14$

1-PS limit. Set

$$\lambda_{14}(t) = \operatorname{diag}\bigl(t^2,\, t^2,\, 1,\, t^{-1},\, t^{-1},\, t^{-1},\, t^{-1}\bigr), \qquad t \in \mathbb{G}_m.$$

For a generic f_{14} as in Section 3, the 1-PS limit is

$$\begin{aligned}\phi_{14} := \lim_{t\to 0} \lambda_{14}(t)\cdot f_{14} = {} & a_1x_2^3 + a_2x_0x_3^2 + a_3x_1x_3^2 + a_4x_0x_3x_4 + a_5x_1x_3x_4 + a_6x_0x_4^2 + a_7x_1x_4^2 \\ & + a_8x_0x_3x_5 + a_9x_1x_3x_5 + a_{10}x_0x_4x_5 + a_{11}x_1x_4x_5 + a_{12}x_0x_5^2 + a_{13}x_1x_5^2 \\ & + a_{14}x_0x_3x_6 + a_{15}x_1x_3x_6 + a_{16}x_0x_4x_6 + a_{17}x_1x_4x_6 \\ & + a_{18}x_0x_5x_6 + a_{19}x_1x_5x_6 + a_{20}x_0x_6^2 + a_{21}x_1x_6^2.\end{aligned}$$

H and $C_G(H)$. Let $H = \lambda_{14}(\mathbb{G}_m)$. The H-weights on $\langle x_0, \dots, x_6\rangle$ are $(2, 2, 0, -1, -1, -1, -1)$ with block multiplicities $(2, 1, 4)$. Hence,

$$\begin{aligned}C_G(H) &= \Bigl\{\, A \oplus \beta \oplus B :\ A \in \mathrm{GL}(2),\ \beta \in \mathbb{G}_m,\ B \in \mathrm{GL}(4),\ \det(A)\,\beta\,\det(B) = 1 \,\Bigr\} \\ &\cong \bigl(\mathrm{GL}(2) \times \mathbb{G}_m \times \mathrm{GL}(4)\bigr) \cap \mathrm{SL}(7), \qquad \dim C_G(H) = 20.\end{aligned}$$

Every monomial of ϕ_{14} has H-weight 0; hence, $\phi_{14} \in W^H$.

Polystability (Luna + Casimiro–Florentino). By Luna's reduction, the closedness of the SL(7)-orbit of ϕ_{14} is equivalent to polystability for the $C_G(H)$-action on W^H. After conjugating inside the GL(2)- and GL(4)-blocks, any $\lambda \in Y\bigl(C_G(H)\bigr)$ may be taken with

$$\mathrm{wt}(x_0, \dots, x_6) = (a+u,\, a-u,\, b,\, c+v_1,\, c+v_2,\, c+v_3,\, c+v_4),$$

where $v_1 + v_2 + v_3 + v_4 = 0$, and with the SL-constraint (Convention 4.6)

$$S := 2a + b + 4c = 0.$$

A direct computation yields the positive identity

$$10w(x_2^3) + 3\left[\sum_{j=3}^{6}\bigl(w(x_0x_j^2) + w(x_1x_j^2)\bigr) + \sum_{3\le k<l\le 6} w(x_0x_kx_l) + w(x_1x_kx_l)\right] = 30S \tag{18}$$

If $\lambda \in \Lambda_{\phi_{14}}$, then all 21 weights are ≥ 0 and $S = 0$; hence, by (18) they all vanish. Solving gives

$$b = 0, \qquad u = 0, \qquad v_1 = v_2 = v_3 = v_4 = 0, \qquad a + 2c = 0.$$

Writing $c = -k$ with $k \in \mathbb{Z}$, we obtain

$$\mu_k(t) = \operatorname{diag}\bigl(t^{2k},\, t^{2k},\, 1,\, t^{-k},\, t^{-k},\, t^{-k},\, t^{-k}\bigr), \qquad \Lambda_{\phi_{14}} = \{\mu_k \mid k \in \mathbb{Z}\}.$$

Thus $\Lambda_{\phi_{14}}$ is symmetric, and by the Casimiro–Florentino criterion ϕ_{14} is polystable; in particular $\mathrm{SL}(7)\cdot\phi_{14}$ is closed.

Normal form and component dimension. We display W^H in block form:

$$W^H = \underbrace{\mathrm{Sym}^3\langle x_2\rangle}_{\text{(I)}} \oplus \underbrace{\langle x_0\rangle \otimes \mathrm{Sym}^2\langle x_3,x_4,x_5,x_6\rangle}_{\text{(II)}} \oplus \underbrace{\langle x_1\rangle \otimes \mathrm{Sym}^2\langle x_3,x_4,x_5,x_6\rangle}_{\text{(III)}}.$$

Equivalently, writing $U := \langle x_3,x_4,x_5,x_6\rangle$ and $V := \langle x_0,x_1\rangle$, any element of W^H can be written as

$$\phi \;=\; a\,x_2^3 + x_0Q_0(x_3,x_4,x_5,x_6) + x_1Q_1(x_3,x_4,x_5,x_6), \qquad Q_0,Q_1 \in \mathrm{Sym}^2U^\vee.$$

Using the $\mathbb{G}_m$-factor on x_2 together with projective scaling, we normalize $a = 1$.

Diagonalization of the quadratic pencil. Consider the pencil $\langle Q_0,Q_1\rangle$, i.e. the binary quartic

$$\Delta(s,t) := \det(sQ_0 + tQ_1), \qquad [s:t] \in \mathbb{P}^1.$$

On the dense open set where Δ has four distinct roots (equivalently, the pencil is *regular*, i.e. it lies off the discriminant locus), we may assume Q_0 is nondegenerate. Acting by $\mathrm{GL}(U) \cong \mathrm{GL}(4)$, we take

$$Q_0 \sim q := x_3^2 + x_4^2 + x_5^2 + x_6^2.$$

With respect to q, the residual group is $O(U,q)$, and for a regular pencil the q-selfadjoint endomorphism A defined by $Q_1(u,v) = q(Au,v)$ has four distinct eigenvalues. Hence, after an $O(U,q)$-change of coordinates,

$$Q_1 \sim \lambda_3x_3^2 + \lambda_4x_4^2 + \lambda_5x_5^2 + \lambda_6x_6^2, \qquad \lambda_3,\lambda_4,\lambda_5,\lambda_6 \text{ pairwise distinct.}$$

Reduction to a one-parameter normal form (cross-ratio). The $\mathrm{GL}(V) \cong \mathrm{GL}(2)$-action on (x_0,x_1) replaces (Q_0,Q_1) by

$$(Q_0',Q_1') = (\alpha Q_0 + \beta Q_1,\ \gamma Q_0 + \delta Q_1), \qquad \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \mathrm{GL}(2),$$

and (after re-normalizing Q_0' back to q by a diagonal rescaling in U) this transforms the eigenvalues by the fractional-linear map

$$\lambda \;\longmapsto\; \frac{\gamma + \delta\lambda}{\alpha + \beta\lambda}.$$

Thus the regular pencil is determined, up to permutation of the diagonal coordinates in U, by the unordered set $\{\lambda_3,\lambda_4,\lambda_5,\lambda_6\} \subset \mathbb{P}^1$ modulo PGL_2, i.e. by a single cross-ratio parameter.

To match the normal form displayed below, we normalize three eigenvalues to $1,-1,2$ and denote the fourth by τ. Using the 3-transitivity of PGL_2 on $\mathbb{P}^1$ and then permuting (x_3,x_4,x_5,x_6), we may assume

$$(\lambda_3,\lambda_4,\lambda_5,\lambda_6) = (1,\tau,-1,2).$$

Consequently (on the dense open set $\tau \notin \{1,-1,2\}$; and restricting further to $\tau \in \mathbb{C}^\times$ if we also require Q_1 nondegenerate), a convenient normal form is

$$\phi_{14}^{\mathrm{nf}}(\tau) = x_2^3 + x_0\big(x_3^2+x_4^2+x_5^2+x_6^2\big) + x_1\big(x_3^2+\tau x_4^2-x_5^2+2x_6^2\big), \qquad \tau \in \mathbb{C}^\times\backslash\{1,-1,2\}.$$

(For the special values $\tau \in \{1, -1, 2\}$ the pencil acquires repeated eigenvalues and the stabilizer jumps; these form a proper closed subset of the same one-dimensional family.)

As $\dim W^H = 21$ and $\dim C_G(H) = 20$, and since $H \simeq \mathbb{G}_m$ acts trivially on W^H, the effective action has dimension 19; after projectivizing we obtain $21 - 19 - 1 = 1$. *For a general parameter τ the residual stabilizer is finite; hence, the corresponding component Φ_{14} of the moduli is one-dimensional.*

4.15 Case $k = 15$

1-PS limit. Set

$$\lambda_{15}(t) = \operatorname{diag}\bigl(t^2,\, t,\, t,\, 1,\, 1,\, t^{-2},\, t^{-2}\bigr), \qquad t \in \mathbb{G}_m.$$

For a generic f_{15} as in Section 3, the 1-PS limit is

$$\begin{aligned}\phi_{15} := \lim_{t\to 0} \lambda_{15}(t)\cdot f_{15} = {} & a_1x_3^3 + a_2x_3^2x_4 + a_3x_3x_4^2 + a_4x_4^3 + a_5x_1^2x_5 + a_6x_1x_2x_5 + a_7x_2^2x_5 \\ & + a_8x_0x_3x_5 + a_9x_0x_4x_5 + a_{10}x_1^2x_6 + a_{11}x_1x_2x_6 + a_{12}x_2^2x_6 \\ & + a_{13}x_0x_3x_6 + a_{14}x_0x_4x_6.\end{aligned}$$

All monomials have H-weight 0, so $\phi_{15} \in W^H$ for $H = \lambda_{15}(\mathbb{G}_m)$.

H and $C_G(H)$. The H-weights on $\langle x_0, \dots, x_6\rangle$ are $(2, 1, 1, 0, 0, -2, -2)$ with block decomposition $\langle x_0\rangle \oplus \langle x_1, x_2\rangle \oplus \langle x_3, x_4\rangle \oplus \langle x_5, x_6\rangle$. Hence,

$$\begin{aligned}C_G(H) &= \Bigl\{\, \operatorname{diag}(\alpha)\oplus A\oplus B\oplus C :\ \alpha\in\mathbb{G}_m,\ A,B,C\in \mathrm{GL}(2),\ \alpha\det(A)\det(B)\det(C) = 1 \,\Bigr\} \\ &\cong \bigl(\mathbb{G}_m \times \mathrm{GL}(2)\times\mathrm{GL}(2)\times\mathrm{GL}(2)\bigr)\cap \mathrm{SL}(7), \qquad \dim C_G(H) = 12.\end{aligned}$$

Polystability (Luna + Casimiro–Florentino). By Luna's reduction, the closedness of the $\mathrm{SL}(7)$-orbit of ϕ_{15} is equivalent to polystability for the $C_G(H)$-action on W^H. After conjugating inside the three $\mathrm{GL}(2)$-blocks, any $\lambda \in Y\bigl(C_G(H)\bigr)$ may be taken with

$$\mathrm{wt}(x_0, \dots, x_6) = (\alpha,\ s{+}u,\ s{-}u,\ t{+}v,\ t{-}v,\ \gamma{+}w,\ \gamma{-}w), \qquad S := \alpha{+}2s{+}2t{+}2\gamma = 0.$$

A direct computation yields the positive identity

$$\begin{aligned}2\bigl[w(x_3^3) + w(x_3^2x_4) + w(x_3x_4^2) + w(x_4^3)\bigr] + 3\Bigl[\bigl(w(x_1^2x_5) + 2w(x_1x_2x_5) + w(x_2^2x_5)\bigr) \\ + \bigl(w(x_1^2x_6) + 2w(x_1x_2x_6) + w(x_2^2x_6)\bigr)\Bigr] \\ + 6\bigl[w(x_0x_3x_5) + w(x_0x_4x_5) + w(x_0x_3x_6) + w(x_0x_4x_6)\bigr] \;=\; 24\,S.\end{aligned} \tag{19}$$

If $\lambda \in \Lambda_{\phi_{15}}$, then all 14 weights above are ≥ 0 and $S = 0$; hence, by (19) they all vanish. Solving gives

$$u = 0, \quad v = 0, \quad w = 0, \quad t = 0, \quad \alpha + \gamma = 0, \quad 2s + \gamma = 0.$$

Writing $s = k \in \mathbb{Z}$ we obtain

$$\mu_k(t) := \operatorname{diag}\bigl(t^{2k},\, t^k,\, t^k,\, 1,\, 1,\, t^{-2k},\, t^{-2k}\bigr), \qquad \Lambda_{\phi_{15}} = \{\mu_k \mid k \in \mathbb{Z}\}.$$

Thus $\Lambda_{\phi_{15}}$ is symmetric; by the Casimiro–Florentino criterion ϕ_{15} is polystable, so $\mathrm{SL}(7)\cdot\phi_{15}$ is closed.

Normal form and component dimension. We have

$$W^H = \underbrace{\mathrm{Sym}^3\langle x_3, x_4\rangle}_{\text{(I)}} \oplus \underbrace{\big(\mathrm{Sym}^2\langle x_1, x_2\rangle\big) \otimes \langle x_5, x_6\rangle}_{\text{(II)}} \oplus \underbrace{x_0 \otimes \langle x_3, x_4\rangle \otimes \langle x_5, x_6\rangle}_{\text{(III)}},$$

of respective dimensions 4, 6, and 4 (total $\dim W^H = 14$).

Reduction to normal form. We now normalize φ_{15} under the action of $C_G(H)$ on W^H, using only: (i) the left GL(2) on $\langle x_3, x_4\rangle$, (ii) the left $\mathrm{Sym}^2\mathrm{GL}(2)$ on $\mathrm{Sym}^2\langle x_1, x_2\rangle$, (iii) the right GL(2) on $\langle x_5, x_6\rangle$, (iv) diagonal tori (subject to $\det = 1$) and projective rescaling. We proceed block by block.

(I) The binary cubic block $\mathrm{Sym}^3\langle x_3, x_4\rangle$. A general binary cubic is GL(2)-equivalent (after one overall scalar) to

$$x_3^2 x_4 + \tau\, x_3 x_4^2, \qquad \tau \in \mathbb{C}^\times,$$

which fixes the Sym^3-part up to the single modulus τ.

(III) The bilinear 2×2 block $x_0 \otimes \langle x_3, x_4\rangle \otimes \langle x_5, x_6\rangle$. Write this part as $x_0\,(x_3, x_4)\,M\,(x_5, x_6)^\top$ with $M \in M_{2\times 2}$. Using the left GL(2) action on $\langle x_3, x_4\rangle$ and the right GL(2) action on $\langle x_5, x_6\rangle$ (an SVD-type reduction), we bring M to the identity; a diagonal torus and projective rescaling normalize the two coefficients to 1:

$$x_0 x_3 x_5 + x_0 x_4 x_6.$$

(II) The 3×2 block $\big(\mathrm{Sym}^2\langle x_1, x_2\rangle\big) \otimes \langle x_5, x_6\rangle$. Choose bases $\{x_1^2,\ x_1x_2,\ x_2^2\}$ and $\{x_5, x_6\}$. The left action of $\mathrm{Sym}^2\mathrm{GL}(2)$ on $\mathrm{Sym}^2\langle x_1, x_2\rangle$ together with the right action of GL(2) on $\langle x_5, x_6\rangle$ allows a simultaneous reduction that eliminates the mixed x_1x_2 row and diagonalizes the remaining two rows. After using diagonal tori and an overall scale, we obtain

$$x_1^2 x_5 + \rho\, x_2^2 x_6, \qquad \rho \in \mathbb{C}^\times.$$

Normal form. Hence, a closed-orbit representative is

$$\phi_{15}^{\mathrm{nf}}(\tau, \rho) \;=\; x_3^2 x_4 + \tau\, x_3 x_4^2 + x_0 x_3 x_5 + x_0 x_4 x_6 + x_1^2 x_5 + \rho\, x_2^2 x_6, \qquad (\tau, \rho) \in (\mathbb{C}^\times)^2. \tag{20}$$

For general (τ, ρ) the residual stabilizer in $C_G(H)$ is finite, so the parameters (τ, ρ) record genuine moduli on the closed stratum.

As $\dim W^H = 14$ and $\dim C_G(H) = 12$, and since $H \simeq G_m$ acts trivially on W^H, the effective action has dimension 11; after projectivizing we obtain

$$\dim \Phi_{15} = 14 - 11 - 1 = 2.$$

Hence, the corresponding component Φ_{15} of the moduli is two-dimensional.

4.16 Case $k = 16$

1-PS limit. Set

$$\lambda_{16}(t) = \operatorname{diag}\big(t^2,\, t,\, 1,\, 1,\, 1,\, t^{-1},\, t^{-2}\big), \qquad t \in \mathbb{G}_m.$$

For a generic f_{16} as in Section 3, the 1-PS limit is

$$\begin{aligned}\phi_{16} := \lim_{t\to 0} \lambda_{16}(t)\cdot f_{16} = {} & a_1x_2^3 + a_2x_2^2x_3 + a_3x_2x_3^2 + a_4x_3^3 + a_5x_2^2x_4 + a_6x_2x_3x_4 + a_7x_3^2x_4 \\ & + a_8x_2x_4^2 + a_9x_3x_4^2 + a_{10}x_4^3 + a_{11}x_1x_2x_5 + a_{12}x_1x_3x_5 + a_{13}x_1x_4x_5 \\ & + a_{14}x_0x_5^2 + a_{15}x_1^2x_6 + a_{16}x_0x_2x_6 + a_{17}x_0x_3x_6 + a_{18}x_0x_4x_6.\end{aligned}$$

(All monomials have H-weight 0, so $\phi_{16} \in W^H$ for $H = \lambda_{16}(\mathbb{G}_m)$.)

H and $C_G(H)$. The H-weights on $\langle x_0, \dots, x_6\rangle$ are $(2, 1, 0, 0, 0, -1, -2)$ with block decomposition $\langle x_0\rangle \oplus \langle x_1\rangle \oplus \langle x_2, x_3, x_4\rangle \oplus \langle x_5\rangle \oplus \langle x_6\rangle$. Hence,

$$\begin{aligned}C_G(H) = \Big\{ & \operatorname{diag}(\alpha) \oplus \operatorname{diag}(\beta) \oplus B \oplus \operatorname{diag}(\delta) \oplus \operatorname{diag}(\varepsilon) \;:\; \alpha, \beta, \delta, \varepsilon \in \mathbb{G}_m,\ B \in \mathrm{GL}(3), \\ & \alpha\,\beta\,\det(B)\,\delta\,\varepsilon = 1 \Big\} \\ \cong\ & \big(\mathbb{G}_m \times \mathbb{G}_m \times \mathrm{GL}(3) \times \mathbb{G}_m \times \mathbb{G}_m\big) \cap \mathrm{SL}(7), \qquad \dim C_G(H) = 12.\end{aligned}$$

Polystability (Luna + Casimiro–Florentino). By Luna's reduction, the closedness of the SL(7)-orbit of ϕ_{16} is equivalent to polystability for the $C_G(H)$-action on W^H. After conjugating within the GL(3)-block on $\langle x_2, x_3, x_4\rangle$, any $\lambda \in Y\big(C_G(H)\big)$ may be taken with

$$\operatorname{wt}(x_0, \dots, x_6) = (\alpha,\, \beta,\, c + v_1,\, c + v_2,\, c + v_3,\, \delta,\, \varepsilon), \qquad v_1 + v_2 + v_3 = 0,$$

and, by Convention 4.6, with the SL-constraint

$$S := \alpha + \beta + 3c + \delta + \varepsilon = 0.$$

Writing $w(\cdot)$ for the λ-weight of a monomial, a direct calculation yields the positive identity

$$\sum_{\substack{i+j+k=3\\ i,j,k\geq 0}} w(x_2^i x_3^j x_4^k) \;+\; 6\,w(x_0x_5^2) \;+\; 6\,w(x_1^2x_6) \;+\; 2\sum_{i=2}^{4} w(x_0x_ix_6) \;=\; 12S. \tag{21}$$

If $\lambda \in \Lambda_{\phi_{16}}$, then all 18 weights on the left are ≥ 0 and $S = 0$; hence, by (21) they all vanish. From the ten cubics, we obtain

$$c = 0, \qquad v_1 = v_2 = v_3 = 0,$$

and from the remaining terms

$$\alpha + 2\delta = 0, \qquad 2\beta + \varepsilon = 0, \qquad \alpha + \varepsilon = 0.$$

Thus $\alpha = 2\beta,\ \delta = -\beta,\ \varepsilon = -2\beta$. Writing $\beta = k \in \mathbb{Z}$ gives

$$\mu_k(t) = \operatorname{diag}\big(t^{2k},\, t^k,\, 1,\, 1,\, 1,\, t^{-k},\, t^{-2k}\big), \qquad \Lambda_{\phi_{16}} = \{\mu_k \mid k \in \mathbb{Z}\}.$$

As $\Lambda_{\phi_{16}}$ is symmetric, ϕ_{16} is polystable by the Casimiro–Florentino criterion; in particular, $\mathrm{SL}(7)\cdot\phi_{16}$ is closed.

Normal form and component dimension. On the H–fixed slice we have the block decomposition

$$W^H = \underbrace{\mathrm{Sym}^3\langle x_2, x_3, x_4\rangle}_{\text{(I)}} \oplus \underbrace{x_1 \otimes \langle x_2, x_3, x_4\rangle \otimes x_5}_{\text{(II)}} \oplus \underbrace{\langle x_0 x_5^2\rangle}_{\text{(III)}} \oplus \underbrace{\langle x_1^2 x_6\rangle}_{\text{(IV)}} \oplus \underbrace{x_0 \otimes \langle x_2, x_3, x_4\rangle \otimes x_6}_{\text{(V)}}.$$

The centralizer is

$$CG(H) \simeq (\mathbb{G}_m)^4 \times \mathrm{GL}(\langle x_2, x_3, x_4\rangle),$$

acting blockwise (subject to the determinant-one condition). We normalize the blocks one by one.

Step 1: Normalization of the rank-one blocks (II) **and** (V)**.**
The blocks

$$\text{(II)} = \{x_1x_2x_5,\ x_1x_3x_5,\ x_1x_4x_5\}, \qquad \text{(V)} = \{x_0x_2x_6,\ x_0x_3x_6,\ x_0x_4x_6\}$$

correspond to two linear functionals on

$$U^\vee = \langle x_2, x_3, x_4\rangle^\vee.$$

For a general member, these two 3-vectors are linearly independent; hence there exists a single $g \in \mathrm{GL}(U)$ sending them to the coordinate covectors:

$$\text{(II)} \ \sim\ x_1x_2x_5, \qquad \text{(V)} \ \sim\ x_0x_3x_6.$$

Using the diagonal tori on (x_1, x_5) and (x_0, x_6) (and projective scaling), we normalize the coefficients to 1.

Step 2: Normalization of the one-dimensional blocks (III) **and** (IV)**.**
The $\mathbb{G}_m$-factors on x_0 and x_1 allow us to set

$$\text{(III)} \sim x_0x_5^2, \qquad \text{(IV)} \sim x_1^2x_6.$$

Step 3: Normalization of the ternary cubic in (I)**.**
Let

$$F(x_2, x_3, x_4) \in \mathrm{Sym}^3\langle x_2, x_3, x_4\rangle.$$

After Step 1, the residual subgroup of $\mathrm{GL}(\langle x_2, x_3, x_4\rangle)$ preserving (II) and (V) consists of

$$x_2 \mapsto s\,x_2, \qquad x_3 \mapsto t\,x_3, \qquad x_4 \mapsto u\,x_4 + \alpha x_2 + \beta x_3, \qquad (s,t,u) \in (\mathbb{G}_m)^3,\ (\alpha,\beta) \in \mathbb{A}^2.$$

(a) Elimination of $x_2x_4^2$ and $x_3x_4^2$. The substitution $x_4 \mapsto x_4 + \alpha x_2 + \beta x_3$ yields

$$(x_4 + \alpha x_2 + \beta x_3)^3 = x_4^3 + 3\alpha\, x_2x_4^2 + 3\beta\, x_3x_4^2 + \cdots,$$

so choosing (α, β) suitably (with the x_4^3 coefficient nonzero), we kill both coefficients.

(b) Normalization of pure cubes. Scaling by (s, t, u) (and compensating using the diagonal tori on (x_1, x_5) and (x_0, x_6) so that the coefficients fixed in Steps 1–2 remain equal to 1), we may assume

$$F = x_2^3 + x_3^3 + \sigma x_4^3 + \cdots, \qquad \sigma \in \mathbb{C}^\times.$$

(c) Remaining mixed terms. After (a) and (b), the possible mixed terms are

$$x_2^2x_3,\ x_2x_3^2,\ x_2^2x_4,\ x_3^2x_4,\ x_2x_3x_4.$$

Note that the residual subgroup preserving the normalizations in Step 1 does *not* contain the shear $x_3 \mapsto x_3 + \gamma x_2$ (it would reintroduce an $x_0x_2x_6$ term in the (V)-block). Hence the coefficient of $x_2x_3^2$ cannot be eliminated in general. Therefore we write

$$F(x_2, x_3, x_4) = x_2^3 + x_3^3 + \sigma x_4^3 + \rho\, x_2x_3x_4 + \kappa\, x_2^2x_3 + \nu\, x_2x_3^2 + \mu\, x_3^2x_4 + \lambda\, x_2^2x_4,$$

with $\rho, \kappa, \nu, \mu, \lambda \in \mathbb{C}^\times$.

Collecting all blocks.
Hence a closed-orbit representative is

$$\begin{aligned}\phi_{16}^{\mathrm{nf}} = {} & x_2^3 + x_3^3 + \sigma x_4^3 + \rho\, x_2x_3x_4 + \kappa\, x_2^2x_3 + \nu\, x_2x_3^2 + \mu\, x_3^2x_4 + \lambda\, x_2^2x_4 \\ & + x_1x_2x_5 + x_0x_3x_6 + x_0x_5^2 + x_1^2x_6,\end{aligned}$$

with six free parameters

$$(\sigma, \rho, \kappa, \nu, \mu, \lambda) \in (\mathbb{C}^\times)^6.$$

Component dimension. We compute

$$\dim W^H = 10 + 3 + 1 + 1 + 3 = 18.$$

Since $C_G(H)$ has dimension 12 and the $H \simeq \mathbb{G}_m$ factor acts trivially on W^H, the effective dimension is

$$\dim_{\mathrm{eff}} C_G(H) = 11.$$

Projectivization subtracts one more dimension, giving

$$\dim \Phi_{16} = 18 - 11 - 1 = 6,$$

agreeing with the six independent parameters of ϕ_{16}^{nf}.

4.17 Case $k = 17$

1-PS limit. Set

$$\lambda_{17}(t) = \mathrm{diag}\big(t,\, t,\, t,\, 1,\, 1,\, t^{-1},\, t^{-2}\big), \qquad t \in \mathbb{G}_m.$$

For a generic f_{17} as in Section 3, the 1-PS limit is

$$\begin{aligned}\phi_{17} := \lim_{t\to 0} \lambda_{17}(t)\cdot f_{17} = {} & a_1\, x_3^3 + a_2\, x_3^2x_4 + a_3\, x_3x_4^2 + a_4\, x_4^3 \\ & + a_5\, x_0x_3x_5 + a_6\, x_1x_3x_5 + a_7\, x_2x_3x_5 \\ & + a_8\, x_0x_4x_5 + a_9\, x_1x_4x_5 + a_{10}\, x_2x_4x_5 \\ & + a_{11}\, x_0^2x_6 + a_{12}\, x_0x_1x_6 + a_{13}\, x_1^2x_6 \\ & + a_{14}\, x_0x_2x_6 + a_{15}\, x_1x_2x_6 + a_{16}\, x_2^2x_6.\end{aligned}$$

H **and** $C_G(H)$**.** Let $H = \lambda_{17}(\mathbb{G}_m)$. The H-weights on $\langle x_0, \dots, x_6\rangle$ are $(1, 1, 1, 0, 0, -1, -2)$ with block multiplicities $\langle x_0, x_1, x_2\rangle$, $\langle x_3, x_4\rangle$, $\langle x_5\rangle$, $\langle x_6\rangle$. Thus

$$\begin{aligned} C_G(H) &= \Big\{ A \oplus B \oplus \operatorname{diag}(\gamma) \oplus \operatorname{diag}(\delta) :\ A \in \mathrm{GL}(3),\ B \in \mathrm{GL}(2),\ \gamma, \delta \in \mathbb{G}_m, \\ &\qquad\qquad \det(A)\det(B)\gamma\delta = 1 \Big\} \\ &\cong \big(\mathrm{GL}(3) \times \mathrm{GL}(2) \times \mathbb{G}_m \times \mathbb{G}_m\big) \cap \mathrm{SL}(7). \end{aligned}$$

Each monomial of ϕ_{17} has H-weight 0. Moreover, $\dim C_G(H) = 14$.

Polystability (Luna + Casimiro–Florentino). By Luna's reduction, the closedness of the $\mathrm{SL}(7)$-orbit of ϕ_{17} is equivalent to polystability for the $C_G(H)$-action on the H-fixed subspace. After conjugating inside the GL(3)- and GL(2)-blocks, any $\lambda \in Y\big(C_G(H)\big)$ may be taken with

$$\mathrm{wt}(x_0, \dots, x_6) = (a + u_1,\ a + u_2,\ a + u_3,\ b + v,\ b - v,\ c,\ d), \quad u_1 + u_2 + u_3 = 0,$$

and the SL(7)-condition (Convention 4.6)

$$S := 3a + 2b + c + d = 0.$$

A direct calculation yields the positive linear identity

$$\begin{aligned} &\big[w(x_3^3) + w(x_3^2x_4) + w(x_3x_4^2) + w(x_4^3)\big] \\ &\quad + 2\big[w(x_0x_3x_5) + w(x_1x_3x_5) + w(x_2x_3x_5) + w(x_0x_4x_5) + w(x_1x_4x_5) + w(x_2x_4x_5)\big] \\ &\quad + 2\big[w(x_0^2x_6) + w(x_0x_1x_6) + w(x_1^2x_6) + w(x_0x_2x_6) + w(x_1x_2x_6) + w(x_2^2x_6)\big] \\ &\quad = 12\,S. \end{aligned} \tag{22}$$

If $\lambda \in \Lambda_{\phi_{17}}$, then all 16 monomial weights are ≥ 0 and $S = 0$; by (22) they must all vanish. Solving gives

$$b = 0, \quad v = 0, \quad u_1 = u_2 = u_3 = 0, \quad c = -a, \quad d = -2a.$$

Writing $a = k$ ($k \in \mathbb{Z}$) yields

$$\Lambda_{\phi_{17}} = \{\mu_k \mid k \in \mathbb{Z}\}, \qquad \mu_k(t) = \operatorname{diag}\big(t^k,\ t^k,\ t^k,\ 1,\ 1,\ t^{-k},\ t^{-2k}\big).$$

Thus, $\Lambda_{\phi_{17}}$ is symmetric, and by the Casimiro–Florentino criterion, ϕ_{17} is polystable; in particular, $\mathrm{SL}(7) \cdot \phi_{17}$ is closed.

Normal form and component dimension. On the H-fixed slice

$$W^H = \underbrace{\mathrm{Sym}^3\langle x_3, x_4\rangle}_{\text{(I)}} \ \oplus\ \underbrace{\langle x_0, x_1, x_2\rangle \otimes \langle x_3, x_4\rangle \otimes x_5}_{\text{(II)}} \ \oplus\ \underbrace{\mathrm{Sym}^2\langle x_0, x_1, x_2\rangle \otimes x_6}_{\text{(III)}},$$

of total dimension $4 + 6 + 6 = 16$.

(III) **The quadratic block** $\mathrm{Sym}^2\langle x_0, x_1, x_2\rangle \otimes x_6$**.** By the GL_3-action, one diagonalizes the ternary quadratic, and using diagonal tori together with projective rescaling, the coefficients of $x_0^2x_6$ and $x_1^2x_6$ are normalized to 1, leaving

$$x_0^2x_6 \;+\; x_1^2x_6 \;+\; \rho\, x_2^2x_6, \qquad \rho \in \mathbb{C}^\times.$$

Importantly, after the reduction of (II) below, the subgroup of GL_3 that preserves (II) is block-diagonal on $\langle x_0, x_1\rangle \oplus \langle x_2\rangle$ (up to an $\mathrm{O}(2)$ on $\langle x_0, x_1\rangle$ and an independent scaling on x_2). Hence, the relative scale of x_2 cannot be absorbed, and the parameter ρ cannot be removed.

(II) **The** 3×2 **block** $\langle x_0, x_1, x_2\rangle \otimes \langle x_3, x_4\rangle \otimes x_5$**.** Using the left GL_3 and right GL_2 actions (an SVD-type reduction), a generic rank-2 element is brought to diagonal form; scaling x_5 then fixes the two nonzero entries to 1:

$$x_0x_3x_5 \;+\; x_1x_4x_5.$$

(I) **The binary cubic** $\mathrm{Sym}^3\langle x_3, x_4\rangle$**.** Acting by GL_2 on $\langle x_3, x_4\rangle$ puts a general binary cubic into a two-term form (sending three roots to $\{0, \infty, -\tau\}$), and a residual diagonal scaling fixes the first coefficient to 1:

$$x_3^2x_4 \;+\; \tau\, x_3x_4^2, \qquad \tau \in \mathbb{C}^\times.$$

Collecting (I)-(III), we obtain the closed-orbit representative

$$\phi_{17}^{\mathrm{nf}}(\tau,\rho) = x_3^2x_4 + \tau\, x_3x_4^2 + x_0x_3x_5 + x_1x_4x_5 + x_0^2x_6 + x_1^2x_6 + \rho\, x_2^2x_6, \qquad (\tau,\rho) \in (\mathbb{C}^\times)^2.$$

Lastly, $\dim W^H = 16$ and $\dim C_G(H) = 14$. As the $H \simeq \mathbb{G}_m$-factor acts trivially on W^H, the effective group dimension is $14 - 1 = 13$. After projectivizing, we get

$$\dim(\text{component}) = \dim W^H - 13 - 1 = 2,$$

so (τ, ρ) are the two genuine moduli of the closed stratum. Hence, the corresponding component Φ_{17} of the moduli is two-dimensional.

4.18 Case $k = 18$

1-PS limit. Set

$$\lambda_{18}(t) = \mathrm{diag}\big(t,\, t,\, 1,\, 1,\, 1,\, t^{-1},\, t^{-1}\big), \qquad t \in \mathbb{G}_m.$$

For a generic f_{18} as in Section 3, the 1-PS limit is

$$\begin{aligned}\phi_{18} := \lim_{t\to 0} \lambda_{18}(t)\cdot f_{18} = {} & a_1\, x_2^3 + a_2\, x_2^2x_3 + a_3\, x_2x_3^2 + a_4\, x_3^3 + a_5\, x_2^2x_4 + a_6\, x_2x_3x_4 + a_7\, x_3^2x_4 \\ & + a_8\, x_2x_4^2 + a_9\, x_3x_4^2 + a_{10}\, x_4^3 + a_{11}\, x_0x_2x_5 + a_{12}\, x_1x_2x_5 + a_{13}\, x_0x_3x_5 \\ & + a_{14}\, x_1x_3x_5 + a_{15}\, x_0x_4x_5 + a_{16}\, x_1x_4x_5 + a_{17}\, x_0x_2x_6 + a_{18}\, x_1x_2x_6 \\ & + a_{19}\, x_0x_3x_6 + a_{20}\, x_1x_3x_6 + a_{21}\, x_0x_4x_6 + a_{22}\, x_1x_4x_6.\end{aligned}$$

H **and** $C_G(H)$**.** Let $H = \lambda_{18}(\mathbb{G}_m)$. The H-weights on $\langle x_0, \dots, x_6\rangle$ are $(1, 1, 0, 0, 0, -1, -1)$ with block decomposition

$$\langle x_0, x_1\rangle \oplus \langle x_2, x_3, x_4\rangle \oplus \langle x_5, x_6\rangle.$$

Hence

$$\begin{aligned} C_G(H) &= \Big\{ A \oplus B \oplus C :\ A \in \mathrm{GL}(2),\ B \in \mathrm{GL}(3),\ C \in \mathrm{GL}(2), \\ &\qquad\qquad \det(A)\det(B)\det(C) = 1 \Big\} \\ &\cong \big(\mathrm{GL}(2) \times \mathrm{GL}(3) \times \mathrm{GL}(2)\big) \cap \mathrm{SL}(7), \end{aligned}$$

so $\dim C_G(H) = 16$. Each monomial of ϕ_{18} has H-weight 0; hence $\phi_{18} \in W^H$.

Polystability (Luna + Casimiro–Florentino). By Luna's reduction, the closedness of the SL(7)-orbit of ϕ_{18} is equivalent to polystability for the $C_G(H)$-action on the H-fixed subspace. After conjugating within the three blocks, any $\lambda \in Y\big(C_G(H)\big)$ may be taken with

$$\mathrm{wt}(x_0, \dots, x_6) = (a+u,\ a-u,\ b+v_1,\ b+v_2,\ b+v_3,\ c+w,\ c-w),$$

where $a, b, c, u, w, v_i \in \mathbb{Z}$ and $v_1 + v_2 + v_3 = 0$, and with the SL-constraint

$$S := 2a + 3b + 2c = 0.$$

Let $w(\cdot)$ denote the λ-weight of a monomial. Then

$$\sum_{\text{all 10 cubics in } x_2,x_3,x_4} w \ = \ 30\, b,$$

and

$$\sum_{i=2}^{4}[w(x_0x_ix_5)+w(x_1x_ix_5)] = 6(a+b+c+w), \qquad \sum_{i=2}^{4}[w(x_0x_ix_6)+w(x_1x_ix_6)] = 6(a+b+c-w).$$

Hence the positive identity

$$\begin{aligned} &\Big[\sum_{\text{all 10 cubics in } x_2,x_3,x_4} w\Big] + 5\sum_{i=2}^{4}\big[w(x_0x_ix_5) + w(x_1x_ix_5)\big] \\ &\quad + 5\sum_{i=2}^{4}\big[w(x_0x_ix_6) + w(x_1x_ix_6)\big] \ = \ 30\, S. \end{aligned} \tag{23}$$

If $\lambda \in \Lambda_{\phi_{18}}$, then all 22 monomial weights are ≥ 0 and $S = 0$; hence, by (23) they must all vanish. Solving yields

$$b = 0, \quad v_1 = v_2 = v_3 = 0, \quad u = 0, \quad w = 0, \quad a + c = 0.$$

Writing $a = k$ ($k \in \mathbb{Z}$) gives

$$\Lambda_{\phi_{18}} = \{\mu_k \mid k \in \mathbb{Z}\}, \qquad \mu_k(t) = \mathrm{diag}\big(t^k,\, t^k,\, 1,\, 1,\, 1,\, t^{-k},\, t^{-k}\big).$$

Thus $\Lambda_{\phi_{18}}$ is symmetric, and by the Casimiro–Florentino criterion, ϕ_{18} is polystable.

Normal form and component dimension. Put

$$V := \langle x_0, x_1 \rangle, \qquad U := \langle x_2, x_3, x_4 \rangle, \qquad W := \langle x_5, x_6 \rangle.$$

Then the H–fixed slice decomposes as

$$W^H = \mathrm{Sym}^3 U \ \oplus \ (V \otimes U \otimes x_5) \ \oplus \ (V \otimes U \otimes x_6) \ \cong \ \mathrm{Sym}^3 U \ \oplus \ (V \otimes U \otimes W),$$

and $C_G(H) = \{A \oplus B \oplus C : \ A \in GL(V), \ B \in GL(U), \ C \in GL(W), \ \det(A)\det(B)\det(C) = 1\}$ acts blockwise (with the *same* $B \in GL(U)$ acting simultaneously on the cubic block and on the mixed block).

To avoid any ambiguity about this shared $GL(U)$–action, we normalize in two stages: first we fix a convenient basis of U using the cubic block, and *then* we reduce the mixed block using only those U–changes that preserve the chosen cubic shape (namely permutations and diagonal scalings of x_2, x_3, x_4).

(I) The ternary cubic block (fixing the U–basis). Write the cubic part as $F \in \mathrm{Sym}^3 U$. On the dense open subset where F is a smooth plane cubic with nonvanishing Hessian, the classical Hesse normal form gives a change of basis $B \in GL(U)$ and a scalar such that (after rescaling and renaming the coordinates on U) we may assume

$$F \ \sim \ x_2^3 + \sigma_1 x_3^3 + \sigma_2 x_4^3 + \rho\, x_2 x_3 x_4, \qquad (\sigma_1, \sigma_2, \rho) \in (\mathbb{C}^\times)^3. \tag{24}$$

From this point on, the basis (x_2, x_3, x_4) is *fixed*. The remaining $GL(U)$–freedom that preserves the 4–term shape in (24) is the monomial subgroup: permutations of (x_2, x_3, x_4) and diagonal scalings $x_i \mapsto t_i x_i$ (which only rescale $(\sigma_1, \sigma_2, \rho)$).

(II)+(III) The mixed block as a 2×3 matrix pencil (reduced relative to the fixed U–basis). Write the mixed part as

$$x_5\, L_5 \ + \ x_6\, L_6, \qquad L_5, L_6 \in V \otimes U,$$

and represent L_5, L_6 by 2×3 matrices M_5, M_6 with respect to the fixed bases (x_0, x_1) of V and (x_2, x_3, x_4) of U:

$$L_j = (x_0, x_1)\, M_j\, (x_2, x_3, x_4)^{\mathsf{T}} \qquad (j = 5, 6).$$

The $GL(V)$–factor acts by left multiplication on both M_5, M_6, the $GL(W)$–factor changes the basis of the pencil (so it replaces (M_5, M_6) by another spanning pair), and the residual monomial subgroup in $GL(U)$ permutes/rescales the three columns.

We work on the dense open set where the pencil $\langle M_5, M_6 \rangle$ contains a rank-2 matrix and where suitable 2×2 minors in the fixed column basis are nonzero (an open condition). Using $GL(W)$ we choose the basis (x_5, x_6) so that M_5 has rank 2. After possibly permuting (x_2, x_3, x_4) (allowed by the monomial subgroup), we may assume the 2×2 minor of M_5 in the (x_2, x_3)–columns is invertible. Then, multiplying on the left by its inverse (an element of $GL(V)$), we achieve

$$M_5 \ \sim \ \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \end{pmatrix}, \qquad \alpha \in \mathbb{C}^\times,$$

where the remaining entry in the (x_4)–column is recorded as the modulus α (we do *not* use column additions, only the allowed row operations, column permutation/diagonal scaling, and basis change in W).

Next, keeping M_5 fixed, we use the subgroup of $GL(W)$ stabilizing the first basis vector x_5 (i.e. upper–triangular matrices) to replace M_6 by $M_6 + \lambda M_5$, and use $GL(V)$–row operations that preserve the pivot normalization of M_5. On the same open set one can arrange that M_6 takes the form

$$M_6 \;\sim\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & \gamma \end{pmatrix}, \qquad (\beta,\gamma) \in (\mathbb{C}^\times)^2,$$

after additionally allowing diagonal rescalings of (x_2, x_3, x_4) (which remain inside the 4–term family (24) and merely rescale $(\sigma_1, \sigma_2, \rho)$).

Translating back to monomials, the mixed block becomes

$$x_0x_2x_5{+}x_1x_3x_5{+}\alpha\, x_0x_4x_5 \,+\, x_0x_2x_6{+}\beta\, x_1x_3x_6{+}\gamma\, x_1x_4x_6, \qquad (\alpha,\beta,\gamma) \in (\mathbb{C}^\times)^3.$$

Collecting (I)–(III). Combining (24) with the above reduction of the mixed block, we obtain the convenient normal form

$$\phi_{18}^{\mathrm{nf}} = x_2^3{+}\sigma_1 x_3^3{+}\sigma_2 x_4^3{+}\rho\, x_2x_3x_4{+}x_0x_2x_5{+}x_1x_3x_5{+}\alpha\, x_0x_4x_5{+}x_0x_2x_6{+}\beta\, x_1x_3x_6{+}\gamma\, x_1x_4x_6,$$

with $(\sigma_1, \sigma_2, \rho, \alpha, \beta, \gamma) \in (\mathbb{C}^\times)^6$. The corresponding component Φ_{18} of the moduli is six-dimensional.

4.19 Case $k = 19$

1-PS limit. Set

$$\lambda_{19}(t) = \mathrm{diag}\big(t^2,\, t^2,\, t^2,\, 1,\, t^{-1},\, t^{-1},\, t^{-4}\big), \qquad t \in \mathbb{G}_m.$$

For a generic f_{19} as in Section 3, the 1-PS limit is

$$\begin{aligned} \phi_{19} := \lim_{t\to 0} \lambda_{19}(t)\cdot f_{19} = {} & a_1\, x_3^3 + a_2\, x_0x_4^2 + a_3\, x_1x_4^2 + a_4\, x_2x_4^2 \\ & + a_5\, x_0x_4x_5 + a_6\, x_1x_4x_5 + a_7\, x_2x_4x_5 \\ & + a_8\, x_0x_5^2 + a_9\, x_1x_5^2 + a_{10}\, x_2x_5^2 \\ & + a_{11}\, x_0^2x_6 + a_{12}\, x_0x_1x_6 + a_{13}\, x_1^2x_6 \\ & + a_{14}\, x_0x_2x_6 + a_{15}\, x_1x_2x_6 + a_{16}\, x_2^2x_6. \end{aligned}$$

H and $C_G(H)$. Let $H = \lambda_{19}(\mathbb{G}_m)$. The H-weights on $\langle x_0, \ldots, x_6\rangle$ are $(2, 2, 2, 0, -1, -1, -4)$ with block decomposition

$$\langle x_0, x_1, x_2\rangle \;\oplus\; \langle x_3\rangle \;\oplus\; \langle x_4, x_5\rangle \;\oplus\; \langle x_6\rangle.$$

Hence,

$$\begin{aligned} C_G(H) = \Big\{ & A \oplus \mathrm{diag}(\beta) \oplus B \oplus \mathrm{diag}(\delta) :\; A \in \mathrm{GL}(3),\; B \in \mathrm{GL}(2), \\ & \qquad \beta, \delta \in \mathbb{G}_m,\; \det(A)\,\beta\,\det(B)\,\delta = 1 \Big\} \\ \cong {} & \big(\mathrm{GL}(3) \times \mathrm{GL}(2) \times \mathbb{G}_m \times \mathbb{G}_m\big) \cap \mathrm{SL}(7), \end{aligned}$$

so $\dim C_G(H) = 14$. Each monomial of ϕ_{19} has H-weight 0.

Polystability (Luna + Casimiro–Florentino). By Luna's reduction, the closedness of the SL(7)-orbit of ϕ_{19} is equivalent to polystability for the $C_G(H)$-action on the H-fixed subspace. After conjugating inside the blocks, any $\lambda \in Y\big(C_G(H)\big)$ may be taken with

$$\mathrm{wt}(x_0,\dots,x_6) = (s+u_1,\, s+u_2,\, s+u_3,\, b,\, t+v,\, t-v,\, c),$$

where $s,t,b,c,u_i,v \in \mathbb{Z}$, $u_1+u_2+u_3=0$, and the SL(7)-condition (Convention 4.6) is

$$S := 3s+b+2t+c = 0.$$

A direct computation yields the positive identity

$$\begin{aligned}\sum_{i=0}^{2}\Big[w\big(x_i x_4^2\big) + 2\,w\big(x_i x_4 x_5\big) + w\big(x_i x_5^2\big)\Big]\\ + 2\Big[w(x_0^2x_6) + w(x_0x_1x_6) + w(x_1^2x_6) + w(x_0x_2x_6) + w(x_1x_2x_6) + w(x_2^2x_6)\Big]\\ + 4\,w(x_3^3) \;=\; 12\,S. \end{aligned} \tag{25}$$

If $\lambda \in \Lambda_{\phi_{19}}$, then all 16 monomial weights are ≥ 0 and $S=0$; by (25) they must all vanish. Solving gives

$$b=0, \quad v=0, \quad s+u_r+2t=0 \ (r=1,2,3), \quad 2s+c=0.$$

Using $u_1+u_2+u_3=0$ we obtain $s+2t=0$ and $u_1=u_2=u_3=0$. Writing $s=2k$ with $k\in\mathbb{Z}$ yields

$$\Lambda_{\phi_{19}} = \{\mu_k \mid k\in\mathbb{Z}\}, \qquad \mu_k(t) = \mathrm{diag}\big(t^{2k},\, t^{2k},\, t^{2k},\, 1,\, t^{-k},\, t^{-k},\, t^{-4k}\big).$$

Thus $\Lambda_{\phi_{19}}$ is symmetric, and by the Casimiro–Florentino criterion ϕ_{19} is polystable; in particular $\mathrm{SL}(7)\cdot\phi_{19}$ is closed.

Normal form and component dimension. In the H-fixed subspace

$$W^H = \ \langle x_3^3\rangle \ \oplus \ \langle x_0,x_1,x_2\rangle \otimes \mathrm{Sym}^2\langle x_4,x_5\rangle \ \oplus \ \mathrm{Sym}^2\langle x_0,x_1,x_2\rangle \otimes x_6,$$

using the GL(3)-action on $\langle x_0,x_1,x_2\rangle$ to identify the basis of $\langle x_0,x_1,x_2\rangle$ with the standard basis $\{x_4^2, x_4x_5, x_5^2\}$ of $\mathrm{Sym}^2\langle x_4,x_5\rangle$ (ensuring full rank), and using the residual stabilizer (isomorphic to GL(2)) to normalize the third block, a generic element is brought to

$$\phi_{19}^{\mathrm{nf}}(\rho,\sigma) = x_3^3 + (x_0x_4^2 + x_1x_4x_5 + x_2x_5^2) + x_6(x_0^2 + x_2^2 + \rho\,x_1^2 + \sigma\,x_0x_2),$$

with $(\rho,\sigma)\in\mathbb{C}^{\times 2}$ general. As $\dim W^H = 16$ and $\dim C_G(H) = 14$ (with H acting trivially), the effective action has dimension 13; after projectivizing we obtain

$$\dim \Phi_{19} = 16 - 13 - 1 = 2.$$

The residual T-stabilizer is finite; hence, the corresponding component Φ_{19} of the moduli is two-dimensional.

4.20 Case $k = 20$

1-PS limit. Set

$$\lambda_{20}(t) = \operatorname{diag}\big(t,\, t,\, t,\, t,\, 1,\, t^{-2},\, t^{-2}\big), \quad t \in \mathbb{G}_m.$$

For a generic f_{20} as in Section 3, the 1-PS limit is

$$\begin{aligned}\phi_{20} := \lim_{t\to 0} \lambda_{20}(t)\cdot f_{20} = {} & a_1\, x_4^3 + a_2\, x_0^2x_5 + a_3\, x_0x_1x_5 + a_4\, x_1^2x_5 + a_5\, x_0x_2x_5 + a_6\, x_1x_2x_5 \\ & + a_7\, x_2^2x_5 + a_8\, x_0x_3x_5 + a_9\, x_1x_3x_5 + a_{10}\, x_2x_3x_5 + a_{11}\, x_3^2x_5 \\ & + a_{12}\, x_0^2x_6 + a_{13}\, x_0x_1x_6 + a_{14}\, x_1^2x_6 + a_{15}\, x_0x_2x_6 + a_{16}\, x_1x_2x_6 \\ & + a_{17}\, x_2^2x_6 + a_{18}\, x_0x_3x_6 + a_{19}\, x_1x_3x_6 + a_{20}\, x_2x_3x_6 + a_{21}\, x_3^2x_6.\end{aligned}$$

H and $C_G(H)$. Let $H = \lambda_{20}(\mathbb{G}_m)$. The H-weights on $\langle x_0, \dots, x_6\rangle$ are $(1, 1, 1, 1, 0, -2, -2)$ with block multiplicities $\langle x_0, x_1, x_2, x_3\rangle$, $\langle x_4\rangle$, $\langle x_5, x_6\rangle$. Hence,

$$\begin{aligned}C_G(H) &= \Big\{ A \oplus \operatorname{diag}(\beta) \oplus C :\ A \in \mathrm{GL}(4),\ \beta \in \mathbb{G}_m,\ C \in \mathrm{GL}(2), \\ &\qquad\qquad \det(A)\,\beta\,\det(C) = 1 \Big\} \\ &\cong \big(\mathrm{GL}(4) \times \mathbb{G}_m \times \mathrm{GL}(2)\big) \cap \mathrm{SL}(7),\end{aligned}$$

so $\dim C_G(H) = 20$. Every monomial of ϕ_{20} has H-weight 0.

Polystability (Luna + Casimiro–Florentino). By Luna's reduction, the closedness of the SL(7)-orbit of ϕ_{20} is equivalent to polystability for the $C_G(H)$-action on the H-fixed subspace. After conjugating inside the GL(4)- and GL(2)-blocks, any $\lambda \in Y(C_G(H))$ may be taken with weights

$$\mathrm{wt}(x_0, \dots, x_6) = \big(a + u_0,\, a + u_1,\, a + u_2,\, a + u_3,\, b,\, c + w,\, c - w\big),$$

where $a, b, c, u_i, w \in \mathbb{Z}$, $u_0 + u_1 + u_2 + u_3 = 0$, and, by Convention 4.6,

$$S := 4a + b + 2c = 0.$$

Let $w(\cdot)$ denote the λ-weight of a monomial. Then

$$\begin{aligned}&w(x_4^3) = 3b, \\ &w(x_i^2x_5) = 2(a + u_i) + c + w, \qquad w(x_i^2x_6) = 2(a + u_i) + c - w \quad (i = 0, 1, 2, 3), \\ &w(x_ix_jx_5) = 2a + (u_i + u_j) + c + w, \quad w(x_ix_jx_6) = 2a + (u_i + u_j) + c - w \ (0 \le i < j \le 3).\end{aligned}$$

A direct computation yields the positive identity

$$\begin{aligned}3\sum_{i=0}^{3}\big[w(x_i^2x_5) + w(x_i^2x_6)\big] + 3 \sum_{0\le i<j\le 3}\big[w(x_ix_jx_5) + w(x_ix_jx_6)\big& \\ + 10\, w(x_4^3) \;=\; 30\, S.& \qquad (26)\end{aligned}$$

If $\lambda \in \Lambda_{\phi_{20}}$, then all 21 monomial weights are ≥ 0 and $S = 0$; by (26) they must all vanish. Solving gives

$$b = 0, \quad w = 0, \quad 2(a+u_i)+c = 0 \ (i = 0, 1, 2, 3), \quad 2a+(u_i+u_j)+c = 0 \ (i < j),$$

Hence, $u_0 = u_1 = u_2 = u_3 = 0$ and $2a + c = 0$. Writing $a = k \in \mathbb{Z}$ yields

$$\Lambda_{\phi_{20}} = \{\mu_k \mid k \in \mathbb{Z}\}, \qquad \mu_k(t) = \operatorname{diag}\bigl(t^k,\, t^k,\, t^k,\, t^k,\, 1,\, t^{-2k},\, t^{-2k}\bigr).$$

As $\Lambda_{\phi_{20}}$ is symmetric, ϕ_{20} is polystable by the Casimiro–Florentino criterion; in particular, $\mathrm{SL}(7) \cdot \phi_{20}$ is closed.

Normal form and component dimension. We have

$$W^H = \underbrace{\langle x_4^3 \rangle}_{\text{(I)}} \oplus \underbrace{\operatorname{Sym}^2\langle x_0, x_1, x_2, x_3\rangle \otimes \langle x_5\rangle}_{\text{(II)}} \oplus \underbrace{\operatorname{Sym}^2\langle x_0, x_1, x_2, x_3\rangle \otimes \langle x_6\rangle}_{\text{(III)}},$$

so a general element of W^H is

$$\varphi = a\, x_4^3 + x_5 Q_5(x_0, \ldots, x_3) + x_6 Q_6(x_0, \ldots, x_3), \qquad Q_5, Q_6 \in \operatorname{Sym}^2\langle x_0, \ldots, x_3\rangle^\vee.$$

Using the central torus factor together with projective scaling, we normalize $a = 1$.

Diagonalization of the quadratic pencil. Consider the pencil $\langle Q_5, Q_6\rangle$ and its discriminant

$$\Delta(s,t) := \det(sQ_5 + tQ_6), \qquad [s:t] \in \mathbb{P}^1.$$

On the dense open set where Δ has four distinct roots (equivalently, the pencil is regular, i.e. it lies off the discriminant locus), we may assume Q_5 is nondegenerate. Acting by $\mathrm{GL}\langle x_0, x_1, x_2, x_3\rangle \cong \mathrm{GL}(4)$, we take

$$Q_5 \sim q := x_0^2 + x_1^2 + x_2^2 + x_3^2.$$

With respect to q, the residual group on $\langle x_0, \ldots, x_3\rangle$ is $O(4)$. For a regular pencil, the q-selfadjoint endomorphism A defined by $Q_6(u,v) = q(Au, v)$ has four distinct eigenvalues, hence after an $O(4)$-change of coordinates

$$Q_6 \sim \lambda_0 x_0^2 + \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2, \qquad \lambda_i \text{ pairwise distinct.}$$

Reduction to a one-parameter normal form (cross-ratio). The $\mathrm{GL}\langle x_5, x_6\rangle \cong \mathrm{GL}(2)$-action replaces (Q_5, Q_6) by

$$(Q_5', Q_6') = (\alpha Q_5 + \beta Q_6,\ \gamma Q_5 + \delta Q_6), \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}(2),$$

and, after re-normalizing Q_5' back to q, this acts on the eigenvalues by the fractional-linear transformation

$$\lambda \;\longmapsto\; \frac{\gamma + \delta\lambda}{\alpha + \beta\lambda}.$$

Thus a regular pencil is determined (up to permuting the diagonal coordinates) by the unordered set $\{\lambda_0, \lambda_1, \lambda_2, \lambda_3\} \subset \mathbb{P}^1$ modulo PGL_2, i.e. by one cross-ratio parameter. Using 3-transitivity of PGL_2 on $\mathbb{P}^1$, we may normalize three eigenvalues to $1, -1, 2$ and denote the fourth by τ. After permuting (x_0, x_1, x_2, x_3) if necessary, we assume

$$(\lambda_0, \lambda_1, \lambda_2, \lambda_3) = (1, \tau, -1, 2), \qquad \tau \notin \{1, -1, 2\}.$$

Consequently, on the open set $\tau \in \mathbb{C}^\times \setminus \{1, -1, 2\}$, a convenient normal form is

$$\phi_{20}^{\mathrm{nf}}(\tau) = x_4^3 + x_5(x_0^2+x_1^2+x_2^2+x_3^2) + x_6(x_0^2+\tau x_1^2-x_2^2+2x_3^2), \qquad \tau \in \mathbb{C}^\times \setminus \{1, -1, 2\}.$$

(For the excluded values, the pencil acquires repeated eigenvalues and the stabilizer jumps; these form a proper closed subset.)

Component dimension. Here $\dim W^H = 21$ and $\dim C_G(H) = 20$, and $C_G(H)$ contains a one-dimensional central torus acting trivially on W^H. Hence the effective action has dimension 19, and after projectivizing we obtain

$$\dim(\Phi_{20}) = \dim W^H - 19 - 1 = 1,$$

in agreement with the single modulus τ for a regular pencil.

4.21 Case $k = 21$

1-PS limit. Set

$$\lambda_{21}(t) = \mathrm{diag}\big(t,\, t,\, 1,\, 1,\, 1,\, 1,\, t^{-2}\big), \qquad t \in \mathbb{G}_m.$$

For a generic f_{21} as in Section 3, the 1-PS limit is

$$\phi_{21} := \lim_{t\to 0} \lambda_{21}(t) \cdot f_{21} = F(x_2, x_3, x_4, x_5) + x_6\, Q(x_0, x_1),$$

where $F \in \mathrm{Sym}^3\langle x_2, x_3, x_4, x_5\rangle$ and $Q \in \mathrm{Sym}^2\langle x_0, x_1\rangle$.

H and $C_G(H)$. Let $H = \lambda_{21}(\mathbb{G}_m)$. The H-weights on $\langle x_0, \dots, x_6\rangle$ are $(1, 1, 0, 0, 0, 0, -2)$ with block decomposition $\langle x_0, x_1\rangle \oplus \langle x_2, x_3, x_4, x_5\rangle \oplus \langle x_6\rangle$. Hence,

$$\begin{aligned} C_G(H) &= \Big\{\ A \oplus B \oplus \gamma\ :\ A \in \mathrm{GL}(2),\ B \in \mathrm{GL}(4),\ \gamma \in \mathbb{G}_m,\ \det(A)\det(B)\gamma = 1\ \Big\} \\ &\cong \big(\mathrm{GL}(2) \times \mathrm{GL}(4) \times \mathbb{G}_m\big) \cap \mathrm{SL}(7), \qquad \dim C_G(H) = 20. \end{aligned}$$

Every monomial of ϕ_{21} has H-weight 0, so $\phi_{21} \in W^H$.

Polystability (Luna + Casimiro–Florentino). By Luna's reduction, the closedness of the SL(7)-orbit of ϕ_{21} is equivalent to polystability for the $C_G(H)$-action on W^H. After conjugating inside the GL(2)- and GL(4)-blocks, any $\lambda \in Y\big(C_G(H)\big)$ may be taken with

$$\mathrm{wt}(x_0, \dots, x_6) = (\alpha + u,\, \alpha - u,\, \beta + v_1,\, \dots,\, \beta + v_4,\, \gamma),$$

where $\sum v_i = 0$, and with the SL-constraint $S := 2\alpha + 4\beta + \gamma = 0$.

Summing the λ-weights of the 20 cubic monomials in F gives

$$w(F) = 15\big((\beta + v_1) + (\beta + v_2) + (\beta + v_3) + (\beta + v_4)\big) = 60\beta,$$

while the weights of the three quadratic monomials in $x_6 Q$ sum to

$$w(x_6 Q) = (\gamma + 2(\alpha + u)) + (\gamma + 2\alpha) + (\gamma + 2(\alpha - u)) = 3\gamma + 6\alpha.$$

Using $S = 0$, we obtain the positive linear identity

$$w(F) + 5\, w(x_6 Q) \ =\ 60\beta + 5(3\gamma + 6\alpha) \ =\ 15(2\alpha + 4\beta + \gamma) \ =\ 15\, S. \qquad (27)$$

If $\lambda \in \Lambda_{\phi_{21}}$, then every monomial weight occurring in F and x_6Q is ≥ 0 and $S = 0$; hence (27) forces

$$w(F) = 0, \qquad w(x_6Q) = 0,$$

so all those monomial weights vanish individually. From the cubic monomials $x_2^3, \ldots, x_5^3$ we get $\beta + v_i = 0$ for all i, hence $\beta = 0$ and $v_i = 0$. From the quadratic monomials $x_6x_0^2$ and $x_6x_1^2$ we get

$$\gamma + 2\alpha + 2u = 0, \qquad \gamma + 2\alpha - 2u = 0,$$

hence $u = 0$ and $\gamma + 2\alpha = 0$. Therefore λ is of the form

$$\lambda(t) = \operatorname{diag}\big(t^k,\, t^k,\, 1,\, 1,\, 1,\, 1,\, t^{-2k}\big) = \lambda_{21}(t^k) \qquad (k \in \mathbb{Z}),$$

and consequently

$$\Lambda_{\phi_{21}} = \{\lambda_{21}(t^k) \mid k \in \mathbb{Z}\}.$$

By Lemma 4.5, $\Lambda_{\phi_{21}}$ is symmetric; hence, by the Casimiro–Florentino criterion, ϕ_{21} is polystable.

Normal form and component dimension. We normalize generic elements under the action of the centralizer $C_G(H)$.

(1) **Quadratic part.** Under the GL(2)-action on $\langle x_0, x_1\rangle$, a generic binary quadratic Q is normalized to $x_0^2 + x_1^2$. The stabilizer of this form contains the orthogonal group O(2), contributing **1 dimension** to the stabilizer of the generic point.

(2) **Cubic part.** Under the GL(4)-action on $\langle x_2, x_3, x_4, x_5\rangle$, a generic cubic surface F can be written as the sum of five cubes of linear forms (**Sylvester's Pentahedron Theorem**). Let $L_0, \ldots, L_4$ be fixed linear forms defining a generic pentahedron in $\mathbb{C}^4$. Then $F = \sum_{i=0}^4 c_i L_i^3$.

Thus, the normal form is:

$$\phi_{21}^{\mathrm{nf}}(c) = \sum_{i=0}^{4} c_i L_i(x_2, x_3, x_4, x_5)^3 + x_6(x_0^2 + x_1^2),$$

where the parameters are $[c_0 : \cdots : c_4] \in \mathbb{P}^4$.

The dimension of the component is computed as follows: $\dim W^H = 20(\text{cubic}) + 3(\text{quadratic}) = 23$. Since H acts trivially on W^H, the effective group is $C_G(H)/H$, of dimension $20 - 1 = 19$. Since the generic stabilizer has dimension 1 (due to O(2)), the orbit dimension is $19 - 1 = 18$.

$$\dim(\Phi_{21}) = 23 - 18 - 1 = 4.$$

Table 2: Summary of closed orbit representatives, criteria used, and component dimensions for $k = 1, \ldots, 21$. C-H means the Convex-hull criterion and C-F means the Casimiro–Florentino criterion.

k	Criterion	Normal form	Parameters	Dim
1	C-H	$x_2x_3^2 + x_1x_3x_4 + x_2^2x_5 + x_0x_5^2 + x_1^2x_6 + x_0x_4x_6$	none	0
2	C-H	$x_2^2x_4 + x_1x_4^2 + x_1x_3x_5 + x_0x_5^2 + x_1x_2x_6 + x_0x_3x_6$	none	0
3	C-F	$x_1x_3^2 + x_1x_3x_4 + x_1x_4^2 + x_2^2x_5 + x_0x_5^2 + x_1x_2x_6 + x_0x_3x_6 + x_0x_4x_6$	none	0
4	C-H	$x_3^3 + x_2x_3x_4 + x_1x_4^2 + x_2^2x_5 + x_1x_3x_5 + x_0x_4x_5 + \alpha x_1x_2x_6 + \beta x_0x_3x_6$	$\alpha, \beta \in \mathbb{C}^\times$	2
5	C-H	$x_3^3+x_2x_3x_4+x_1x_4^2+x_2^2x_5+x_1x_3x_5+x_0x_5^2+\alpha x_1^2x_6+\beta x_0x_3x_6$	$\alpha, \beta \in \mathbb{C}^\times$	2
6	C-H	$x_2x_4^2 + x_2^2x_5 + x_1x_3x_5 + x_0x_4x_5 + x_1^2x_6 + x_0x_3x_6$	none	0
7	C-H	$x_3^2x_4 + x_1x_4^2 + x_2x_3x_5 + x_0x_5^2 + x_1^2x_6 + x_0x_2x_6$	none	0
8	C-F	$x_3^3+x_0x_4^2+x_0x_5^2+x_0x_6^2+x_1^2x_4+\rho\, x_1x_2x_5+\sigma\, x_2^2x_6$	$(\rho, \sigma) \in (\mathbb{C}^\times)^2$	2
9	C-F	$x_2^2x_3 + \tau\, x_2x_3^2 + x_0x_4^2 + \rho\, x_0x_5^2 + x_1x_4^2 + x_1x_5^2 + x_0x_2x_6 + x_1x_3x_6$	$(\tau, \rho) \in (\mathbb{C}^\times)^2$	2
10	C-F	$x_2^3 + \tau\, x_2^2x_3 + \rho\, x_2x_3^2 + x_3^3 + x_1x_2x_5 + x_1x_3x_4 + x_0x_4x_5 + x_0x_6^2$	$(\tau, \rho) \in (\mathbb{C}^\times)^2$	2
11	C-F	$\sum_{i=0}^4 c_iL_i(x_1, x_2, x_3, x_4)^3 + x_0(x_5^2 + x_6^2)$	$(c_0 : c_1 : c_2 : c_3 : c_4) \in \mathbb{P}^4$	4
12	C-F	$x_1x_4^2 + x_2^2x_5 + x_3^2x_5 + x_0x_4x_5 + x_1^2x_6 + x_0x_2x_6$	none	0
13	C-F	$x_3^3 + x_0x_4^2 + x_0x_5^2 + x_1x_3x_4 + \rho\, x_2x_3x_5 + x_1^2x_6 + \sigma\, x_2^2x_6 + x_0x_3x_6$	$(\rho, \sigma) \in (\mathbb{C}^\times)^2$	2
14	C-F	$x_2^3+x_0\big(x_3^2+x_4^2+x_5^2+x_6^2\big)+x_1\big(x_3^2+\tau x_4^2-x_5^2+2x_6^2\big)$	$\tau \in \mathbb{C}^\times \setminus \{-1, 1, 2\}$	1
15	C-F	$x_3^2x_4 + \tau\, x_3x_4^2 + x_0x_3x_5 + x_0x_4x_6 + x_1^2x_5 + \rho\, x_2^2x_6$	$(\tau, \rho) \in (\mathbb{C}^\times)^2$	2
16	C-F	$x_2^3+x_3^3+\sigma x_4^3+\rho\, x_2x_3x_4+x_1x_2x_5+x_0x_3x_6+x_0x_5^2+x_1^2x_6+\kappa\, x_2^2x_3+\nu\, x_2x_3^2+\mu\, x_3^2x_4+\lambda\, x_2^2x_4$	$(\sigma, \rho, \kappa, \nu, \mu, \lambda) \in (\mathbb{C}^\times)^6$	6
17	C-F	$x_3^2x_4 + \tau\, x_3x_4^2 + x_0x_3x_5 + x_1x_4x_5 + x_0^2x_6 + x_1^2x_6 + \rho\, x_2^2x_6$	$(\tau, \rho) \in (\mathbb{C}^\times)^2$	2
18	C-F	$x_2^3 + \sigma_1x_3^3 + \sigma_2x_4^3 + \rho\, x_2x_3x_4 + x_0x_2x_5 + x_1x_3x_5 + \alpha\, x_0x_4x_5 + x_0x_2x_6 + \beta\, x_1x_3x_6 + \gamma\, x_1x_4x_6$	$(\sigma_1, \sigma_2, \rho, \alpha, \beta, \gamma) \in (\mathbb{C}^\times)^6$	6
19	C-F	$x_3^3 + (x_0x_4^2 + x_1x_4x_5 + x_2x_5^2) + x_6(x_0^2 + x_2^2 + \rho\, x_1^2 + \sigma\, x_0x_2)$	$(\rho, \sigma) \in (\mathbb{C}^\times)^2$	2
20	C-F	$x_4^3+x_5(x_0^2+x_1^2+x_2^2+x_3^2)+x_6(x_0^2+\tau x_1^2-x_2^2+2x_3^2)$	$\tau \in \mathbb{C}^\times \setminus \{-1, 1, 2\}$	1
21	C-F	$\sum_{i=0}^4 c_iL_i(x_2, x_3, x_4, x_5)^3 + x_6(x_0^2 + x_1^2)$	$(c_0 : c_1 : c_2 : c_3 : c_4) \in \mathbb{P}^4$	4

5 Singular loci of 21 polystable cubic fivefolds

In this section, we determine the singular loci of general closed–orbit representatives ϕ_k^{nf} constructed in Section 4. For each $k = 1, \ldots, 21$, we set

$$X_k := V(\phi_k^{\mathrm{nf}}) \subset \mathbb{P}^6$$

and compute

$$\mathrm{Sing}(X_k) \;=\; V\big(J(\phi_k^{\mathrm{nf}})\big) \subset \mathbb{P}^6, \qquad J(\varphi) := \left(\frac{\partial\varphi}{\partial x_0}, \ldots, \frac{\partial\varphi}{\partial x_6}\right) : (x_0, \ldots, x_6)^\infty,$$

i.e. the scheme cut out by the saturated Jacobian ideal. We give a set–theoretic description of $\mathrm{Sing}(X_k)$ in each case, and when isolated points occur, we also record the corank and the local invariants (Milnor and Tjurina number, which agree for our quasi–homogeneous normal forms). Table 3 presents a compact summary—listing the type and degree of the top–dimensional part and indicating the presence (or absence) of isolated points. Detailed case–by–case statements are recorded as Propositions 5.8–5.28.

Note that for the boundary families with moduli (parameters), the descriptions of the singular loci in the following propositions and the minimal exponent computations in Section 6 apply to general parameters. At certain special values of the parameters, the singularities may degenerate further, and the minimal exponent may strictly drop.

Isolated singularities appearing in this paper

We define the isolated singularities that appear in the investigation of the singularities of ϕ_k^{nf}. Since we could not locate an explicit normal form in the standard tables (e.g. [AGV12]), we fix notation here.

Definition 5.1. *We write* $\mathrm{QH}(9,6,5;24)_{19}$ *for the isolated hypersurface singularity analytically equivalent to* $X^2Y + Y^4 + XZ^3$. *It is quasi-homogeneous of total degree* 24 *with respect to weights* $(w_X, w_Y, w_Z) = (9,6,5)$. *Its corank, Milnor number and Tjurina number are* 3, 19 *and* 19 *respectively.*

Definition 5.2. *We write* $\mathrm{QH}(2,3,5;12)_{21}(t)$ *for the isolated hypersurface singularity analytically equivalent to* $tX^6 + X^3Y^2 + X^2YZ + XZ^2 + Y^4$, *where* $t \in \mathbb{C}^\times$ *is a parameter. It is quasi-homogeneous of total degree* 12 *with respect to weights* $(w_X, w_Y, w_Z) = (2,3,5)$. *Its corank, Milnor number and Tjurina number are* 3, 21 *and* 21 *respectively.*

Definition 5.3. *We write* $\mathrm{QH}(3,3,4;12)_{18}(t)$ *for the isolated hypersurface singularity analytically equivalent to* $Z^3+X^4+tX^2Y^2+Y^4$. *It is quasi-homogeneous of total degree* 12 *with respect to weights* $(w_X, w_Y, w_Z) = (3,3,4)$. *Its corank, Milnor number and Tjurina number are* 3, 18 *and* 18 *respectively.*

Definition 5.4. *We write* $\mathrm{QH}(1,1;6)_{25}$ *for the isolated hypersurface singularity analytically equivalent to a homogeneous sextic*

$$\Psi(X,Y) = \prod_{i=1}^{6} L_i(X,Y),$$

where $L_1, \dots, L_6$ are pairwise distinct linear forms in (X, Y). Equivalently, after a linear change of coordinates, one may write $\Psi(X, Y) = \prod_{i=1}^{6}(X - \lambda_i Y)$ with λ_i pairwise distinct in $\mathbb{P}^1$.

It is quasi-homogeneous of total degree 6 with respect to weights $(w_X, w_Y) = (1, 1)$. Its corank, Milnor number and Tjurina number are 2, 25 and 25 respectively.

Definition 5.5. *We write $QH(1,1,1,1;3)_{16}([c])$ for the isolated hypersurface singularity analytically equivalent to a homogeneous cubic*

$$G(X_1, X_2, X_3, X_4) = \Sigma_{i=0}^{4} c_i L_i^3$$

where $L_i = L_i(X_1, X_2, X_3, X_4)$ are generic linear forms and $[c] = (c_0 : c_1 : c_2 : c_3 : c_4) \in \mathbb{P}^4$. Its corank, Milnor number and Tjurina number are 4, 16 and 16. It has four-dimensional moduli.

Definition 5.6. $\mathrm{QH}(1,2,2;6)_{20}(t)$

$$F(X, Y, Z) = Y^3 + Z^3 + X^6 + t_1 X^2 Y^2 + t_2 X^2 YZ + t_3 Y^2 Z + t_4 X^4 Y + t_5 X^4 Z,$$

where $t = (t_1, t_2, \cdots, t_5)$. Its corank, Milnor number and Tjurina number are 3, 20 and 20.

Remark 5.7 (Remark for Definition 5.4)**.** *The germ $\Psi(X, Y) = \prod_{i=1}^{6} L_i(X, Y)$ with pairwise distinct linear forms defines an* ordinary sextuple point *(also called an ordinary 6-fold point): it has 6 smooth branches meeting transversely at the origin, with 6 distinct tangent directions in $\mathbb{P}^1$.*

We will frequently use the following shorthand labels for isolated singularity types:

Shorthand Label	*Label*	Defining equation
E_{19}	$\mathrm{QH}(9,6,5;24)_{19}$	$X^2Y + Y^4 + XZ^3 = 0$
E_{21}	$\mathrm{QH}(2,3,5;12)_{21}(t)$	$tX^6 + X^3Y^2 + X^2YZ + XZ^2 + Y^4 = 0$
E_{18}	$\mathrm{QH}(3,3,4;12)_{18}(t)$	$Z^3 + X^4 + tX^2Y^2 + Y^4 = 0$
$\mathcal{O}_6$	$\mathrm{QH}(1,1;6)_{25}$	$\prod_{i=1}^{6}(X - \lambda_i Y) = 0$
$\mathcal{C}_{cub}$	$\mathrm{QH}(1,1,1,1;3)_{16}([c])$	$\Sigma_{i=0}^{4} c_i L_i^3 = 0$
E_{20}	$\mathrm{QH}(1,2,2;6)_{20}(t)$	$Y^3+Z^3+X^6+t_1X^2Y^2+t_2X^2YZ+t_3Y^2Z+t_4X^4Y+t_5X^4Z=0$

5.1 Case $k = 1$

Proposition 5.8. *Let $X_1 = V(\phi_1^{\mathrm{nf}}) \subset \mathbb{P}^6$. The set-theoretic singular locus is*

$$\mathrm{Sing}(X_1) = C \cup \{P\},$$

where

$$C = \{\, x_0 = x_1 = x_2 = x_3 = 0,\ x_5^2 + x_4 x_6 = 0 \,\}$$

is a smooth conic in the plane $\{x_0 = x_1 = x_2 = x_3 = 0\} \cong \mathbb{P}^2$ (hence, $\deg C = 2$), and

$$P = (1 : 0 : 0 : 0 : 0 : 0 : 0)$$

is an isolated singular point.

The isolated singular point is of type E_{19}.

5.2 Case $k = 2$

Proposition 5.9. *Let $X_2 = V(\phi_2^{\mathrm{nf}}) \subset \mathbb{P}^6$. Then*

$$\mathrm{Sing}(X_2) = L_{01} \cup C,$$

with $L_{01} = V(x_2, x_3, x_4, x_5, x_6) \simeq \mathbb{P}^1$, and

$$C = V(x_0, x_1, x_2,\ x_5^2 + x_3x_6,\ x_4^2 + x_3x_5) \subset \Pi, \quad \Pi = \{x_0 = x_1 = x_2 = 0\} \simeq \mathbb{P}^3.$$

Here C is a complete intersection $\mathrm{CI}(2,2)$ of degree 4, with $L_{01} \cap C = \varnothing$.

5.3 Case $k = 3$

Proposition 5.10. *Let $X_3 = V(\phi_3^{\mathrm{nf}}) \subset \mathbb{P}^6$. Then*

$$\mathrm{Sing}(X_3) = L_{01} \cup C,$$

with $L_{01} = V(x_2, x_3, x_4, x_5, x_6) \simeq \mathbb{P}^1$, and

$$C = V(x_0, x_1, x_2, x_5^2 + x_3x_6 + x_4x_6, x_3^2 + x_3x_4 + x_4^2) \subset \Pi, \quad \Pi = \{x_0 = x_1 = x_2 = 0\} \simeq \mathbb{P}^3.$$

Here C is a complete intersection $\mathrm{CI}(2,2)$ of degree 4, with $L_{01} \cap C = \varnothing$.

5.4 Case $k = 4$

Proposition 5.11. *Let $X_4 = V(\phi_4^{\mathrm{nf}}) \subset \mathbb{P}^6$. Then*

$$\mathrm{Sing}(X_4) = L_{01} \cup L_{56},$$

with $L_{01} = V(x_2, x_3, x_4, x_5, x_6) \simeq \mathbb{P}^1$ and $L_{56} = V(x_0, x_1, x_2, x_3, x_4) \simeq \mathbb{P}^1$. In particular, $L_{01} \cap L_{56} = \varnothing$ and no isolated singular points occur.

5.5 Case $k = 5$

Proposition 5.12. *Let $X_5 = V(\phi_5^{\mathrm{nf}}) \subset \mathbb{P}^6$. Then*

$$\mathrm{Sing}(X_5) = \{P\} \cup \{Q\},$$

where $P = (1:0:0:0:0:0:0)$, $Q = (0:0:0:0:0:0:1)$ are isolated singular points.

Then (X_5, P) and (X_5, Q) are isolated hypersurface singularities of corank 3, and they are analytically equivalent. The isolated singular points P, Q are of type E_{21} and have same parameter t.

5.6 Case $k = 6$

Proposition 5.13. *Let $X_6 = V(\phi_6^{\mathrm{nf}}) \subset \mathbb{P}^6$. Then*

$$\mathrm{Sing}(X_6) = C \cup L_{56}, \quad L_{56} = V(x_0, x_1, x_2, x_3, x_4),$$

$$C = V(x_4, x_5, x_6,\ x_2^2 + x_1x_3,\ x_1^2 + x_0x_3) \subset \Pi = \{x_4 = x_5 = x_6 = 0\}.$$

C is a $\mathrm{CI}(2,2)$ of degree 4; $C \cap L_{56} = \varnothing$.

5.7 Case $k = 7$

Proposition 5.14. *Let $X_7 = V(\phi_7^{\mathrm{nf}}) \subset \mathbb{P}^6$. Then*

$$\mathrm{Sing}(X_7) = C \cup \{P\},$$

where

$$C = V(x_3, x_4, x_5, x_6,\ x_1^2 + x_0x_2), \quad P = (0:0:0:0:0:0:1).$$

The isolated singular point P is of type E_{19}.

5.8 Case $k = 8$

Proposition 5.15. *Let $X_8 := V(\phi_8^{\mathrm{nf}}) \subset \mathbb{P}^6$. Then the singular locus has two irreducible components:*

$$\mathrm{Sing}(X_8) = C \cup \{P\}$$

, where

$$C := V\big(x_0, x_1, x_2, x_3,\ x_4^2 + x_5^2 + x_6^2\big) \subset \Pi := \{x_0 = x_1 = x_2 = x_3 = 0\} \cong \mathbb{P}^2$$

is a (smooth) conic of degree 2 (hence $\mathrm{codim}_{\mathbb{P}^6} C = 5$), and

$$P := V\big(x_1, x_2, x_3, x_4, x_5, x_6\big) = (1:0:0:0:0:0:0)$$

is an isolated point (hence $\mathrm{codim}_{\mathbb{P}^6} P = 6$, $\deg P = 1$). In particular, $C \cap \{P\} = \varnothing$. The isolated singular point P is of type E_{18}.

5.9 Case $k = 9$

Proposition 5.16. *Let $X_9 := V(\phi_9^{\mathrm{nf}}) \subset \mathbb{P}^6$. Then the singular locus has two irreducible components:*

$$\mathrm{Sing}(X_9) = L_{01} \cup \{P\},$$

where $L_{01} = V(x_2, x_3, x_4, x_5, x_6)$ and $P = (0:0:0:0:0:0:1)$. The isolated point P is of type $\mathcal{O}_6$.

5.10 Case $k = 10$

Proposition 5.17. *Let $X_{10} := V(\phi_{10}^{\mathrm{nf}}) \subset \mathbb{P}^6$. Then the singular locus has two irreducible components:*

$$\mathrm{Sing}(X_{10}) = L_{01} \cup C,$$

where $L_{01} = V(x_2, x_3, x_4, x_5, x_6)$ and $C = V(x_0, x_1, x_2, x_3, x_4x_5 + x_6^2)$ which is a curve of degree 2. We have $L_{01} \cap C = \varnothing$.

5.11 Case $k = 11$

Proposition 5.18. *Let $X_{11} := V(\phi_{11}^{\mathrm{nf}}) \subset \mathbb{P}^6$. Then the singular locus has three irreducible components:*

$$\mathrm{Sing}(X_{11}) = \{P_1\} \cup \{P_2\} \cup \{P_3\},$$

where P_1, P_2 and P_3 are isolated singular points of type $\mathcal{C}_{cub}$.

5.12 Case $k = 12$

Proposition 5.19. *Let $X_{12} := V(\phi_{12}^{\mathrm{nf}}) \subset \mathbb{P}^6$. Then the singular locus has two irreducible components:*

$$\mathrm{Sing}(X_{12}) = L_{56} \cup C,$$

where $L_{56} = V(x_0, x_1, x_2, x_3, x_4)$ is a projective line and $C = V(x_2^2 + x_3^2, x_1^2 + x_0x2, x_4, x_5, x_6)$ is a $(2,2)$-complete intersection. We have $L_{56} \cap C = \varnothing$.

5.13 Case $k = 13$

Proposition 5.20. *Let $X_{13} := V(\phi_{13}^{\mathrm{nf}}(\rho, \sigma)) \subset \mathbb{P}^6$. Then the singular locus has two connected components:*

$$\mathrm{Sing}(X_{13}) = C_1 \cup C_2,$$

where $C_1 = V(x_1^2 + \sigma x_2^2, x_3, x_4, x_5, x_6)$ and $C_2 = V(x_0, x_1, x_2, x_3, x_4^2 + x_5^2)$. Each C_i consists of two lines which intersect at one point. We have $C_1 \cap C_2 = \varnothing$.

5.14 Case $k = 14$

Proposition 5.21. *Let $X_{14} := V(\phi_{14}^{\mathrm{nf}}(\tau)) \subset \mathbb{P}^6$. Then the singular locus has two irreducible components:*

$$\mathrm{Sing}(X_{14}) = L_{01} \cup C,$$

where $L_{01} = V(x_2, x_3, x_4, x_5, x_6)$ and

$$C = V(x_0, x_1, x_2, (\tau-1)x_4^2 - 2x_5^2 + x_6^2, (\tau-1)x_3^2 + (\tau+1)x_5^2 + (\tau-2)x_6^2).$$

We have $L_{01} \cap C = \varnothing$.

5.15 Case $k = 15$

Proposition 5.22. *Let $X_{15} := V(\phi_{15}^{\mathrm{nf}}(\tau, \sigma)) \subset \mathbb{P}^6$. Then the singular locus consists of three irreducible components:*

$$\mathrm{Sing}(X_{15}) = C_1 \cup C_2 \cup L,$$

where $C_1 = V(x_1, x_3, x_6, \tau x_4^2 + x_0x_5, \sigma x_2^2 + x_0x_4)$, $C_2 = V(x_2, x_4, x_5, x_3^2 + x_0x_6, x_1^2 + x_0x_3)$, and $L = V(x_0, x_1, x_2, x_3, x_4)$. Here, C_1 and C_2 are curves of degree 4 (complete intersections of two quadrics in $\mathbb{P}^3$), and L is a line.

5.16 Case $k = 16$

Proposition 5.23. *Let $X_{16} := V(\phi_{16}^{\mathrm{nf}}) \subset \mathbb{P}^6$. Then the singular locus has two irreducible components:*

$$\mathrm{Sing}(X_{16}) = \{P\} \cup \{Q\},$$

$P = (1:0:0:0:0:0:0)$, $Q = (0:0:0:0:0:0:1)$, where P, Q are isolated singular points of type E_{20} with the same modulus.

5.17 Case $k = 17$

Proposition 5.24. *Let $X_{17} := V(\phi_{17}^{\mathrm{nf}}(\tau, \sigma)) \subset \mathbb{P}^6$. Then the singular locus consists of two disjoint components:*

$$\mathrm{Sing}(X_{17}) = C \cup L_{56},$$

where $C = V(x_3, x_4, x_5, x_6, x_0^2 + x_1^2 + \sigma x_2^2)$ and $L_{56} = V(x_0, x_1, x_2, x_3, x_4)$. Here, C is a smooth conic (degree 2) and L is a line (degree 1). We have $C \cap L_{56} = \varnothing$.

5.18 Case $k = 18$

Proposition 5.25. *Let $X_{18} := V(\phi_{18}^{\mathrm{nf}}) \subset \mathbb{P}^6$. Then the singular locus consists of two disjoint components:*

$$\mathrm{Sing}(X_{18}) = L_1 \cup L_2,$$

where $L_1 = V(x_2, x_3, x_4, x_5, x_6)$ and $L_2 = V(x_0, x_1, x_2, x_3, x_4)$. Here, both L_1 and L_2 are lines (degree 1). We have $L_1 \cap L_2 = \varnothing$.

5.19 Case $k = 19$

Proposition 5.26. *Let $X_{19} := V(\phi_{19}^{\mathrm{nf}}) \subset \mathbb{P}^6$. Then the singular locus has two irreducible components:*

$$\mathrm{Sing}(X_{19}) = \{P\} \cup C,$$

$P = (0:0:0:0:0:0:1)$, $C = V(x_0^2 + \rho x_1^2 + \sigma x_0 x_2 + x_2^2, x_3, x_4, x_5, x_6)$. Here P is an isolated singular point of type E_{18}.

5.20 Case $k = 20$

Proposition 5.27. *Let $X_{20} := V(\phi_{20}^{\mathrm{nf}}(\tau)) \subset \mathbb{P}^6$. Then the singular locus consists of two disjoint components:*

$$\mathrm{Sing}(X_{20}) = C \cup L,$$

where $C = V(x_4, x_5, x_6, (\tau-1)x_1^2 - 2x_2^2 + x_3^2, (\tau-1)x_0^2 + (\tau+1)x_2^2 + (\tau-2)x_3^2)$ and $L = V(x_0, x_1, x_2, x_3, x_4)$. Here, C is a curve of degree 4 (a complete intersection of two quadrics, which is an elliptic curve) and L is a line (degree 1). We have $C \cap L = \varnothing$.

5.21 Case $k = 21$

Proposition 5.28. *Let $X_{21} := V(\phi_{21}^{\mathrm{nf}}) \subset \mathbb{P}^6$. Then the singular locus has three irreducible components:*

$$\mathrm{Sing}(X_{21}) = \{P_1\} \cup \{P_2\} \cup \{P_3\},$$

where P_1, P_2 and P_3 are isolated singular points of type C_{cub}.

Table 3: Singular loci of the 21 closed-orbit representatives (Section 5)

k	Singular locus (notation of §5)	Type / degree of top-dimensional part	Isolated point(s) / invariants
1	$C \cup \{P\}$ (C smooth conic)	conic (deg 2)	one point P of type E_{19}
2	$L_{01} \cup C$ ($L_{01} \cong \mathbb{P}^1$, C a CI$(2,2)$)	line + quartic curve (deg 1+4)	—
3	$L_{01} \cup C$ ($L_{01} \cong \mathbb{P}^1$, C a CI$(2,2)$)	line + quartic curve (deg 1+4)	—
4	$L_{01} \cup L_{56}$ (two disjoint lines)	two disjoint lines (deg 1+1)	—
5	$\{P\} \cup \{Q\}$	two points	P, Q of type E_{21} (same modulus)
6	$C \cup L_{56}$ (C a CI$(2,2)$, L_{56} line)	line + quartic curve (deg 1+4)	—
7	$\{P\} \cup C$ (C conic)	conic (deg 2)	one point P of type E_{19}
8	$C \cup \{P\}$ (C smooth conic)	conic (deg 2)	one point P of type E_{18}
9	$L_{01} \cup \{P\}$ (L_{01} line)	line (deg 1)	one point P of type $\mathcal{O}_6$
10	$L_{01} \cup C$ (L_{01} line, C conic)	line + conic (deg 1+2)	—
11	$\{P_1\} \cup \{P_2\} \cup \{P_3\}$	three points	three points P_i of type $\mathcal{C}_{cub}$
12	$L_{56} \cup C$ (L_{56} line, C a CI$(2,2)$)	line + quartic curve (deg 1+4)	—
13	$C_1 \cup C_2$ (disjoint)	two pairs of intersecting lines (deg 1+1+1+1)	—
14	$L_{01} \cup C$ (L_{01} line, C a CI$(2,2)$)	line + quartic curve (deg 1+4)	—
15	$C_1 \cup C_2 \cup L$ (two CI$(2,2)$s + line)	two quartics + line (deg 4+4+1)	—
16	$\{P\} \cup \{Q\}$	two points	P, Q of type E_{20} (same modulus)
17	$C \cup L$ (C smooth conic, L line)	conic + line (deg 2+1)	—
18	$L_1 \cup L_2$ (two disjoint lines)	two disjoint lines (deg 1+1)	—
19	$C \cup \{P\}$ (C smooth conic)	conic (deg 2)	one point P of type E_{18}
20	$C \cup L$ (C elliptic curve, L line)	elliptic curve + line (deg 4+1)	—
21	$\{P_1\} \cup \{P_2\} \cup \{P_3\}$	three points	three points P_i of type $\mathcal{C}_{cub}$

Table 4: List of isolated singularities

Symbol	Type	Defining Equation (Normal Form)
E_{19}	$\mathrm{QH}(9,6,5;24)_{19}$	$X^2Y + Y^4 + XZ^3 = 0$
E_{21}	$\mathrm{QH}(2,3,5;12)_{21}(t)$	$tX^6 + X^3Y^2 + X^2YZ + XZ^2 + Y^4 = 0$
E_{18}	$\mathrm{QH}(3,3,4;12)_{18}(t)$	$Z^3 + X^4 + tX^2Y^2 + Y^4 = 0$
$\mathcal{O}_6$	$\mathrm{QH}(1,1;6)_{25}$	$\prod_{i=1}^{6}(X - \lambda_i Y) = 0$
$\mathcal{C}_{cub}$	$\mathrm{QH}(1,1,1,1;3)_{16}([c])$	$\sum_{i=0}^{4} c_i L_i^3 = 0$
E_{20}	$\mathrm{QH}(1,2,2;6)_{20}(t)$	$Y^3+Z^3+X^6+t_1X^2Y^2+t_2X^2YZ+t_3Y^2Z+t_4X^4Y+t_5X^4Z=0$

6 The minimal exponents

6.1 Minimal exponents of isolated singularities

In Definitions 5.1–5.6 we introduced six quasi-homogeneous isolated hypersurface singularities that occur as the isolated points on the GIT boundary. The next theorem records a uniform numerical feature of these singularities, and explains why they sit exactly at the critical value in Park's stability criterion.

Theorem 6.1. *Let (X, x) be one of the six isolated boundary singularities from Definitions 5.1–5.6. Then the local minimal exponent is*

$$\widetilde{\alpha}_x(X) = \frac{7}{3}.$$

Proof. By the holomorphic splitting lemma, the local defining equation of X at an isolated singular point x is analytically equivalent to

$$u_1^2 + \cdots + u_{6-c}^2 \ + \ g(z_1, \ldots, z_c),$$

where $c = \operatorname{corank}_x(X)$ and g is one of the weighted-homogeneous germs from Definitions 5.1–5.6

By the Thom-Sebastiani type theorem for minimal exponents (see, for example, [Par25, Theorem 2.3]),

$$\widetilde{\alpha}_x(X) = \widetilde{\alpha}_0\big(u_1^2 + \cdots + u_{6-c}^2\big) + \widetilde{\alpha}_0(g) = \frac{6-c}{2} + \widetilde{\alpha}_0(g).$$

For a weighted-homogeneous isolated hypersurface singularity of type $\mathrm{QH}(w_1, \ldots, w_c; D)$, we have $\widetilde{\alpha}_0(g) = \frac{w_1+\cdots+w_c}{D}$. In our list, this yields

$$\widetilde{\alpha}_0(g) = \begin{cases} \frac{5}{6}, & g \in \{\mathrm{QH}(9,6,5;24)_{19},\ \mathrm{QH}(2,3,5;12)_{21}(t),\ \mathrm{QH}(3,3,4;12)_{18}(t),\ \mathrm{QH}(1,2,2;6)_{20}(t)\}, \\ \frac{1}{3}, & g = \mathrm{QH}(1,1;6)_{25}, \\ \frac{4}{3}, & g = \mathrm{QH}(1,1,1,1;3)_{16}([c]). \end{cases}$$

The corresponding coranks are $c = 3, 2, 4$ respectively (cf. the definitions), hence in each case

$$\widetilde{\alpha}_x(X) = \frac{6-c}{2} + \widetilde{\alpha}_0(g) = \frac{7}{3}.$$

□

Remark 6.2 (Significance via Park's threshold)**.** *For cubic fivefolds $(n, d) = (6, 3)$, Park's GIT criterion gives the sharp threshold $\frac{n+1}{d} = \frac{7}{3}$: a degree-$d$ hypersurface $X \subset \mathbb{P}^n$ is GIT stable (resp. semistable) if $\widetilde{\alpha}(X) > \frac{n+1}{d}$ (resp. $\geq \frac{n+1}{d}$) [Par25, Theorem A]. Hence, if X has a point with local minimal exponent $7/3$, then $\widetilde{\alpha}(X) \leq 7/3$, so X cannot lie in the strictly stable locus. This explains why the isolated singularities in Theorem 6.1 occur precisely on the GIT boundary.*

Moreover, Park observed that the classical explicit GIT analyses for cubic hypersurfaces with $n \leq 5$ can be reformulated uniformly in terms of the same threshold $\frac{n+1}{3}$ for the global minimal exponent [Par25, Remark 7.6]; Theorem 6.1 shows that, for cubic fivefolds ($n = 6$), the equality

$$\widetilde{\alpha}_x(X) = \frac{7}{3} = \frac{n+1}{3}$$

is attained along the strictly semistable boundary.

Definition 6.3 (Extremal and 7/3-singularities)**.** *We call the six isolated analytic types appearing in Theorem 6.1* extremal cubic fivefold singularities*. More generally, an isolated cubic fivefold singularity with local minimal exponent* 7/3 *is called a* 7/3-singularity*.*

6.2 The global minimal exponents of general members of the boundary families

For families whose singular locus $\mathrm{Sing}(X_k)$ has positive-dimensional components (curves or lines), we must compute the local minimal exponent both at a general point and at the special points along these strata in order to determine the global minimal exponent

$$\widetilde{\alpha}(X_k) \;=\; \min_{x\in \mathrm{Sing}(X_k)} \widetilde{\alpha}_x(X_k).$$

Our computations for all 21 closed-orbit normal forms reveal a striking uniformity that matches Park's threshold:

Theorem 6.4. *For each $k \in \{1,\dots,21\}$, the global minimal exponent of a general strictly semistable closed-orbit representative $X_k = V(\phi_k^{\mathrm{nf}}) \subset \mathbb{P}^6$ in the component Φ_k is*

$$\widetilde{\alpha}(X_k) \;=\; \frac{7}{3}.$$

Proof. We prove the theorem by a uniform case analysis organized by the structure of the one-dimensional components of $\mathrm{Sing}(X_k)$. More precisely, for each k with $\dim \mathrm{Sing}(X_k) = 1$, we classify every irreducible curve component $\Gamma \subset \mathrm{Sing}(X_k)$ into one of finitely many pattern types according to the behavior of the local minimal exponent $p \mapsto \widetilde{\alpha}_p(X_k)$ along Γ (see Table 5).

In each pattern type, the minimum of $\widetilde{\alpha}_p(X_k)$ along Γ equals 7/3. For types A1 and A2, this minimum is attained at an explicitly described finite subset of special points, whereas for types B and C, the minimal exponent is identically 7/3 everywhere along Γ. Consequently, $\min_{p\in\Gamma} \widetilde{\alpha}_p(X_k) = \frac{7}{3}$ for every curve component $\Gamma \subset \mathrm{Sing}(X_k)$.

If $\mathrm{Sing}(X_k)$ has only isolated singular points (namely $k \in \{5, 11, 16, 21\}$), then $\widetilde{\alpha}(X_k) = 7/3$ follows directly from the computation of the six isolated extremal singularity types recorded in Table 4. Taking the minimum over all singular points, we obtain $\widetilde{\alpha}(X_k) = 7/3$ for every $k = 1,\dots,21$. □

component type	Type A1	Type A2	Type B	Type C
line	$k=2$: L_{01} $k=3$: L_{01} $k=4$: L_{01}, L_{56} $k=6$: L_{56} $k=10$: L_{01} $k=12$: L_{56} $k=17$: L_{56}	–	$k=18$: L_{01}, L_{56}	$k=9$: L_{01} $k=14$: L_{01} $k=15$: L $k=20$: L
smooth conic	$k=1$: C $k=7$: C	–	$k=10$: C $k=17$: C	$k=8$: C $k=19$: C
twisted quartic	$k=6$: C $k=12$: C	–	–	–
elliptic quartic	–	–	$k=14$: C $k=20$: C	–
reducible conic	$k=13$: C_1, C_2	–	–	–
CI(2,2)	$k=2$: C $k=3$: C	$k=15$: C_1, C_2	–	–

Table 5: Index of one-dimensional singular-locus components by minimal-exponent pattern type.

Pattern types for one-dimensional singular components

Let $X \subset \mathbb{P}^6$ be one of the closed-orbit normal forms $X_k = V(\phi_k^{\mathrm{nf}})$, and let $\Gamma \subset \mathrm{Sing}(X)$ be an irreducible curve component. We classify Γ according to the behavior of the function

$$\Gamma \ni p \longmapsto \widetilde{\alpha}_p(X)$$

along Γ. In all cases appearing in this paper, the only values that occur on Γ are $5/2$ and $7/3$. We say that Γ is of one of the following pattern types.

Type A1 (one special point). Γ is of Type A1 if there exists a unique point $p_0 \in \Gamma$ such that

$$\widetilde{\alpha}_{p_0}(X) = \frac{7}{3}, \qquad \widetilde{\alpha}_p(X) = \frac{5}{2} \ \text{ for all } p \in \Gamma \setminus \{p_0\}.$$

Equivalently, $\widetilde{\alpha}_p(X) = 5/2$ at a general point of Γ, and the minimum along Γ is attained at a single explicitly described "special" point p_0.

Type A2 (two special points). Γ is of Type A2 if there exist exactly two points $p_1, p_2 \in \Gamma$ such that

$$\widetilde{\alpha}_{p_1}(X) = \widetilde{\alpha}_{p_2}(X) = \frac{7}{3}, \qquad \widetilde{\alpha}_p(X) = \frac{5}{2} \ \text{ for all } p \in \Gamma \setminus \{p_1, p_2\}.$$

Equivalently, $\widetilde{\alpha}_p(X) = 5/2$ at a general point of Γ, and the minimum along Γ is attained at two explicitly described special points.

Type B (uniform $7/3$ without rank drop). Γ is of Type B if

$$\widetilde{\alpha}_p(X) = \frac{7}{3} \quad \text{for all } p \in \Gamma,$$

and moreover the transverse quadratic part to Γ has constant rank along Γ (i.e. there are no rank-drop points of the Hessian along Γ). In particular, the minimum of $\widetilde{\alpha}_p(X)$ on Γ equals $7/3$ and is attained everywhere on Γ.

Type C (uniform $7/3$ with rank drop). Γ is of Type C if

$$\widetilde{\alpha}_p(X) = \frac{7}{3} \quad \text{for all } p \in \Gamma,$$

but the transverse quadratic part degenerates at finitely many points of Γ (equivalently, the Hessian rank drops along Γ at a finite set). Thus Type C shares the same uniform value $7/3$ as Type B, but differs in that certain special points on Γ exhibit additional quadratic degenerations, which nevertheless do not change the minimal exponent.

In all cases in this paper, every curve component $\Gamma \subset \mathrm{Sing}(X_k)$ is of one of the types A1, A2, B, or C; see Table 5.

7 Adjacency relations among strictly semistable components

In this section we record the *codimension-one wall adjacency* among the closed strata $\{\Phi_k\}_{k=1}^{21} \subset \mathbb{P}(W)^{\mathrm{ss}}$, where $W = \mathrm{Sym}^3\mathbb{C}^7$. Throughout Section 7, we use the term "adjacent" *only* in the sense of *wall adjacency* (i.e. codimension-one incidence in the Hilbert–Mumford weight fan).

Sign convention and why we match $\pm r_j$. A wall is determined by a hyperplane, hence it is insensitive to the orientation of a normal vector. Consequently, when we compare slices of the form

$$I(r)_{\geq 0} \cap I(r_k)_{=0} \subset I(r_k)_{=0},$$

the same geometric wall may be represented either by r or by $-r$. For this reason, in the computational matching against the reduced list $r_1, \ldots, r_{21}$ (Section 2), we always test *both* signs $\pm r_j$. This eliminates the spurious "one-way" phenomena that occur if one only matches the fixed sign representatives r_j.

For background on wall-crossing in GIT and Kirwan's stratification, see [Kir84, DH98, Tha96].

Definition 7.1 (Adjacency via codimension-one walls)**.** *Let Φ_i, Φ_j be closed strata in the strictly semistable locus. We say that Φ_i and Φ_j are* adjacent *if the associated maximal cones $I(r_i)_{\geq 0}$ and $I(r_j)_{\geq 0}$ meet along a codimension-one face (a* wall*) in the Hilbert–Mumford weight fan. Equivalently, there exists a codimension-one wall on which the Hilbert–Mumford numerical functions for the corresponding maximally destabilizing 1-PS's tie, and the wall separates the chambers corresponding to r_i and r_j.*

7.1 Maximal slices and the computational criterion

Fix an index k. Inside the hyperplane $I(r_k)_{=0} \subset I$ we consider subsets of the form

$$I(r)_{\geq 0} \cap I(r_k)_{=0} \qquad (r \in \mathbb{Z}^7_{(0)},\ r \not\parallel r_k).$$

Such a subset corresponds to a codimension-one face of the cone $I(r_k)_{\geq 0}$ precisely when it is *inclusion-maximal* among all subsets of this form. Following the terminology used in the accompanying Sage computations, we call these inclusion-maximal subsets the *maximal slices* in $I(r_k)_{=0}$.

Directed slice-matching relation. For bookkeeping, we introduce a directed relation on $\{1, \ldots, 21\}$: we write $k \rightsquigarrow j$ if *some* maximal slice in $I(r_k)_{=0}$ equals

$$I(\pm r_j)_{\geq 0} \cap I(r_k)_{=0}$$

for at least one choice of sign. (Here $\pm r_j$ indicates that we match against both orientations of r_j, as explained above.)

By Definition 7.1, wall adjacency is symmetric. In our combinatorial model this means:

$$\Phi_i \text{ and } \Phi_j \text{ are adjacent} \quad \Longleftrightarrow \quad (i \rightsquigarrow j) \text{ and } (j \rightsquigarrow i).$$

In other words, the adjacency graph is obtained from the directed slice-matching digraph by retaining only the mutual pairs.

Implementation. We compute maximal slices by the same linear-algebraic method as in Section 2, adapted to the codimension-one setting: we enumerate 4-tuples of points in $I(r_k)_{=0}$, extract candidate normals from the right kernel (modulo r_k), build the corresponding slices, and keep only inclusion-maximal ones. We then match each maximal slice against the list $\{\pm r_1, \ldots, \pm r_{21}\}$. (See the accompanying Sage scripts and logs for the full output.)

7.2 Codimension-one adjacency graph

The main output of the codimension-one computation is the following.

Theorem 7.2 (Codimension-one wall adjacency)**.** *Among the closed strata Φ_k in the strictly semistable locus, the wall-adjacency graph (codimension-one adjacency in the sense of Definition 7.1) has* 21 *vertices and* 56 *edges. Equivalently, for each k the set*

$$N(k) \;:=\; \{\, j \neq k \mid \Phi_k \textit{ is adjacent to } \Phi_j \,\}$$

is given by Table 6.

Remark 7.3 (One-sided slice matches and higher codimension)**.** *In the directed slice-matching digraph one also encounters* one-sided *arrows $k \rightsquigarrow j$ without the reverse arrow $j \rightsquigarrow k$. These do* not *represent codimension-one wall adjacency in the sense of Definition 7.1; rather, they indicate that a facet of $I(r_k)_{\geq 0}$ is detected by $\pm r_j$ when restricted to $I(r_k)_{=0}$, while from the j-side the same incidence is not codimension-one. Empirically, such one-sided phenomena are accounted for by higher-codimension incidence among faces (codimension ≥ 2), as can be explored by extending the same slice computation to intersections of multiple hyperplanes $I(r_{k_1})_{=0} \cap \cdots \cap I(r_{k_c})_{=0}$.*

Table 6: Codimension-one wall adjacency: for each k, the neighbor set $N(k)$ of indices j such that Φ_k and Φ_j are adjacent (Definition 7.1).

k	$N(k)$	k	$N(k)$
1	$3, 5, 7, 10$	12	$3, 4, 5, 6, 7, 15, 17$
2	$3, 4, 5, 9$	13	$4, 8, 18, 19$
3	$1, 2, 4, 5, 9, 10, 12$	14	$9, 10, 11, 18$
4	$2, 3, 5, 6, 12, 13, 18$	15	$6, 8, 12, 17, 18, 20$
5	$1, 2, 3, 4, 6, 7, 12$	16	$8, 18, 19$
6	$4, 5, 12, 15$	17	$7, 12, 15, 18, 19, 20, 21$
7	$1, 5, 12, 17$	18	$4, 9, 10, 13, 14, 15, 16, 17, 20$
8	$10, 11, 13, 15, 16, 19$	19	$8, 9, 13, 16, 17, 21$
9	$2, 3, 10, 14, 18, 19$	20	$15, 17, 18, 21$
10	$1, 3, 8, 9, 11, 14, 18$	21	$17, 19, 20$
11	$8, 10, 14$		

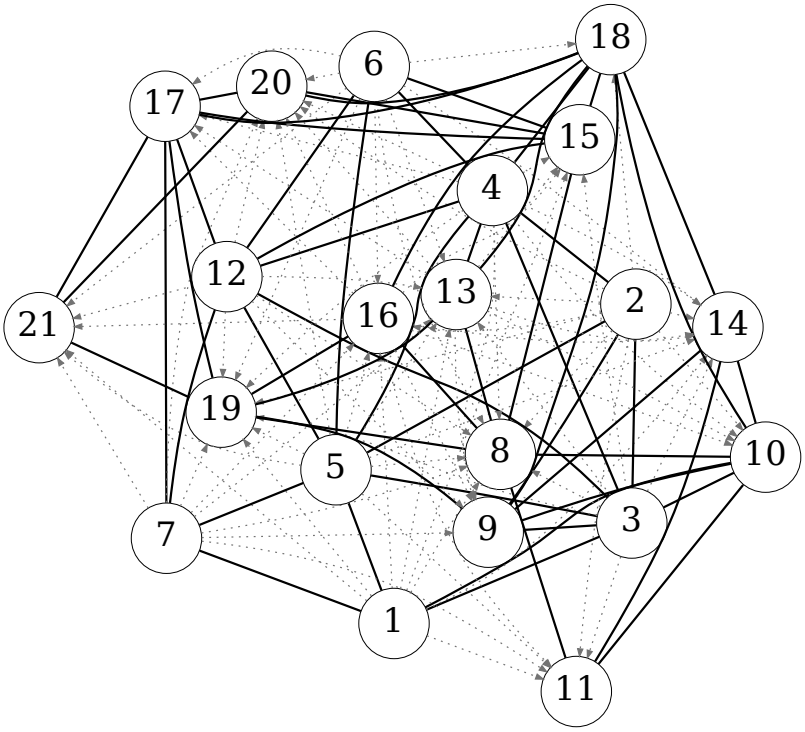

Figure 2: Codimension-one slice-matching graph for the strata $\Phi_1, \ldots, \Phi_{21}$. Vertices are indexed by k and represent Φ_k. A dotted arrow $i \to j$ indicates a *one-sided* slice match: there exists an inclusion-maximal slice $S \subset I(r_i)_{=0}$ of the form $S = I(\pm r_j)_{\geq 0} \cap I(r_i)_{=0}$, while no such match occurs in the reverse direction. A solid (undirected) edge between i and j indicates that slice matches occur in both directions ($i \to j$ and $j \to i$); equivalently, Φ_i and Φ_j are adjacent in the sense of Definition 7.1.

8 Non-inclusions and Filters

In this section, we prove that there are no inclusion relations among the 21 boundary families of strictly semistable cubic fivefolds constructed in the previous sections. Concretely, for any distinct indices $k, \ell \in \{1, \dots, 21\}$ we show

$$\Phi_k \not\subseteq \Phi_\ell.$$

Equivalently, we must rule out all $21 \cdot 20 = 420$ ordered pairs (k, ℓ) with $k \neq \ell$.

Rather than checking these pairs directly, we use a *sieve* (or "filter") method: we impose a sequence of *necessary* conditions for an inclusion $\Phi_k \subseteq \Phi_\ell$. Each condition is monotone under specialization (typically by upper semicontinuity), so any pair violating the condition is discarded. Applying these filters successively shrinks the candidate list until it becomes empty.

The filters are applied in the following order.

(1) **Filter 1: Dimension** (§8.1). If $\Phi_k \subseteq \Phi_\ell$ and $k \neq \ell$, then necessarily

$$\dim \Phi_k \ < \ \dim \Phi_\ell.$$

(2) **Filter 2: Apolar Betti numbers** (§8.2). Within a fixed Hilbert-function stratum, the graded Betti numbers of the apolar algebra can only increase under specialization; hence an inclusion forces the entrywise Betti-table inequalities

$$\beta_{i,j}\big(A_{f_k}\big) \ \geq \ \beta_{i,j}\big(A_{f_\ell}\big) \qquad \text{for all } (i,j).$$

(3) **Filter 3: Singular-locus Hilbert functions** (§8.3). For a cubic f, the Hilbert function of the singular scheme $\mathrm{Sing}(X_f)$ is upper semicontinuous in families, so an inclusion forces the degreewise inequalities

$$\mathrm{HF}_{\mathrm{Sing}(X_{f_k})}(d) \ \geq \ \mathrm{HF}_{\mathrm{Sing}(X_{f_\ell})}(d) \qquad \text{for all } d \geq 0.$$

(4) **Filter 4: Singular-locus degrees (leading term)** (§8.4). The scheme-theoretic degree of the one-dimensional part of the singular locus is monotonic under specialization. An inclusion therefore forces the inequality of the leading coefficients of the Hilbert polynomials:

$$\deg_1 \mathrm{Sing}(X_{f_k}) \ \geq \ \deg_1 \mathrm{Sing}(X_{f_\ell}).$$

(5) **Filter 5: Minimal exponents and limits of 1-cycles** (§8.5). Under a flat specialization $X_\ell \rightsquigarrow X_k$, the fundamental 1-cycle of the positive-dimensional singular locus specializes to an effective 1-cycle supported on $\mathrm{Sing}(X_k)$. By the lower semicontinuity of the local minimal exponent $\widetilde{\alpha}$, this limit cycle must be supported on components having a generic minimal exponent no greater than that of the original curve.

(6) **Filter 6: Generic stabilizer tori** (§8.6). If $\Phi_k \subset \Phi_\ell$, then the connected stabilizer of a general point of Φ_ℓ must be conjugate to a subgroup of the connected stabilizer of a general point of Φ_k in $G = \mathrm{SL}(7)$. For the four pairs surviving Filters 1–5, these generic stabilizers are one-dimensional tori coming from the distinguished 1-PS's λ_k, and the corresponding weight multisets are incompatible.

8.1 Filter 1: Dimension

We begin with the most basic necessary condition coming from dimension monotonicity.

Lemma 8.1 (Dimension monotonicity)**.** *Let $Y \subset Z$ be irreducible closed subsets of an algebraic variety. If the inclusion is proper ($Y \subsetneq Z$), then $\dim Y < \dim Z$. Equivalently, if $\dim Y \geq \dim Z$, then $Y \not\subsetneq Z$.*

Proof. If $Y \subsetneq Z$ are irreducible, then any maximal chain of irreducible closed subsets in Y extends to a strictly longer chain in Z, so the Krull dimension strictly increases. (See any standard reference on dimension theory in algebraic geometry.) □

In Section 4 (Table 2) we computed the dimensions of the 21 families Φ_k. Since we are only interested in *proper* inclusions $\Phi_k \subsetneq \Phi_\ell$ with $k \neq \ell$, Lemma 8.1 implies the strict inequality

$$\Phi_k \subsetneq \Phi_\ell \quad \Longrightarrow \quad \dim \Phi_k < \dim \Phi_\ell. \tag{28}$$

Thus, any ordered pair (k, ℓ) with $\dim \Phi_k \geq \dim \Phi_\ell$ is discarded immediately.

For bookkeeping, set

$$\mathcal{D}_d \;:=\; \{\, k \in \{1, \ldots, 21\} \mid \dim \Phi_k = d \,\}.$$

From Table 2 we have the partition

$$\begin{aligned}
\mathcal{D}_6 &= \{16, 18\},\\
\mathcal{D}_4 &= \{11, 21\},\\
\mathcal{D}_2 &= \{4, 5, 8, 9, 10, 13, 15, 17, 19\},\\
\mathcal{D}_1 &= \{14, 20\},\\
\mathcal{D}_0 &= \{1, 2, 3, 6, 7, 12\}.
\end{aligned}$$

There are $21 \cdot 20 = 420$ ordered pairs (k, ℓ) with $k \neq \ell$. After imposing (28), the *only* dimension jumps that can survive are

$$0 \to 1, 2, 4, 6; \qquad 1 \to 2, 4, 6; \qquad 2 \to 4, 6; \qquad 4 \to 6.$$

Hence the number of pairs surviving Filter 1 is

$$\begin{aligned}
|\mathcal{D}_0|\,(|\mathcal{D}_1| + |\mathcal{D}_2| + |\mathcal{D}_4| + |\mathcal{D}_6|) \;+\; |\mathcal{D}_1|\,(|\mathcal{D}_2| + |\mathcal{D}_4| + |\mathcal{D}_6|)& \\
+\; |\mathcal{D}_2|\,(|\mathcal{D}_4| + |\mathcal{D}_6|) \;+\; |\mathcal{D}_4|\,|\mathcal{D}_6|& \\
= 6 \cdot 15 \;+\; 2 \cdot 13 \;+\; 9 \cdot 4 \;+\; 2 \cdot 2& \\
= 156.&
\end{aligned}$$

Therefore Filter 1 eliminates $420 - 156 = 264$ pairs, and the remaining 156 candidates are passed to Filter 2.

8.2 Filter 2: Apolar Betti numbers

Let

$$S := \mathbb{C}[x_0, \dots, x_6], \qquad R := \mathbb{C}[\partial_0, \dots, \partial_6],$$

where R acts on S by differentiation $\partial_i = \frac{\partial}{\partial x_i}$. For a cubic form $f \in S_3$, the apolar ideal is

$$\mathrm{Ann}(f) := \{D \in R \mid D \cdot f = 0\} \subset R,$$

and the corresponding apolar algebra is the graded Artinian Gorenstein algebra

$$A_f := R/\mathrm{Ann}(f).$$

Write $\beta_{i,j}(A_f)$ for the graded Betti numbers of the minimal graded free resolution of A_f over R.

Lemma 8.2 (Upper semicontinuity of graded Betti numbers)**.** *Let $\mathcal{A}$ be a flat family of finitely generated graded R-modules over an integral base B. Then for every pair (i, j) the function*

$$b \longmapsto \beta_{i,j}(\mathcal{A}_b) \qquad (b \in B)$$

is upper semicontinuous. In particular, if b_0 is a specialization of the generic point b_η, then $\beta_{i,j}(\mathcal{A}_{b_0}) \geq \beta_{i,j}(\mathcal{A}_{b_\eta})$ for all (i, j).

Proof. For each (i, j) one has $\beta_{i,j}(M) = \dim_{\mathbb{C}} \mathrm{Tor}_i^R(M, \mathbb{C})_j$. Moreover $\mathrm{Tor}_i^R(\mathcal{A}, \mathbb{C})_j$ is the fiber of a coherent sheaf on B obtained from a graded free resolution of $\mathcal{A}$; fiber dimensions of coherent sheaves are upper semicontinuous, and flatness ensures compatibility with specialization. □

Filter 8.3 (Apolar Betti numbers)**.** *Fix $k \neq \ell$. Suppose $\Phi_k \subseteq \Phi_\ell$. Choose generic points $f_k \in \Phi_k$ and $f_\ell \in \Phi_\ell$ such that A_{f_k} and A_{f_ℓ} lie in the same (Artinian) Hilbert-function stratum. Then*

$$\beta_{i,j}(A_{f_k}) \geq \beta_{i,j}(A_{f_\ell}) \qquad \text{for all } (i, j).$$

Consequently, if there exists (i, j) with $\beta_{i,j}(A_{f_k}) < \beta_{i,j}(A_{f_\ell})$, then $\Phi_k \not\subseteq \Phi_\ell$.

Computed generic apolar Betti tables. For each family Φ_k we computed the graded Betti table of A_f for a numerically generic element $f \in \Phi_k$ (by fixing the weight–0 normal form and adding random positive-weight terms; in each case we verified that $\mathrm{HF}(A_f) = (1, 7, 7, 1)$). The resulting Betti tables fall into the following types.

For $m \in \{0, 4, 6, 8, 10, 12\}$ set

$B(m) :=$

	0	1	2	3	4	5	6	7
total	1	21	64	$70+m$	$70+m$	64	21	1
0	1	.	.	.	.	.	.	.
1	.	21	64	70	m	.	.	.
2	.	.	.	m	70	64	21	.
3	.	.	.	.	.	.	.	1

.

Then we obtain:

$$B(0) \text{ occurs for } k \in \{4, 13, 18\},$$
$$B(4) \text{ occurs for } k \in \{2, 5, 6, 7, 12, 15, 16, 17, 19\},$$
$$B(6) \text{ occurs for } k \in \{1, 3\},$$
$$B(8) \text{ occurs for } k \in \{9, 10\},$$
$$B(10) \text{ occurs for } k = 8,$$
$$B(12) \text{ occurs for } k \in \{14, 20\}.$$

The remaining two cases are:

$B_{21} :=$

	0	1	2	3	4	5	6	7
total	1	21	65	87	87	65	21	1
0	1	.	.	.	.	.	.	.
1	.	21	64	71	16	1	.	.
2	.	.	1	16	71	64	21	.
3	.	.	.	.	.	.	.	1

, $B_{11} :=$

	0	1	2	3	4	5	6	7
total	1	21	67	99	99	67	21	1
0	1	.	.	.	.	.	.	.
1	.	21	64	73	26	3	.	.
2	.	.	3	26	73	64	21	.
3	.	.	.	.	.	.	.	1

.

In particular, these Betti tables are totally ordered entrywise:

$$B(0) \le B(4) \le B(6) \le B(8) \le B(10) \le B(12) \le B_{21} \le B_{11}.$$

Therefore, for any pair (k, ℓ) we can discard $\Phi_k \subseteq \Phi_\ell$ as soon as the Betti type of k is strictly smaller than the Betti type of ℓ.

Proposition 8.4 (Effect of Filter 2)**.** *Applying Filter 8.3 to the* 156 *ordered pairs surviving Filter 1 eliminates* 66 *further pairs, leaving* 90 *candidates for the subsequent filters. More precisely, the following* 66 *pairs are discarded by the apolar Betti criterion:*

$$k \in \{1, 2, 3, 7, 12\}, \quad \ell \in \{8, 9, 10, 11, 14, 20, 21\};$$
$$k = 6, \quad \ell \in \{8, 9, 10, 11, 14, 20, 21\};$$
$$k \in \{4, 13\}, \quad \ell \in \{11, 16, 21\};$$
$$k \in \{5, 8, 9, 10, 14, 15, 17, 19, 20\}, \quad \ell \in \{11, 21\}.$$

Proof. By Lemma 8.2, in a specialization within a fixed Hilbert-function stratum the graded Betti numbers can only increase. Since $\mathrm{HF}(A_f) = (1, 7, 7, 1)$ for all computed generic representatives, the comparison is valid across all families. Using the explicit Betti types listed above (which form a single chain), any candidate pair (k, ℓ) with Betti type$(k) <$ Betti type(ℓ) is excluded. Intersecting this condition with the 156 pairs from Filter 1 yields exactly the list displayed, consisting of 66 pairs. □

8.3 Filter 3: Hilbert functions of singular loci

Keep the notation $S = \mathbb{C}[x_0, \ldots, x_6]$. For a cubic form $f \in S_3$, set $X_f := V(f) \subset \mathbb{P}^6$ and recall that the (projective) singular scheme is cut out by the saturated Jacobian ideal

$$J(f) := \Big(\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_6}\Big) : (x_0, \ldots, x_6)^\infty \subset S, \qquad \mathrm{Sing}(X_f) = V\big(J(f)\big) \subset \mathbb{P}^6.$$

For $d \ge 0$ define the singular-locus Hilbert function in degree d by

$$\mathrm{HF}_{\mathrm{Sing}(X_f)}(d) := \dim_{\mathbb{C}}(S/J(f))_d.$$

Lemma 8.5 (Upper semicontinuity of Hilbert functions in each degree)**.** *Let B be an integral variety and let $Z \subset \mathbb{P}^6 \times B$ be a family of closed subschemes with ideal sheaf $\mathcal{I}_Z$. Then for every $d \geq 0$ the function*

$$b \longmapsto \dim_{\mathbb{C}}(S/I_b)_d \qquad (b \in B),$$

where $I_b \subset S$ is the saturated homogeneous ideal of the fiber $Z_b \subset \mathbb{P}^6$, is upper semicontinuous. In particular, if b_0 is a specialization of the generic point b_η, then $\mathrm{HF}_{Z_{b_0}}(d) \geq \mathrm{HF}_{Z_{b_\eta}}(d)$ for all $d \geq 0$.

Proof. Fix $d \geq 0$. The vector space $(S/I_b)_d$ is the fiber in degree d of a coherent quotient of the trivial bundle $S_d \otimes \mathcal{O}_B$, hence its fiber dimension is upper semicontinuous. The stated inequality follows for specializations. □

Filter 8.6 (Singular-locus Hilbert functions)**.** *Fix $k \neq \ell$. If $\Phi_k \subseteq \Phi_\ell$, then for generic choices $f_k \in \Phi_k$ and $f_\ell \in \Phi_\ell$ one has*

$$\mathrm{HF}_{\mathrm{Sing}(X_{f_k})}(d) \geq \mathrm{HF}_{\mathrm{Sing}(X_{f_\ell})}(d) \qquad \textit{for all } d \geq 0.$$

Consequently, if there exists d with $\mathrm{HF}_{\mathrm{Sing}(X_{f_k})}(d) < \mathrm{HF}_{\mathrm{Sing}(X_{f_\ell})}(d)$, then $\Phi_k \not\subseteq \Phi_\ell$.

Computed (minimal) singular-locus Hilbert functions. For each $k = 1, \ldots, 21$ we generated 5 random perturbations of the weight–0 normal form ϕ_k^{nf} (as in Filter 2) and computed $\mathrm{HF}_{\mathrm{Sing}(X_f)}(d)$ up to degree 40. For each d we record the componentwise minimum

$$h_k(d) := \min\{\mathrm{HF}_{\mathrm{Sing}(X_f)}(d) : f \text{ among the samples in } \Phi_k\}.$$

In the comparison below we apply Filter 8.6 using these values $h_k(d)$.

Proposition 8.7 (Effect of Filter 3)**.** *Applying Filter 8.6 to the* 90 *ordered pairs surviving Filter 2 eliminates* 67 *further pairs, leaving* 23 *candidates for the subsequent filters. More precisely, the following* 67 *pairs are discarded by the*

singular-locus Hilbert-function criterion:

$$
\begin{aligned}
&k = 1,\ \ell \in \{4, 5, 16\};\\
&k = 2,\ \ell \in \{4, 5, 13, 16, 19\};\\
&k = 3,\ \ell \in \{4, 5, 13, 16, 19\};\\
&k = 4,\ \ell = 18;\\
&k = 5,\ \ell = 18;\\
&k = 6,\ \ell \in \{4, 13, 18\};\\
&k = 7,\ \ell \in \{4, 5, 13, 15, 16, 17, 18\};\\
&k = 8,\ \ell \in \{16, 18\};\\
&k = 9,\ \ell = 18;\\
&k = 10,\ \ell \in \{16, 18\};\\
&k = 11,\ \ell \in \{16, 18\};\\
&k = 12,\ \ell \in \{4, 5, 13, 15, 16, 18, 19\};\\
&k = 14,\ \ell \in \{4, 5, 8, 9, 10, 13, 15, 16, 17, 18, 19\};\\
&k = 15,\ \ell = 16;\\
&k = 17,\ \ell \in \{16, 18\};\\
&k = 19,\ \ell \in \{16, 18\};\\
&k = 20,\ \ell \in \{4, 5, 8, 9, 10, 13, 15, 16, 17, 19\};\\
&k = 21,\ \ell \in \{16, 18\}.
\end{aligned}
$$

Hence the only ordered pairs (k, l) not ruled out by Filters 1–3 are:

(1, 13), (1, 15), (1, 17), (1, 18), (1, 19),
(2, 15), (2, 17), (2, 18), (3, 15), (3, 17), (3, 18),
(5, 16), (6, 5), (6, 15), (6, 16), (6, 17), (6, 19),
(7, 19), (9, 16), (12, 17), (13, 18), (15, 18), (20, 18).

8.4 Filter 4: Singular-locus degrees (leading term)

In this filter we extract a coarse but robust numerical invariant from the singular-locus Hilbert function, namely the *scheme-theoretic degree of the one-dimensional part*. This ignores any isolated (possibly nonreduced) points, since such components contribute only to the constant term of the Hilbert polynomial.

(A) Scheme-theoretic degree via Hilbert polynomial. Let $Z \subset \mathbb{P}^6$ be a closed subscheme with $\dim Z \leq 1$, and write

$$HF_Z(d) := \dim_{\mathbb{C}}(S/I_Z)_d$$

for its Hilbert function. For $d \gg 0$, $HF_Z(d)$ agrees with a linear Hilbert polynomial

$$P_Z(d) \ = \ \deg_1(Z)\, d + \chi(\mathcal{O}_Z),$$

where $\deg_1(Z) \in \mathbb{Z}_{\geq 0}$ is the coefficient of d. Equivalently,

$$\deg_1(Z) \ = \ \lim_{d \to \infty} \big(HF_Z(d) - HF_Z(d-1)\big).$$

By construction, $\deg_1(Z)$ is *scheme-theoretic*: it counts multiplicities along 1-dimensional components, and it is insensitive to embedded/isolated 0-dimensional structure.

(B) Practical extraction from the computed $h_k(d)$. For a cubic f we apply this to $Z = \mathrm{Sing}(X_f)$ and write

$$\deg_1 \mathrm{Sing}(X_f) \;:=\; \lim_{d\to\infty} \Big(HF_{\mathrm{Sing}(X_f)}(d) - HF_{\mathrm{Sing}(X_f)}(d-1) \Big).$$

In our computations of §8.3 we recorded the componentwise minima $h_k(d)$ among random perturbations in Φ_k; the corresponding degree can be read off from the stabilized first difference $\Delta h_k(d) = h_k(d) - h_k(d-1)$ for $d \gg 0$.

Lemma 8.8 (Leading-coefficient monotonicity)**.** *Let $Z, Z' \subset \mathbb{P}^6$ be closed subschemes with $\dim Z, \dim Z' \le 1$. If $HF_Z(d) \ge HF_{Z'}(d)$ for all $d \ge 0$, then $\deg_1(Z) \ge \deg_1(Z')$.*

Proof. For $d \gg 0$ we have $HF_Z(d) = P_Z(d)$ and $HF_{Z'}(d) = P_{Z'}(d)$, with $P_Z(d) = \deg_1(Z)\, d + \chi(\mathcal{O}_Z)$ and $P_{Z'}(d) = \deg_1(Z')\, d + \chi(\mathcal{O}_{Z'})$. If $\deg_1(Z) < \deg_1(Z')$, then $P_Z(d) < P_{Z'}(d)$ for all $d \gg 0$, contradicting the assumed inequality $HF_Z(d) \ge HF_{Z'}(d)$ in large degree. □

Filter 8.9 (Singular-locus degrees)**.** *Fix $k \ne \ell$. If $\Phi_k \subseteq \Phi_\ell$, then for generic choices $f_k \in \Phi_k$ and $f_\ell \in \Phi_\ell$ one has*

$$\deg_1 \mathrm{Sing}(X_{f_k}) \;\ge\; \deg_1 \mathrm{Sing}(X_{f_\ell}).$$

Consequently, if $\deg_1 \mathrm{Sing}(X_{f_k}) < \deg_1 \mathrm{Sing}(X_{f_\ell})$, then $\Phi_k \not\subseteq \Phi_\ell$.

Corollary 8.10 (Six immediate non-inclusions)**.** *The following six ordered pairs are ruled out by Filter 8.9:*

$$(1,13),\ (1,15),\ (1,17),\ (2,15),\ (3,15),\ (6,15).$$

Proof. By Table 3, the degrees of the top-dimensional singular-locus components are

$$\deg_1 \mathrm{Sing}(X_1) = 2,\ \deg_1 \mathrm{Sing}(X_{13}) = 4,$$

$$\deg_1 \mathrm{Sing}(X_{15}) = 9,\ \deg_1 \mathrm{Sing}(X_{17}) = 3,$$

$$\deg_1 \mathrm{Sing}(X_2) = \deg_1 \mathrm{Sing}(X_3) = \deg_1 \mathrm{Sing}(X_6) = 5.$$

Each of the listed pairs has $\deg_1$ strictly increasing from k to ℓ, so $\Phi_k \subseteq \Phi_\ell$ is impossible by Filter 8.9. □

Proposition 8.11 (Updated candidate list after Filter 4)**.** *Applying Filter 8.9 to the 23 pairs remaining after Proposition 8.7 eliminates 6 further pairs. Hence the only ordered pairs not ruled out by Filters 1–4 are:*

$$(1,18),\ (1,19),\ (2,17),\ (2,18),\ (3,17),\ (3,18),\ (5,16),\ (6,5),\ (6,16),\ (6,17),$$

$$(6,19),\ (7,19),\ (9,16),\ (12,17),\ (13,18),\ (15,18),\ (20,18).$$

8.5 Filter 5: Minimal Exponents and Limits of 1-Cycles

In Filters 2–4, we utilized numerically generic points $f \in \Phi$ to extract stable algebraic invariants (such as Betti numbers), which tend to jump and become noisy at the highly symmetric 1-PS limits. However, to efficiently rule out the majority of the remaining candidate pairs, we directly compare the geometric invariants of the polystable normal forms ϕ^{nf}. This is theoretically justified by the following lemma.

Lemma 8.12 (Semicontinuity on singular 1-cycles)**.** *Let $X_\ell \rightsquigarrow X_k$ be a flat specialization of cubic fivefolds in $\mathbb{P}^6$, realized by a flat family $\mathcal{X} \to \Delta$ over a smooth pointed curve $(\Delta, 0)$ with generic fiber X_ℓ and special fiber X_k. Let $\Gamma_\ell \subset \mathrm{Sing}(X_\ell)$ be an irreducible curve component and define*

$$\widetilde{\alpha}_{\mathrm{gen}}(\Gamma_\ell) \;:=\; \widetilde{\alpha}_p(X_\ell) \qquad \text{for a general closed point } p \in \Gamma_\ell.$$

Let $\mathrm{Sing}(\mathcal{X}/\Delta) \subset \mathcal{X}$ be the relative singular scheme, and let $\overline{\Gamma} \subset \mathrm{Sing}(\mathcal{X}/\Delta)$ be the scheme-theoretic closure of Γ_ℓ. Write $(\overline{\Gamma} \cap X_k)^{(1)}$ for the union of the 1-dimensional components of $\overline{\Gamma} \cap X_k$ with their scheme structures, and set

$$Z_k \;:=\; \big[(\overline{\Gamma} \cap X_k)^{(1)}\big]$$

for its fundamental 1-cycle.

Then:

(1) Z_k is an effective 1-cycle supported on $\mathrm{Sing}(X_k)$, and $\deg Z_k = \deg \Gamma_\ell$.

(2) There exists an irreducible curve component $\Gamma_k \subset \mathrm{Supp}(Z_k)$ such that

$$\widetilde{\alpha}_{\mathrm{gen}}(\Gamma_k) \;\le\; \widetilde{\alpha}_{\mathrm{gen}}(\Gamma_\ell).$$

(3) If Γ_k is a curve component of $\mathrm{Sing}(X_k)$ with multiplicity $m_k(\Gamma_k)$ in the fundamental 1-cycle $[\mathrm{Sing}(X_k)]^{(1)}$, then the coefficient of Γ_k in Z_k is at most $m_k(\Gamma_k)$.

Proof. Since $\overline{\Gamma} \subset \mathrm{Sing}(\mathcal{X}/\Delta)$, its special fiber $\overline{\Gamma} \cap X_k$ is a closed subscheme of $\mathrm{Sing}(X_k)$. Hence Z_k is effective and supported on $\mathrm{Sing}(X_k)$.

The cycle Z_k is the specialization of the cycle $[\Gamma_\ell]$ inside the proper family $\overline{\Gamma} \to \Delta$. In particular, its degree (equivalently, its class in $A_1(\mathbb{P}^6) \cong \mathbb{Z}$) is preserved under specialization, so $\deg Z_k = \deg \Gamma_\ell$.

Choose an irreducible component Γ_k of $\mathrm{Supp}(Z_k)$ that is dominated by $\overline{\Gamma}$. Let $p_0 \in \Gamma_k$ be a general closed point. After possibly shrinking Δ, we may choose a curve of points $p(t) \in \overline{\Gamma} \cap X_t$ specializing to p_0. By lower semicontinuity of minimal exponents in flat families (see, e.g., [Par25, Theorem 2.3]),

$$\widetilde{\alpha}_{p_0}(X_k) \;\le\; \liminf_{t \to 0} \widetilde{\alpha}_{p(t)}(X_t).$$

For $t \neq 0$ general, $p(t)$ is a general point of Γ_ℓ, hence the right-hand side equals $\widetilde{\alpha}_{\mathrm{gen}}(\Gamma_\ell)$. Since p_0 is general on Γ_k, the left-hand side equals $\widetilde{\alpha}_{\mathrm{gen}}(\Gamma_k)$, proving (2).

Finally, (3) follows from the surjection of local rings at the generic point η_{Γ_k}:

$$\mathcal{O}_{\mathrm{Sing}(X_k),\eta_{\Gamma_k}} \twoheadrightarrow \mathcal{O}_{(\overline{\Gamma}\cap X_k)^{(1)},\eta_{\Gamma_k}},$$

so lengths (hence cycle multiplicities) can only decrease. □

Filter 8.13 (Minimal exponent and cycle degree)**.** *Fix $k \neq \ell$ and suppose $\Phi_k \subseteq \Phi_\ell$. Let $X_\ell = V(\phi_\ell^{\mathrm{nf}})$ and $X_k = V(\phi_k^{\mathrm{nf}})$ be the closed-orbit representatives.*

Let

$$Z_\ell^{BC} := \sum_{\substack{\Gamma \subset \mathrm{Sing}(X_\ell) \\ \Gamma \text{ of Type B or C}}} m_\ell(\Gamma)\,[\Gamma]$$

be the effective 1-cycle supported on the Type B/C curve components of $\mathrm{Sing}(X_\ell)$, where $m_\ell(\Gamma)$ denotes the multiplicity of Γ in the fundamental 1-cycle $[\mathrm{Sing}(X_\ell)]^{(1)}$. Set $D_\ell := \deg Z_\ell^{BC}$.

Then any flat degeneration $X_\ell \rightsquigarrow X_k$ forces the existence of an effective 1-cycle

$$Z_k^{BC} = \sum_{\substack{\Gamma \subset \mathrm{Sing}(X_k) \\ \Gamma \text{ of Type B or C}}} a(\Gamma)\,[\Gamma]$$

supported on the Type B/C components of $\mathrm{Sing}(X_k)$ such that

$$\deg Z_k^{BC} = D_\ell, \qquad 0 \le a(\Gamma) \le m_k(\Gamma) \text{ for all Type B/C components } \Gamma \subset \mathrm{Sing}(X_k),$$

where $m_k(\Gamma)$ is the multiplicity of Γ in $[\mathrm{Sing}(X_k)]^{(1)}$. In particular, if no such choice of coefficients $a(\Gamma)$ exists (e.g. if $D_\ell > 0$ but $\mathrm{Sing}(X_k)$ has no Type B/C component, or if D_ℓ cannot be realized as a degree combination within the available multiplicities), then $\Phi_k \not\subseteq \Phi_\ell$.

Proposition 8.14 (Effect of Filter 5)**.** *Applying Filter 8.13 to the 17 candidate pairs remaining after Filter 4 eliminates 13 pairs. Specifically, the following pairs are discarded:*

$$(1,18),\ (1,19),\ (2,17),\ (2,18),\ (3,17),\ (3,18),\ (6,17),$$

$$(6,19),\ (7,19),\ (12,17),\ (13,18),\ (15,18),\ (20,18).$$

Proof. We analyze the limit of the 1-dimensional components based on the data in Table 3 and Table 5.

• **Pairs with $\ell = 18$:** $(1,18), (2,18), (3,18), (13,18), (15,18), (20,18)$.
X_{18} has two disjoint lines L_1, L_2 of Type B ($\widetilde{\alpha}_{\mathrm{gen}} = 7/3$). Thus it carries a degree 2 cycle with $\widetilde{\alpha}_{\mathrm{gen}} = 7/3$.

- For $k \in \{1,2,3,13\}$, all curve components of X_k are of Type A1 ($\widetilde{\alpha}_{\mathrm{gen}} = 5/2 > 7/3$). They cannot support the limit.
- For $k = 15$, the only component with $\widetilde{\alpha}_{\mathrm{gen}} \le 7/3$ is L (Type C, degree 1). However, L has scheme-theoretic degree 1 in $\mathrm{Sing}(X_{15})$ (as $\deg_1 \mathrm{Sing}(X_{15}) = 9 = 4 + 4 + 1$), which cannot absorb the limit of two lines (degree 2).
- For $k = 20$, the components are C (elliptic quartic, degree 4) and L (line, degree 1). A limit of two lines must be supported on lines, but X_{20} only has a degree 1 line, which is insufficient.

Thus all these pairs are excluded.

• **Pairs with $\ell = 17$:** $(2,17), (3,17), (6,17), (12,17)$.
X_{17} contains a smooth conic C of Type B ($\widetilde{\alpha}_{\mathrm{gen}} = 7/3$, degree 2). For $k \in$

$\{2, 3, 6, 12\}$, all curve components of X_k are of Type A1 ($\widetilde{\alpha}_{\text{gen}} = 5/2 > 7/3$). They cannot support the limit of the conic. Thus these are excluded.

• **Pairs with $\ell = 19$:** $(1, 19), (6, 19), (7, 19)$.
X_{19} contains a smooth conic C of Type C ($\widetilde{\alpha}_{\text{gen}} = 7/3$, degree 2). For $k \in \{1, 6, 7\}$, all curve components of X_k are of Type A1 ($\widetilde{\alpha}_{\text{gen}} = 5/2 > 7/3$). Thus these are excluded. □

Corollary 8.15 (Updated candidate list after Filter 5)**.** *The only ordered pairs not ruled out by Filters 1–5 are the following* 4 *pairs:*

$$(5, 16),\ (6, 5),\ (6, 16),\ (9, 16).$$

8.6 Filter 6: Generic stabilizer tori

We now eliminate the remaining four pairs using only a generic-stabilizer obstruction. By Corollary 8.15, the only ordered pairs not ruled out by Filters 1–5 are

$$(5, 16), \qquad (6, 5), \qquad (6, 16), \qquad (9, 16).$$

Lemma 8.16 (Generic stabilizer type obstruction)**.** *Let G act on a projective variety X, and let $S \subset X$ and $S' \subset X$ be G-stable irreducible subsets. Assume that a general point of S has connected stabilizer a one-dimensional torus $T \simeq \mathbb{G}_m$, and that a general point of S' has connected stabilizer a one-dimensional torus $T' \simeq \mathbb{G}_m$. If $S \subset S'$, then T' is conjugate to T in G. In particular, if both T and T' are diagonal one-parameter subgroups of* SL(7)*, then their multisets of weights coincide.*

Proof. Under specialization, the connected stabilizer can only increase. Therefore, if $S \subset S'$, then the generic connected stabilizer of S' embeds into the generic connected stabilizer of S. Since both are one-dimensional tori, this embedding is an isomorphism. Hence T' and T are conjugate in G. For diagonal one-parameter subgroups of SL(7), conjugacy is equivalent to equality of the weight multiset. □

Proposition 8.17 (Generic connected stabilizers for the remaining four families)**.** *For $k \in \{5, 6, 9, 16\}$, let*

$$H_k := \lambda_k(\mathbb{G}_m) \subset G = \mathrm{SL}(7).$$

Then a general point of Φ_k has connected stabilizer conjugate to H_k.

Proof. It suffices to compute the connected stabilizer on the normal-form slice constructed in Section 4.

For $k = 5$, the diagonal weights of λ_5 are pairwise distinct, so $C_G(H_5) = T$. The action on $P(W^{H_5})$ factors through the effective torus T/H_5, and Section 4 shows that for a point of $P(W^{H_5})$ with all eight coefficients nonzero, the stabilizer in T/H_5 is finite. Hence the generic stabilizer for the effective action is finite, so upstairs the generic connected stabilizer is exactly H_5.

For $k = 6$, again the diagonal weights are pairwise distinct, so $C_G(H_6) = T$. Section 4 states that the residual T-stabilizer of a general representative is finite. Therefore the generic connected stabilizer is exactly H_6.

For $k = 9$, Section 4 gives

$$\dim W^{H_9} = 14, \qquad \dim_{\text{eff}} C_G(H_9) = 11, \qquad \dim \Phi_9 = 2.$$

Thus a general orbit in $P(W^{H_9})$ has dimension

$$\dim P(W^{H_9}) - \dim \Phi_9 = 13 - 2 = 11,$$

which equals the effective group dimension. Hence the generic stabilizer for the effective action is finite, so the generic connected stabilizer upstairs is exactly H_9.

For $k = 16$, Section 4 gives

$$\dim W^{H_{16}} = 18, \qquad \dim_{\text{eff}} C_G(H_{16}) = 11, \qquad \dim \Phi_{16} = 6.$$

Therefore a general orbit in $P(W^{H_{16}})$ has dimension

$$\dim P(W^{H_{16}}) - \dim \Phi_{16} = 17 - 6 = 11,$$

again equal to the effective group dimension. Hence the generic stabilizer for the effective action is finite, so the generic connected stabilizer upstairs is exactly H_{16}. □

Proposition 8.18 (Elimination of the remaining four pairs)**.** *None of the inclusions*

$$\Phi_5 \subset \Phi_{16}, \qquad \Phi_6 \subset \Phi_5, \qquad \Phi_6 \subset \Phi_{16}, \qquad \Phi_9 \subset \Phi_{16}$$

holds.

Proof. By Proposition 8.17, the generic connected stabilizers of $\Phi_5, \Phi_6, \Phi_9, \Phi_{16}$ are the diagonal one-parameter subgroups $\lambda_5, \lambda_6, \lambda_9, \lambda_{16}$, respectively. Their weight multisets are

$$\lambda_5 : \ (4, 2, 1, 0, -1, -2, -4),$$
$$\lambda_6 : \ (5, 3, 2, 1, -1, -4, -6),$$
$$\lambda_9 : \ (2, 2, 0, 0, -1, -1, -2),$$
$$\lambda_{16} : \ (2, 1, 0, 0, 0, -1, -2).$$

These four multisets are pairwise distinct.

If any one of the displayed inclusions were true, then Lemma 8.16 would force the generic connected stabilizer of the larger family to be conjugate to that of the smaller family. For diagonal one-parameter subgroups of SL(7) this would imply equality of the corresponding weight multisets, which does not occur. Hence all four inclusions are impossible. □

Corollary 8.19. *After applying Filters 1–6, there are no remaining candidate inclusions $\Phi_k \subset \Phi_\ell$ with $k \neq \ell$. In particular, Theorem 3.5 holds.*

Proof. By Corollary 8.15, the only candidates surviving Filters 1–5 are the four pairs listed above, and they are excluded by Proposition 8.18. □

9 Concluding Remarks and Future Directions

The results of this paper turn the strictly semistable GIT boundary for cubic fivefolds into an explicit and tractable set of test degenerations. Concretely, we obtain: the list of 21 SL(7)-inequivalent boundary families (Theorem 3.6), explicit polystable normal forms and component dimensions (Section 4 and Table 2), a complete description of the singular loci of the closed-orbit representatives (Section 5 and Table 3), the six isolated analytic types that occur on the boundary (Table 4), the sharp minimal-exponent equalities at the critical threshold (Theorems 6.1 and 6.4), and the sparse Kirwan wall-crossing adjacency graph among boundary components (Section 7 and Theorem 7.2). Together with the non-inclusion statement (Theorem 3.5), this provides a concrete "dictionary" for the boundary geometry of the moduli space.

We close by outlining several directions where such explicit boundary models should be useful. A long-term motivation is to leverage these degenerations in Hodge-theoretic and integrable-systems approaches to the intermediate Jacobians of cubic fivefolds (Section 9.3). To make that program viable, however, one needs a substantially refined understanding of (i) the deformation and moduli theory of the extremal isolated singularities (Section 9.1), (ii) the resulting limit mixed Hodge structures for boundary degenerations (Section 9.2), and (iii) the extent to which the present methodology scales to higher-dimensional cubic hypersurfaces (Section 9.4). We also record a conjectural structural principle suggested by our computations (Section 9.5).

9.1 Detailed analysis of extremal cubic fivefold singularities

Isolated singular points occur only for a proper subset of the boundary families (Table 3), and, on general closed-orbit representatives, every isolated boundary germ belongs to one of the six quasi-homogeneous analytic types listed in Table 4. A key outcome of Section 6 is that each of these isolated types realizes the equality case in Park's threshold: Theorem 6.1 shows

$$\widetilde{\alpha}_x(X) = \frac{7}{3} = \frac{n+1}{d} \qquad \text{for } (n,d) = (6,3)$$

at every isolated boundary point. Motivated by this sharp borderline behavior, we singled out these six analytic types as *extremal* (Definition 6.3).

Beyond their role in the GIT classification, these singularities appear to be a natural and rigid collection of "boundary" local models for cubic fivefolds, and they invite a more intrinsic analysis. Several concrete problems seem especially compelling:

- **Local deformation theory and comparison with wall-crossing.** A systematic study of the miniversal deformation spaces (semi-universal unfoldings) of the six extremal types should clarify the local analytic structure of the moduli space near boundary points. Since the singularities are quasi-homogeneous, one expects the Milnor and Tjurina numbers to coincide, and the deformation space can be approached via explicit Tjurina algebras. A natural goal is to understand how (semi)stability conditions cut out chambers inside these bases and how the resulting local pictures reflect Kirwan wall-crossing (Definition 7.1, Theorem 7.2).

- **Connections to K-stability and K-moduli.** With the rapid development of K-moduli for Fano varieties, it is natural to ask how the classical GIT compactification compares to the K-moduli compactification for cubic fivefolds. While minimal exponents are Hodge-theoretic invariants and do not directly encode discrepancies, the uniform inequality $\widetilde{\alpha}_x(X) = \frac{7}{3} > 1$ for the extremal isolated types suggests that these singularities should be relatively mild from several birational and Hodge-theoretic perspectives. It would therefore be very interesting to determine for each explicit extremal local model whether it is klt, and more broadly to test local K-stability by computing normalized volumes and related stability thresholds.

- **Finer Hodge-theoretic and topological invariants.** Since these extremal germs govern the most degenerate isolated limits on the GIT boundary, their refined invariants (Hodge spectrum, monodromy, Milnor lattice, Brieskorn lattice, etc.) should feed directly into computations of limiting Hodge data for global degenerations (cf. Section 9.2).

9.2 Limit mixed Hodge structures and moduli compactifications

For cubic threefolds and fourfolds, a recurring theme is that explicit GIT compactifications interact in a meaningful way with Hodge-theoretic compactifications of period domains (e.g. [CG72, Laz09]). For cubic fivefolds, understanding the degeneration of periods and intermediate Jacobians [Col86] requires an analysis of the associated limit mixed Hodge structures (LMHS).

From this perspective, two kinds of input are indispensable: one needs explicit local models of singular fibers, and one needs control over how boundary strata meet. The present paper provides precisely this data in a concrete form: explicit closed-orbit boundary representatives (Table 2), their singular loci (Section 5, Table 3), and the adjacency relations among boundary components under Kirwan wall-crossing (Section 7, Theorem 7.2). Combined with the local invariants of the extremal isolated types (Section 9.1), this should make it feasible to construct transverse slices to boundary strata and to compute local Picard–Lefschetz transformations and monodromy cones in explicit families. A natural medium-term objective is therefore a systematic computation of LMHS for selected degenerations built from our 21 boundary models, with an eye toward comparisons between the GIT boundary and Hodge-theoretic compactifications.

9.3 Towards the Schottky problem for cubic fivefolds via integrable systems

Let $X \subset \mathbb{P}^6$ be a smooth cubic fivefold. Its intermediate Jacobian $J(X)$ is a principally polarized abelian variety (PPAV) of dimension 21, and a Torelli-type statement is available in this setting (see [Voi86]). Thus the period map may be viewed as embedding the 35-dimensional moduli space of smooth cubic fivefolds into the 231-dimensional PPAV moduli space $\mathcal{A}_{21}$. A basic and widely open problem is to characterize, or even effectively constrain, this 35-dimensional locus inside $\mathcal{A}_{21}$.

In the classical Schottky problem for Jacobians of curves, integrable systems play a decisive role: Shiota's theorem characterizes Jacobians among PPAVs via the KP hierarchy [Shiota86]. This motivates the question of whether the locus of intermediate Jacobians of cubic fivefolds admits an analogous characterization in terms of modular/differential constraints on theta functions. At present, even formulating a plausible candidate hierarchy is nontrivial, and any integrable-systems approach must confront degenerations.

Here the boundary classification becomes relevant. Theta-functional constraints are typically most informative when one understands their behavior under degeneration, where period matrices and theta functions develop controlled asymptotics. Our 21 explicit boundary families (Theorem 3.6 and Table 2), together with the specialization relations encoded by Theorem 7.2, provide a concrete set of test degenerations and a combinatorial skeleton for how boundary strata meet. A natural open-ended program is therefore: use these explicit degenerations to compute LMHS and limiting period data (as in Section 9.2), and then investigate whether the resulting locus in $\mathcal{A}_{21}$ satisfies distinctive modular or differential constraints of an integrable-systems flavor.

9.4 Extremal singularities on cubic sixfolds and higher dimensions

The computational and structural pipeline used in this paper (maximal strictly semistable supports via convex geometry, Luna's centralizer reduction, and polystability checks via the convex-hull criterion or the Casimiro–Florentino criterion) relies on general features of reductive group actions and is therefore, in principle, portable to higher-dimensional cubic hypersurfaces, albeit with rapidly growing combinatorial complexity.

An immediate target is the GIT compactification for cubic sixfolds ($X \subset \mathbb{P}^7$). Park's stability criterion [Par25, Theorem A] identifies the critical minimal exponent threshold as $\frac{n+1}{d}$ for degree d hypersurfaces in $\mathbb{P}^n$; in particular, for cubic sixfolds $(n, d) = (7, 3)$ the critical value is $\frac{8}{3}$. Our results for cubic fivefolds show that the isolated boundary singularities realize the equality case $\widetilde{\alpha}_x(X) = \frac{7}{3}$ (Theorem 6.1), suggesting a broader principle: in higher dimensions, the strictly semistable boundary may again be governed by a distinguished collection of quasi-homogeneous isolated singularities sitting exactly at the critical threshold. Identifying such "extremal" types for cubic sixfolds would be a natural next step toward a higher-dimensional boundary dictionary.

9.5 A hypersurface version of maximal polystability

One striking uniform phenomenon emerged throughout our explicit classification: for every maximal T-strictly semistable support $I(\mathbf{r}_k)_{\geq 0}$ produced by the convex-geometric algorithm, the associated 1-PS degeneration

$$\phi_k \;=\; \lim_{t \to 0} \lambda_{\mathbf{r}_k}(t) \cdot f_k$$

of a general polynomial f_k with $\mathrm{Supp}(f_k) = I(\mathbf{r}_k)_{\geq 0}$ landed on a *polystable* point for the residual action of the corresponding centralizer. In other words, once the support is maximal among strictly semistable ones, the weight-0 initial form produced by the limiting process appears to have closed orbit automatically.

The fact that this automatic polystability holds flawlessly without a single exception across all 21 highly diverse boundary families computed in Section 4—despite the wide variation in their residual centralizers, ranging from tori to complicated non-abelian groups—provides overwhelming empirical evidence.

Heuristically, the convex-geometric maximality of the support dictates a combinatorial rigidity: one cannot add further monomials without making the support unstable. Intuitively, this extremality forces the barycenter to be strictly "trapped" in the relative interior of the weight polytope for the residual action, which exactly translates to polystability.

Motivated by this evidence, we record the following hypersurface-level formulation, tailored to the natural $\mathrm{SL}(n{+}1)$-action on spaces of homogeneous polynomials.

Conjecture 9.1 (Hypersurface Maximal Polystability)**.** *Fix integers $n \geq 1$ and $d \geq 2$, and let*

$$G = \mathrm{SL}(n{+}1), \qquad W = \mathrm{Sym}^d(\mathbb{C}^{n+1}) \cong \mathbb{C}[x_0, \ldots, x_n]_d$$

with the natural linear G-action. Fix the diagonal maximal torus $T \subset G$. Let

$$I = \mathbb{Z}^{n+1}_{(d)} \cap \mathbb{Z}^{n+1}_{\geq 0}, \qquad \eta = \left(\tfrac{d}{n+1}, \ldots, \tfrac{d}{n+1}\right)$$

be the exponent simplex and its barycenter. For $\mathbf{r} \in \mathbb{Z}^{n+1}_{(0)}$ define

$$I(\mathbf{r})_{\geq 0} = \{\mathbf{i} \in I \mid \mathbf{r} \cdot \mathbf{i} \geq 0\}, \qquad I(\mathbf{r})_{=0} = \{\mathbf{i} \in I \mid \mathbf{r} \cdot \mathbf{i} = 0\}.$$

We say that $I(\mathbf{r})_{\geq 0}$ is a maximal T-strictly semistable support *if $\eta \in \mathrm{Conv}\big(I(\mathbf{r})_{\geq 0}\big)$, $\eta \in \partial\mathrm{Conv}\big(I(\mathbf{r})_{\geq 0}\big)$ (relative boundary inside the affine hyperplane* $\mathrm{wt} = d$*), and $I(\mathbf{r})_{\geq 0}$ is maximal for these properties with respect to inclusion among subsets of I of the form $I(\mathbf{r}')_{\geq 0}$. Let $\lambda_{\mathbf{r}} : \mathbb{G}_m \to T$ be the 1-PS given by $\lambda_{\mathbf{r}}(t) = \mathrm{diag}(t^{r_0}, \ldots, t^{r_n})$. For a general form*

$$f = \sum_{\mathbf{i} \in I} a_{\mathbf{i}} x^{\mathbf{i}} \in W \quad \textit{with} \quad \mathrm{Supp}(f) = I(\mathbf{r})_{\geq 0},$$

consider the 1-PS limit

$$f_0 \;:=\; \lim_{t \to 0} \lambda_{\mathbf{r}}(t) \cdot f \;=\; \sum_{\mathbf{i} \in I(\mathbf{r})_{=0}} a_{\mathbf{i}} x^{\mathbf{i}} \;\in\; W^{\lambda_{\mathbf{r}}(\mathbb{G}_m)}.$$

Then f_0 is polystable.

Possible strategy for a proof. One may hope to prove Conjecture 9.1 by combining Luna's centralizer reduction [Lun75] with the Casimiro–Florentino criterion [CF12] (or, in toric cases, the convex-hull criterion, Theorem 4.1). Luna reduction translates closedness of the G-orbit of the 1-PS limit f_0 into polystability for the residual action of the centralizer $C_G(\lambda(\mathbb{G}_m))$ on the fixed locus $W^{\lambda(\mathbb{G}_m)}$. After choosing a maximal torus in the centralizer, the Casimiro–Florentino/convex-hull criteria reduce polystability to the existence of a strictly positive "balancing" linear identity in the real character space.

In the hypersurface case $W = \mathrm{Sym}^d \mathbb{C}^{n+1}$, this can be phrased as the existence of positive coefficients $a_\alpha > 0$ (summing to 1) such that

$$\sum_{\alpha \in \mathrm{Supp}(f_0)} a_\alpha \, \alpha \;=\; \frac{d}{n+1}(1, \ldots, 1).$$

The conjecture asserts that the maximal-support hypothesis forces such a strictly positive solution, equivalently that the relevant weight polytope already contains the barycenter in its interior for the residual action. If true, this would streamline the search for closed orbits in many hypersurface GIT problems: once maximal strictly semistable supports are enumerated, the residual polystability check would become essentially automatic.

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