Boundary of the Moduli Space of Stable Cubic Fivefolds

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Abstract

Using geometric invariant theory (GIT), we compactify the moduli space of stable cubic fivefolds by adjoining strictly semistable hypersurfaces. We show that the strictly semistable locus decomposes into 21 irreducible components and provide a closed-orbit representative for each. Our analysis of the boundary singularities reveals the presence of wild isolated hypersurface singularities, suggesting that dimension five marks a threshold beyond the ADE/unimodal paradigm observed in lower dimensions. We also determine adjacency relations among boundary components, providing explicit instances of wall-crossing in Kirwan's stratification.

Introduction

The geometric invariant–theoretic (GIT) compactification of the moduli space of cubic hypersurfaces is obtained by adjoining the strictly semistable hypersurfaces as boundary strata. In low dimensions—cubic threefolds ($n = 3$) and cubic fourfolds ($n = 4$)—conventional analyses demonstrate that the boundary is essentially governed by simple (ADE) or, at most, unimodal singularities; in particular, wild isolated hypersurface singularities are not observed at the typical boundary points (see [Yok02] for cubic threefolds and [Laz09] for cubic fourfolds). This study demonstrates that the picture changes qualitatively for cubic fivefolds ($n = 5$). We construct the GIT compactification by adjoining strictly semistable fivefolds and demonstrate that its boundary decomposes into 21 irreducible components, each admitting an explicit closed–orbit representative in normal form (Table 2). An analysis of saturated Jacobian ideals determines the singular loci of these representatives and presents two phenomena that are absent in lower dimensions: first, exactly two closed–orbit representatives (Cases $k = 1, 6$) exhibit a wild isolated hypersurface singularity of type $\mathrm{QH}(3)_{19}$ (quasi–homogeneous, corank 3, $\mu = \tau = 19$). Second, among the positive–dimensional possibilities encountered—besides lines, smooth conics, quadric surfaces (including a rank–3 cone), and space quartics $\mathrm{CI}(2, 2)$—are a *quadric threefold* (Case $k = 20$) and *quadric threefold cone* (Case $k = 14$). In this sense, dimension five marks a threshold beyond the ADE/unimodal paradigm (see Theorem C and Table 3 in Section 5).

Theorem A (Decomposition of the strictly semistable locus). The strictly semistable locus in $\mathbb{P}(W)//\mathrm{SL}(7)$ decomposes into 21 irreducible components. Each component admits a closed $\mathrm{SL}(7)$-orbit represented by an explicit normal form; see Table 2 (Section 4).

Theorem B (Closed–orbit representatives and stability criteria). For each component, we produce a closed–orbit representative via a one–parameter specialization followed by Luna's centralizer reduction. When the centralizer is a torus, *polystability is certified by the convex–hull criterion*; otherwise we apply the *Casimiro–Florentino symmetric–1-PS criterion*. In both cases, we obtain explicit normal forms (see Section 4 and Table 2).

Theorem C (Singularities on the boundary). Let $W = \mathrm{Sym}^3\mathbb{C}^7$ and write $\Phi_1, \ldots, \Phi_{21} \subset \mathbb{P}(W)//\mathrm{SL}(7)$ for the strictly semistable boundary components. For each k, let φ_k^{nf} be the closed-orbit representative from Section 4 and set $X_k = V(\varphi_k^{\mathrm{nf}}) \subset \mathbb{P}^6$. Then:

(1) The saturated Jacobian ideal of φ_k^{nf} computes $\mathrm{Sing}(X_k)$ explicitly; their set-theoretic types are listed in the summary table of Section 5 (Table 3). The positive-dimensional possibilities observed include: a line; a smooth conic; a quadric surface (including the rank-3 cone in Case $k = 10$); a quartic complete intersection $\mathrm{CI}(2, 2)$; linear spaces $\mathbb{P}^2$ or $\mathbb{P}^3$; and also a *quadric threefold* and a *quadric threefold cone* (Cases $k = 20$ and $k = 14$, respectively).

(2) Exactly two closed-orbit representatives have an isolated singular point, namely $k = 1$ and $k = 6$. In these cases, the isolated point is quasi-homogeneous of corank 3 with Milnor and Tjurina numbers $\mu = \tau = 19$; Analytically, it is right-equivalent to $X^2Y + Y^4 + XZ^3$ (type $\mathrm{QH}(3)_{19}$).

(3) For a general point of a boundary component, the singular locus is one of: a line, a smooth conic, a non-degenerate space quartic $\mathrm{CI}(2, 2)$, or an isolated point of corank 3.

Theorem D (Adjacency via wall-crossing). We record pairwise adjacencies among strictly semistable components as wall-crossings in Kirwan's stratification. Aside from isolated components, the only nonempty pairwise intersections occur in the eight pairs

$$\{\Phi_1, \Phi_7\}, \{\Phi_2, \Phi_6\}, \{\Phi_3, \Phi_{12}\}, \{\Phi_8, \Phi_{19}\}, \{\Phi_9, \Phi_{15}\}, \{\Phi_{10}, \Phi_{17}\}, \{\Phi_{11}, \Phi_{21}\}, \{\Phi_{14}, \Phi_{20}\}.$$

See Theorem 6.1 and [Kir84, DH98, Tha96].

Ideas and methods

Our starting point is a convex-geometric analysis of Hilbert–Mumford weights on $W = \mathrm{Sym}^3\mathbb{C}^7$ [MFK94, Dol03]. Fixing a maximal torus $T \subset \mathrm{SL}(7)$ and writing

$\mathrm{Supp}(f) \subset I = \mathbb{Z}^7_{\geq 0}(3)$ for the exponent set of f, we enumerate the *maximal* T*-strictly semistable supports* $I(r)_{\geq 0}$ containing the barycenter $\eta = (3/7, \ldots, 3/7)$. An algorithmic search over the faces of $\mathrm{Conv}(I)$ that pass through η yields 23 candidates up to permutation; exactly one is T-unstable and is discarded, leaving 22 families with respect to T. Modulo $\mathrm{SL}(7)$ there is a single residual identification $f_{21} \sim f_{22}$, and a case-by-case inclusion analysis then demonstrates that no further identifications occur, producing the required 21 $\mathrm{SL}(7)$-inequivalent families (Algorithm 2.2, Proposition 3.4, Theorem 3.6).

For each family, we obtain a closed $\mathrm{SL}(7)$-orbit by considering a one-parameter subgroup limit and applying Luna's centralizer reduction [Lun75]. If the centralizer is a torus, we invoke the convex-hull criterion; if it is non-toric, we use the Casimiro–Florentino criterion [CF12]. This dichotomy uniformly yields closed-orbit representatives, from which the component dimensions follow (Section 4; see also Table 2).

Lastly, to demonstrate non-inclusions among distinct families—and hence to demonstrate that the 21 components vary significantly—we employ Gröbner-basis computations together with the Rabinowitsch trick (Section 7) [Buc65, CLO07].

Organization of the paper

Section 1 recalls the numerical criterion for (semi)stability in convex-geometric language. Section 2 enumerates maximal strictly semistable supports for the maximal torus T (Algorithm 2.2). Section 3 passes from T-data to the $\mathrm{SL}(7)$-action, eliminates redundancies, and arrives at the 21 families $f_1, \ldots, f_{21}$; it also records the non-inclusion statement among distinct families (with the proof deferred to Section 7). Section 4 constructs closed-orbit representatives, proves polystability (via the convex-hull or Casimiro–Florentino criterion), and records component dimensions. Section 5 computes singular loci and lists isolated types, including wild examples. Section 6 records adjacencies as wall-crossings in Kirwan's stratification. Section 7 provides the Gröbner-based certification of non-inclusions. Table 2 presents a compact summary of the normal forms and dimensions.

We present below a table comparing cubic threefolds, fourfolds, and fivefolds.

Cubic threefolds in $\mathbb{P}^4$ (See [Yok02].)
Moduli dimension. Number of monomials $\binom{5+3-1}{3} = 35$, projective dimension 34; $\dim \mathrm{PGL}_5 = 24$. Thus $\dim \mathcal{M}^{\mathrm{GIT}} = 34 - 24 = \mathbf{10}$.

Components of the GIT boundary (polystable closed orbits). Two irreducible components:

(i) a $\mathbb{P}^1$-family $\{\phi_{\alpha,\beta}\}$ with parameter $[\alpha : \beta]$, and

(ii) the isolated point represented by $\phi = vwz + x^3 + y^3$.

Singularity profile (stability). Stable $\iff$ only double points of type A_n with $n \le 4$. Semistable $\iff$ only A_n ($n \le 5$), D_4, or A_∞ double points. Inside the $\mathbb{P}^1$-component, the special member with $\alpha^2 = 4\beta$ is the secant threefold (the singular locus is the rational normal curve).

Adjacency/closures. Both boundary components consist of closed orbits; the $\mathbb{P}^1$-component and the isolated point are distinct boundary components (there is no specialization of one to the other).

Cubic fourfolds in $\mathbb{P}^5$ (See [Laz09, Yok08, Huy23].)

Moduli dimension. Number of monomials $\binom{6+3-1}{3} = 56$, projective dimension 55; $\dim \mathrm{PGL}_6 = 35$. Thus $\dim \mathcal{M}^{\mathrm{GIT}} = \mathbf{20}$.

Boundary (closed orbit) families and their dimensions. There are six types, denoted $[C.1]$-$[C.6]$, of respective dimensions $1, 2, 3, 1, 1, 0$. A convenient set of normal forms is:

[C.1] $u\, q_1(w, x, y, z) + v\, q_2(w, x, y, z)$ with $V(u, v, q_1, q_2)$ smooth.

[C.2] $u(xy + xz + yz + \alpha z^2) + v^2 x + w^2 y + 2vwz$ (generic α).

[C.3] $uy^2 + v^2 z + l_1(w, x)\, uz + 2\, l_2(w, x)\, vy + c(w, x)$ with $l_2^2 \nmid c$, $l_1 \nmid c$.

[C.4] $uvw + c(x, y, z)$ with $V(u, v, w, c)$ smooth.

[C.5] $\alpha\, uy^2 + v^2 z + w^2 x - uxz + 2vwy$ ($\alpha \neq 0$).

[C.6] $uvw + xyz$.

Stability via singularities. A cubic fourfold with only isolated simple (ADE) singularities is GIT stable. Conversely, non-stability occurs if any of the following conditions holds: (1) $\mathrm{Sing}(X)$ contains a conic; (2) $\mathrm{Sing}(X)$ contains a line; (3) $\mathrm{Sing}(X)$ contains the intersection of two quadrics; (4) X has a double point of rank ≤ 2; (5) a rank 3 double point with a hyperplane section whose singular locus is a line with ranks ≤ 2 along it; (6) a rank 3 double point whose tangent-cone singular locus is a 2-plane in X.

Adjacency (specialization) among boundary strata. If we denote the families $[C.1]$ to $[C.6]$ by $C_1, S_2, V_3, C_4, C_5, P_6$, then

$$P_6 \subset \overline{S_2} \cap \overline{V_3}, \qquad P_6 \in \overline{C_1} \cap \overline{C_4} \cap \overline{C_5}.$$

Cubic fivefolds in $\mathbb{P}^6$

Moduli dimension. Number of monomials $\binom{7+3-1}{3} = 84$, projective dimension 83; $\dim \mathrm{PGL}_7 = 48$. Thus $\dim \mathcal{M}^{\mathrm{GIT}} = \mathbf{35}$.

Components of the GIT boundary (strictly semistable locus). There are **21** irreducible boundary components. At the level of closed-orbit representatives, the positive-dimensional possibilities further include a quadric surface (including a rank-3 cone), linear spaces $\mathbb{P}^2$ and $\mathbb{P}^3$, as well as a *quadric threefold* and a *quadric threefold cone*; see Table 3 (Section 5). Exactly two closed-orbit representatives carry an isolated singular point (Cases $k = 1, 6$), which is quasi-homogeneous of corank 3 with Milnor and Tjurina numbers $\mu = \tau = 19$ (type $\mathrm{QH}(3)_{19}$).

Scripts used in this paper

The scripts used in this paper are publicly available at [Shi25]. In particular, the archive includes the scripts for Sections 2, 5, 6, and 7.

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1 Numerical criterion for cubic fivefolds

In this section, we review the numerical criterion for stability or semistability of cubic fivefolds. We use the following notations.

- Let $\mathbb{C}[x_0, \cdots, x_6]_3$ be the set of homogeneous polynomials of degree 3.
- For a vector $\mathbf{x} \in \mathbb{Q}^7$, $\mathrm{wt}(\mathbf{x}) = \sum_{k=0}^{6} x_k$ is called the weight of $\mathbf{x}$.
- We define $\mathbb{Z}^7_{\geq 0} = \{\mathbf{x} = (x_0, x_1, \cdots, x_6) \in \mathbb{Z}^7 | x_k \geq 0 (k = 0, 1, \cdots, 6)\}$,

  $\mathbb{Z}^7_{(d)} = \{\mathbf{x} \in \mathbb{Z}^7 | \mathrm{wt}(\mathbf{x}) = d\}$,

  $\mathbb{I} = \mathbb{Z}^7_{(3)} \cap \mathbb{Z}^7_{\geq 0}$ and it is simply called the simplex.
- For $\mathbf{r} \in \mathbb{Q}^7$, we define $\mathbb{I}(\mathbf{r})_{\geq 0} = \{\mathbf{i} \in \mathbb{I} | \mathbf{r} \cdot \mathbf{i} \geq 0\}$, $\mathbb{I}(\mathbf{r})_{>0} = \{\mathbf{i} \in \mathbb{I} | \mathbf{r} \cdot \mathbf{i} > 0\}$ and $\mathbb{I}(\mathbf{r})_{=0} = \{\mathbf{i} \in \mathbb{I} | \mathbf{r} \cdot \mathbf{i} = 0\}$, here $\cdot$ denotes the standard inner product of vectors.
- For a polynomial $f = \sum_{\mathrm{wt}(\mathbf{i})=3} a_{\mathbf{i}} x^{\mathbf{i}} \in \mathbb{C}[x_0, \cdots, x_6]_3$, we define the support of f by $\mathrm{Supp}(f) = \{\mathbf{i} \in \mathbb{I} | a_{\mathbf{i}} \neq 0\}$

- We set $\eta = (3/7, 3/7, 3/7, 3/7, 3/7, 3/7, 3/7) \in \mathbb{Q}^7$ and it is called the barycenter of the simplex $\mathbb{I}$.
- A vector $\mathbf{r} \in \mathbb{Z}^7$ is said to be reduced when there is no integer α such that $|\alpha| \geq 2$ and $\frac{1}{\alpha}\mathbf{r} \in \mathbb{Z}^7$

We fix a maximal torus T of $\mathrm{SL}(7)$. Consider a one-parameter subgroup (1-PS for short) $\lambda : \mathbb{G}_m \to \mathrm{SL}(7)$ whose image is contained in T. For a suitable basis of $\mathbb{C}^7$, λ can be expressed as a diagonal matrix $\mathrm{diag}(t^{r_0}, t^{r_1}, \cdots, t^{r_6})$ where $t \neq 0$ is a parameter of $\mathbb{G}_m$. Let us choose and fix such basis. Then λ corresponds to an element $\mathbf{r} = (r_0, r_1, \cdots, r_6)$ in $\mathbb{Z}^7_{(0)}$. We can regard an element of $\mathbb{Z}^7_{(0)}$ as a 1-PS of T.

Definition 1.1. *Let s be a subset of $\mathbb{I}$. We say that s is not stable (resp. unstable) with respect to T when $s \subseteq \mathbb{I}(\mathbf{r})_{\geq 0}$ (resp. $s \subseteq \mathbb{I}(\mathbf{r})_{>0}$) for some 1-PS $\mathbf{r}$. For $0 \neq f \in \mathbb{C}[x_0, \cdots, x_6]_3$, we say that f is not stable (resp. unstable) with respect to T when $\mathrm{Supp}(f) \subseteq \mathbb{I}$ is not stable (resp. unstable) with respect to T.*

The following theorem is the numerical criterion for stability via the language of convex geometry.

Theorem 1.2. *The cubic fivefold defined by $f \in \mathbb{C}[x_0, \cdots, x_6]_3$ is not stable (resp. unstable) if and only if there exists an element $\sigma \in \mathrm{SL}(7)$ such that f^σ is not stable (resp. unstable) with respect to T.*

In particular, f is strictly semistable if and only if

(1) *There exist $\sigma \in \mathrm{SL}(7)$ such that f^σ is not stable with respect to T, and*

(2) *For any $\sigma \in \mathrm{SL}(7)$, f^σ is semistable with respect to T.*

Proof. See Theorem 9.1 of [Dol03]. □

2 Maximal strictly semistable cubic fivefolds with respect to the maximal torus$T$

In this section, we list the irreducible components corresponding to strictly semistable cubic fivefolds. For this purpose, we list all strictly semistable cubic fivefolds with respect to the maximal torus T. To solve this problem, we will consider the set of maximal strictly semistable subsets of $\mathbb{I}$. The order in the set of subsets of $\mathbb{I}$ is given by inclusion. For this purpose, we list the set of all maximal elements of $\mathcal{S} = \{\mathbb{I}(\mathbf{r})_{\geq 0} | \mathbf{r} \in \mathbb{Z}^7_{(0)}\}$.

We solve this problem computationally. We need an algorithm which enables us to obtain them in finitely many steps. Before giving such an algorithm, we remark that $\mathbb{I}(\mathbf{r})_{\geq 0}$ and $\mathbb{I}(\mathbf{r}')_{\geq 0}$ might be the same for two different vectors $\mathbf{r}, \mathbf{r}' \in \mathbb{Z}^7_{(0)}$.

Lemma 2.1. *Let $\mathbb{I}(\mathbf{r})_{\geq 0}$ be a maximal element of $\mathcal{S}$, where $\mathbf{r} \in \mathbb{Z}^7_{(0)}$. Then there exist 5 elements $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_5 \in \mathbb{I}$ and a vector $\mathbf{r}' \in \mathbb{Z}^7_{(0)}$ such that they satisfy the following three conditions:*

(1) *The vector subspace W of $\mathbb{Q}^7$ spanned by $\mathbf{x}_1, \cdots, \mathbf{x}_5, \eta$ over $\mathbb{Q}$ has dimension 6*

(2) *The vector $\mathbf{r}'$ is orthogonal to the subspace W of $\mathbb{Q}^7$.*

(3) $\mathbb{I}(\mathbf{r})_{\geq 0} = \mathbb{I}(\mathbf{r}')_{\geq 0}$

Proof. Let us put $C = \mathbb{I}(r) \cup \{\eta\}$. We consider the convex-hull $\check{C}$ of C in $\mathbb{Q}^7$. Let F be a face of $\check{C}$ containing the point η. There is a normal vector $\mathbf{r}'$ of F in $\mathbb{Z}^7_{(0)}$ such that $\check{C} \subseteq \{\mathbf{x} \in \mathbb{Q}^7 | \mathbf{r}' \cdot \mathbf{x} \geq 0\}$. We have $\mathrm{wt}(\mathbf{r}') = 0$ because the hyperplane defined by $\{\mathbf{x} \in \mathbb{Q}^7 | \mathbf{r}' \cdot \mathbf{x} = 0\}$ passes through the point η. By the definition of the faces of a convex set in $\mathbb{Q}^7$, we can take 5 points $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_5$ from the set $\mathbb{I} \cap F$ such that $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_5, \eta$ are linearly independent over $\mathbb{Q}$. In general we have $\mathbb{I}(\mathbf{r})_{\geq 0} \subseteq \mathbb{I}(\mathbf{r}')_{\geq 0}$, and by the assumption that $\mathbb{I}(\mathbf{r})_{\geq 0}$ is maximal in $\mathcal{S}$, we conclude that $\mathbb{I}(\mathbf{r})_{\geq 0} = \mathbb{I}(\mathbf{r}')_{\geq 0}$. $\square$

By this lemma, we can determine the set of maximal elements of $\mathcal{S}$ up to permutations of coordinates in finite steps using the following algorithm.

Algorithm 2.2. *Let $\mathcal{F}$ be the set of five different points of $\mathbb{I}$. We fix a total order on $\mathcal{F}$. As an initial data, we set $\mathcal{S}' = \varnothing$ and $\mathbf{x} = (x_0, \cdots, x_5)$ be the minimum element of $\mathcal{F}$. We will modify $\mathcal{S}'$ using the following algorithm.*

- *Step 1. If the subspace W spanned by $x_0, \cdots, x_5, \eta$ of $\mathbb{Q}^7$ has dimension 6 then take a reduced normal vector $\mathbf{r} = (r_0, \cdots, r_6) \in \mathbb{Z}^7_{(0)}$ of W and go to Step 2, else go to Step 5.*

- *Step 2. If $r = (r_0, \cdots, r_6)$ satisfies the condition $r_0 \geq \cdots \geq r_6$ or $r_0 \leq \cdots \leq r_6$, then go to Step 3, else go to Step 5.*

- *Step 3. If $r_0 \geq \cdots \geq r_6$ (resp. $r_0 \leq \cdots \leq r_6$) add $\mathbb{I}(\mathbf{r})$ (resp. $\mathbb{I}(-\mathbf{r})$) to $\mathcal{S}'$ and go to Step 4.*

- *Step 4. Delete all elements of $\mathcal{S}'$ that are not maximal in $\mathcal{S}'$ and go to Step 5.*

- *Step 5. Replace the element $\mathbf{x}$ with the next element if $\mathbf{x}$ is not the maximum element, and go to Step 1. Otherwise, stop the algorithm.*

We note that Step 2 removes the S_7 symmetry on the variables $x_0, \cdots, x_6$. We also note that Step 4 is not essential but serves as technical measure to save memory. After running this algorithm with the aid of a computer, we find 23 elements $\mathbb{I}(\mathbf{r}_1)_{\geq 0}, \cdots, \mathbb{I}(\mathbf{r}_{23})_{\geq 0}$ in $\mathcal{S}'$, where $\mathbf{r}_k = (r_0, \cdots, r_6) \in \mathbb{Z}^7_{(0)}$ is a reduced vector with $r_0 \geq \cdots \geq r_6$. When we compute the convex-hulls of $\mathbb{I}(\mathbf{r}_1)_{\geq 0}, \cdots, \mathbb{I}(\mathbf{r}_{23})_{\geq 0}$ in $\mathbb{Q}^7$, only one of the convex-hulls of $\mathbb{I}(\mathbf{r}_k)_{\geq 0}$ does not contain η. We denote it as $\mathbb{I}(\mathbf{r}_{23})_{\geq 0}$. As $\mathbb{I}(\mathbf{r}_{23})_{\geq 0}$ is unstable with respect to T, we remove it from the list. Thus, we can conclude that there are 22 maximal strictly semistable cubic fivefolds for the fixed maximal torus T. Because of this algorithm, we have the following proposition.

Proposition 2.3. *The set* $\mathcal{M} = \{\mathbb{I}(\mathbf{r}_1)_{\geq 0}, \cdots, \mathbb{I}(\mathbf{r}_{22})_{\geq 0}\}$ *is given as follows.*

$\mathbf{r}_1 = (8, 3, 2, -1, -2, -4, -6)$	$\mathbf{r}_2 = (6, 4, 1, -1, -2, -3, -5)$
$\mathbf{r}_3 = (4, 2, 1, -1, -1, -2, -3)$	$\mathbf{r}_4 = (3, 2, 1, 0, -1, -2, -3)$
$\mathbf{r}_5 = (4, 2, 1, 0, -1, -2, -4)$	$\mathbf{r}_6 = (5, 3, 2, 1, -1, -4, -6)$
$\mathbf{r}_7 = (6, 4, 2, 1, -2, -3, -8)$	$\mathbf{r}_8 = (4, 1, 1, 0, -2, -2, -2)$
$\mathbf{r}_9 = (2, 2, 0, 0, -1, -1, -2)$	$\mathbf{r}_{10} = (2, 1, 0, 0, -1, -1, -1)$
$\mathbf{r}_{11} = (2, 0, 0, 0, 0, -1, -1)$	$\mathbf{r}_{12} = (3, 2, 1, 1, -1, -2, -4)$
$\mathbf{r}_{13} = (2, 1, 1, 0, -1, -1, -2)$	$\mathbf{r}_{14} = (2, 2, 0, -1, -1, -1, -1)$
$\mathbf{r}_{15} = (2, 1, 1, 0, 0, -2, -2)$	$\mathbf{r}_{16} = (2, 1, 0, 0, 0, -1, -2)$
$\mathbf{r}_{17} = (1, 1, 1, 0, 0, -1, -2)$	$\mathbf{r}_{18} = (1, 1, 0, 0, 0, -1, -1)$
$\mathbf{r}_{19} = (2, 2, 2, 0, -1, -1, -4)$	$\mathbf{r}_{20} = (1, 1, 1, 1, 0, -2, -2)$
$\mathbf{r}_{21} = (1, 1, 0, 0, 0, 0, -2)$	$\mathbf{r}_{22} = (1, 0, 0, 0, 0, 0, -1)$

For example, $\mathbb{I}(\mathbf{r}_1)_{\geq 0}$ is
$\mathbb{I}(\mathbf{r}_1)_{\geq 0} = \{x_0^3, x_0^2x_1, x_0^2x_2, x_0^2x_3, x_0^2x_4, x_0^2x_5, x_0^2x_6, x_0x_1^2, x_0x_1x_2, x_0x_1x_3, x_0x_1x_4, x_0x_1x_5, x_0x_1x_6, x_0x_2^2, x_0x_2x_3, x_0x_2x_4, x_0x_2x_5, x_0x_2x_6, x_0x_3^2, x_0x_3x_4, x_0x_3x_5, x_0x_3x_6, x_0x_4^2, x_0x_4x_5, x_0x_4x_6, x_0x_5^2, x_1^3, x_1^2x_2, x_1^2x_3, x_1^2x_4, x_1^2x_5, x_1^2x_6, x_1x_2^2, x_1x_2x_3, x_1x_2x_4, x_1x_2x_5, x_1x_3^2, x_1x_3x_4, x_2^3, x_2^2x_3, x_2^2x_4, x_2^2x_5, x_2x_3^2\}$.
Here we use the notation $x_0^{i_0}x_1^{i_1}\cdots x_6^{i_6}$ for an element $(i_0, i_1, \cdots, i_6) \in \mathbb{Z}^7_{(3)}$ in order to save space.

Remark 2.4. *The following vectors can serve as* r_{23}*, i.e., there are several vectors that yield the set* $\mathbb{I}(\mathbf{r}_{23})_{\geq 0}$*. For example, we can take* $\mathbf{r}_{23} = (8, 5, 3, 2, -4, -4, -10)$.

Remark 2.5. *The algorithm in this section has been comprehensively generalized by* [GMMS23].

3 21 maximal strictly semistable cubic fivefolds under action of SL(7)

An element $\mathbb{I}(\mathbf{r}_k)_{\geq 0}$ of $\mathcal{M}$ represents a family of cubic fivefolds whose defining polynomial's support is contained in $\mathbb{I}(\mathbf{r}_k)_{\geq 0}$. In this section, we analyze the inclusion relations among $\mathbb{I}(\mathbf{r}_k)_{\geq 0}$ under the action of SL(7). Let f_k be a generic polynomial whose support is $\mathbb{I}(\mathbf{r}_k)_{\geq 0}$. $(k = 1, 2, \cdots, 22)$. If we express f_k directly, it becomes too long, so we introduce notation.

Definition 3.1. *The symbols* c, q, l, α *stand for a cubic form, a quadratic form, a linear form, and a constant term, respectively. Similarly, the symbols* q_i, l_i, α_i *denote the i-th quadratic form, a linear form, and a constant term, respectively.*

The following theorem is a direct consequence of the list in Proposition 2.3.

Theorem 3.2. *Using the above notations, the generic polynomials of* $f_1, \cdots, f_{22}$ *are the following forms.*

- $f_1 = c(x_0, x_1, x_2) + q_1(x_0, x_1, x_2)x_3 + l_1(x_0, x_1, x_2)x_3^2 + \{q_2(x_0, x_1, x_2) + l_2(x_0, x_1)x_3\}x_4 + \alpha_1x_0x_4^2 + \{q_3(x_0, x_1, x_2) + x_0l_3(x_3, x_4)\}x_5 + \alpha_2x_0x_5^2 + \{q_4(x_0, x_1) + x_0l_4(x_2, x_3, x_4)\}x_6$

- $f_2 = c(x_0,x_1,x_2)+q_1(x_0,x_1,x_2)x_3+l_1(x_0,x_1)x_3^2+\{q_2(x_0,x_1,x_2)+l_2(x_0,x_1)x_3\}x_4+l_3(x_0,x_1)x_4^2+\{q_3(x_0,x_1)+l_4(x_0,x_1)x_2+l_5(x_0,x_1)x_3+\alpha_1x_0x_4\}x_5+\alpha_2x_0x_5^2+\{q_4(x_0,x_1)+l_6(x_0,x_1)x_2+\alpha_3x_0x_3\}x_6$

- $f_3 = c(x_0,x_1,x_2)+q_1(x_0,x_1,x_2)x_3+l_1(x_0,x_1)x_3^2+\{q_2(x_0,x_1,x_2)+l_2(x_0,x_1)x_3\}x_4+l_3(x_0,x_1)x_4^2+\{q_3(x_0,x_1,x_2)+x_0l_4(x_3,x_4)\}x_5+\alpha_1x_0x_5^2+\{q_4(x_0,x_1)+l_5(x_0,x_1)x_2+x_0l_6(x_3,x_4)\}x_6$

- $f_4 = c(x_0,x_1,x_2)+\{q_1(x_0,x_1)+l_1(x_0,x_1)x_2\}x_3+l_2(x_0,x_1)x_3^2+\{q_2(x_0,x_1)+l_3(x_0,x_1)x_2+l_4(x_0,x_1)x_3\}x_4+l_5(x_0,x_1)x_4^2+\{q_3(x_0,x_1)+l_6(x_0,x_1)x_2+l_7(x_0,x_1)x_3+l_8(x_0,x_1)x_4\}x_5+l_9(x_0,x_1)x_5^2+\{q_4(x_0,x_1)+l_{10}(x_0,x_1)x_2+l_{11}(x_0,x_1)x_3+l_{12}(x_0,x_1)x_4+l_{13}(x_0,x_1)x_5\}x_6+l_{14}(x_0,x_1)x_6^2$

- $f_5 = c(x_0,x_1,x_2,x_3)+\{q_1(x_0,x_1,x_2)+l_1(x_0,x_1,x_2)x_3\}x_4+l_2(x_0,x_1)x_4^2+\{q_2(x_0,x_1,x_2)+l_3(x_0,x_1)x_3+\alpha_1x_0x_4\}x_5+\{q_3(x_0,x_1)+l_4(x_0,x_1)x_2+\alpha_2x_0x_3\}x_6$

- $f_6 = c(x_0,x_1,x_2,x_3)+\{q_1(x_0,x_1,x_2)+l_1(x_0,x_1,x_2)x_3\}x_4+l_2(x_0,x_1)x_4^2+\{q_2(x_0,x_1,x_2)+l_3(x_0,x_1)x_3+\alpha_1x_0x_4\}x_5+\{q_3(x_0,x_1)+x_0l_4(x_2,x_3)\}x_6$

- $f_7 = c(x_0,x_1,x_2,x_3)+q_1(x_0,x_1,x_2,x_3)x_4+l_1(x_0,x_1,x_2)x_4^2+\{q_2(x_0,x_1,x_2)+l_2(x_0,x_1)x_3+\alpha_1x_0x_4\}x_5+\{q_3(x_0,x_1)+x_0l_3(x_2,x_3)\}x_6$

- $f_8 = c(x_0,x_1,x_2,x_3)+q_1(x_0,x_1,x_2,x_3)x_4+l_1(x_0,x_1)x_4^2+\{q_2(x_0,x_1,x_2)+l_2(x_0,x_1,x_2)x_3+\alpha_1x_0x_4\}x_5+\alpha_2x_0x_5^2+\{q_3(x_0,x_1)+\alpha_3x_0x_2\}x_6$

- $f_9 = c(x_0,x_1,x_2,x_3)+\{q_1(x_0,x_1,x_2)+\alpha_1x_0x_3\}x_4+\alpha_2x_0x_4^2+\{q_2(x_0,x_1,x_2)+x_0l_1(x_3,x_4)\}x_5+\alpha_3x_0x_5^2+\{q_3(x_0,x_1,x_2)+x_0l_2(x_3,x_4,x_5)\}x_6+\alpha_4x_0x_6^2$

- $f_{10} = c(x_0,x_1,x_2,x_3)+\{q_1(x_0,x_1)+l_1(x_0,x_1)x_2+l_2(x_0,x_1)x_3\}x_4+l_3(x_0,x_1)x_4^2+\{q_2(x_0,x_1)+l_4(x_0,x_1)x_2+l_5(x_0,x_1)x_3+l_6(x_0,x_1)x_4\}x_5+l_7(x_0,x_1)x_5^2+\{q_3(x_0,x_1)+l_8(x_0,x_1)x_2+l_9(x_0,x_1)x_3\}x_6$

- $f_{11} = c(x_0,x_1,x_2,x_3)+\{q_1(x_0,x_1)+l_1(x_0,x_1)x_2+l_2(x_0,x_1)x_3\}x_4+\alpha_1x_0x_4^2+\{q_2(x_0,x_1)+l_4(x_0,x_1)x_2+l_5(x_0,x_1)x_3+\alpha_2x_0x_4\}x_5+\alpha_3x_0x_5^2+\{l_7(x_0,x_1)x_2+l_8(x_0,x_1)x_3+x_0l_9(x_4,x_5)\}x_6+\alpha_4x_0x_6^2$

- $f_{12} = c(x_0,x_1,x_2,x_3)+q_1(x_0,x_1,x_2,x_3)x_4+l_1(x_0,x_1)x_4^2+\{q_2(x_0,x_1,x_2,x_3)+\alpha_1x_0x_4\}x_5+\{q_3(x_0,x_1)+x_0l_2(x_2,x_3)\}x_6$

- $f_{13} = c(x_0,x_1,x_2,x_3)+\{q_1(x_0,x_1,x_2)+l_1(x_0,x_1,x_2)x_3\}x_4+\alpha_1x_0x_4^2+\{q_2(x_0,x_1,x_2)+l_2(x_0,x_1,x_2)x_3+\alpha_2x_0x_4\}x_5+\alpha_3x_0x_5^2+\{q_3(x_0,x_1,x_2)+\alpha_4x_0x_3\}x_6$

- $f_{14} = c(x_0,x_1,x_2,x_3)+\{q_1(x_0,x_1,x_2)+l_1(x_0,x_1,x_2)x_3\}x_4+l_2(x_0,x_1,x_2)x_4^2+\{q_2(x_0,x_1,x_2)+l_3(x_0,x_1,x_2)x_3+l_4(x_0,x_1,x_2)x_4\}x_5+l_5(x_0,x_1,x_2)x_5^2+q_3(x_0,x_1,x_2)x_6$

- $f_{15} = c(x_0,x_1,x_2,x_3,x_4)+x_0l_1(x_0,x_1,x_2,x_3,x_4)x_5+\alpha_1x_0x_5^2+x_0l_2(x_0,x_1,x_2,x_3,x_4,x_5)x_6+\alpha_2x_0x_6^2$

- $f_{16} = c(x_0,x_1,x_2,x_3,x_4)+\{q_1(x_0,x_1,x_2)+x_0l_1(x_3,x_4)\}x_5+\{q_2(x_0,x_1,x_2)+x_0l_2(x_3,x_4)\}x_6$
- $f_{17} = c(x_0,x_1,x_2,x_3,x_4)+\{q_1(x_0,x_1)+l_1(x_0,x_1)x_2+l_2(x_0,x_1)x_3+l_3(x_0,x_1)x_4\}x_5+\alpha_1x_0x_5^2+\{q_2(x_0,x_1)+x_0l_4(x_2,x_3,x_4)\}x_6$
- $f_{18} = c(x_0,x_1,x_2,x_3,x_4)+\{q_1(x_0,x_1,x_2)+l_1(x_0,x_1,x_2)x_3+l_2(x_0,x_1,x_2)x_4\}x_5+q_2(x_0,x_1,x_2)x_6$
- $f_{19} = c(x_0,x_1,x_2,x_3,x_4)+\{q_1(x_0,x_1)+l_1(x_0,x_1)x_2+l_2(x_0,x_1)x_3+l_3(x_0,x_1)x_4\}x_5+\{q_2(x_0,x_1)+l_4(x_0,x_1)x_2+l_5(x_0,x_1)x_3+l_6(x_0,x_1)x_4\}x_6$
- $f_{20} = c(x_0,x_1,x_2,x_3,x_4)+q_1(x_0,x_1,x_2,x_3)x_5+q_2(x_0,x_1,x_2,x_3)x_6$
- $f_{21} = c(x_0,x_1,x_2,x_3,x_4,x_5)+q(x_0,x_1)x_6$
- $f_{22} = c(x_0,x_1,x_2,x_3,x_4,x_5)+x_0l(x_0,x_1,x_2,x_3,x_4,x_5)x_6$

For an element σ in $\mathrm{SL}(7)$ and $\mathbb{J} \subseteq \mathbb{I}$, we set $\mathbb{J}^\sigma = \cup_f \mathrm{Supp}(f^\sigma)$, where f runs through all polynomials with $\mathrm{Supp}(f) \subseteq \mathbb{J}$.

Definition 3.3. *We denote*

$$\mathbb{I}(\mathbf{r}_k)_{\geq 0} \subseteq \mathbb{I}(\mathbf{r}_l)_{\geq 0} \text{ mod } \mathrm{SL}(7)$$

when there exists $\sigma \in \mathrm{SL}(7)$ such that $\mathbb{I}(\mathbf{r}_k)^\sigma_{\geq 0} \subseteq \mathbb{I}(\mathbf{r}_l)_{\geq 0}$ and say that $\mathbb{I}(\mathbf{r}_k)_{\geq 0}$ is included in $\mathbb{I}(\mathbf{r}_l)_{\geq 0}$ modulo $\mathrm{SL}(7)$.

We construct a smaller subset $\mathcal{M}'$ of $\mathcal{M}$ such that (1) any element $\mathbb{I}(\mathbf{r}_k)_{\geq 0}$ in $\mathcal{M}$ is included in some element $\mathbb{I}(\mathbf{r}_l)_{\geq 0}$ in $\mathcal{M}'$ mod $\mathrm{SL}(7)$, (2) any element $\mathbb{I}(\mathbf{r}_k)_{\geq 0}$ in $\mathcal{M}'$ is not included in any other $\mathbb{I}(\mathbf{r}_l)_{\geq 0}$ in $\mathcal{M}'$ mod $\mathrm{SL}(7)$ $(1 \leq l \leq 22)$

Proposition 3.4. *There are two relations*

- $\mathbb{I}(\mathbf{r}_{21})_{\geq 0} \subseteq \mathbb{I}(\mathbf{r}_{22})_{\geq 0}$ mod $\mathrm{SL}(7)$
- $\mathbb{I}(\mathbf{r}_{22})_{\geq 0} \subseteq \mathbb{I}(\mathbf{r}_{21})_{\geq 0}$ mod $\mathrm{SL}(7)$.

Proof.
$$\begin{aligned} f_{22} &= c(x_0,\cdots,x_5)+x_0l(x_0,\cdots,x_5)x_6 \\ &\equiv c(x_0,\cdots,x_5)+x_0l(x_0,x_1)x_6 \\ &\equiv c(x_0,\cdots,x_5)+q(x_0,x_1)x_6 \\ &= f_{21} \end{aligned}$$

Here $\equiv$ means equality after an $\mathrm{SL}(7)$ change of coordinates. □

From this proposition, we can remove f_{22} from the list. Thus, we obtain a list of 21 types of cubic fivefolds.

Proposition 3.5. *For any $1 \leq k, l \leq 21$ with $k \neq l$, there is no inclusion*

$$\mathbb{I}(\mathbf{r}_k)_{\geq 0} \subseteq \mathbb{I}(\mathbf{r}_l)_{\geq 0} \quad \text{mod } \mathrm{SL}(7).$$

Proof. The proof is deferred to Section 7. For now, we take this fact for granted and proceed with the argument. Consequently, we obtain the following theorem. □

Theorem 3.6. *The moduli space of strictly semistable cubic fivefolds has* 21 *irreducible components, represented by* $f_1, f_2, \ldots, f_{21}$.

4 Closed orbits of 21 families

In this section, we find the closed orbits in the 21 families of strictly semistable cubic fivefolds. We define $\lambda_k(t)\colon \mathbb{G}_m \to \mathrm{SL}(7)$ as 1-PSs corresponding to $\mathbf{r}_k$. *We shall use the following convex–geometric criterion repeatedly in this section, so we record it at the outset.*

For the following theorem, see [PV94].

Theorem 4.1 (Convex-hull criterion)**.** *Let T be an algebraic torus acting linearly on a finite-dimensional vector space, V, and let $v \in V$. Subsequently, the following conditions are equivalent:*

(1) the T-orbit $T \cdot v$ is closed in V;

(2) 0 is an interior point of the convex-hull of $\mathrm{Supp}(v)$ in $X(T)_{\mathbb{R}}$.

Here, $\mathrm{Supp}(v)$ denotes the set of T-weights that occur in v, and $X(T)_{\mathbb{R}}$ is the real vector space spanned by the character lattice $X(T)$.

The following series of definitions and theorems will be extremely useful for determining polystability in cases where the centralizer is not a torus and the convex-hull criterion cannot be applied. Because they will be used repeatedly from this point on, we state them here beforehand.

Notation 4.2. *Let G be a reductive algebraic group, and let X be an affine variety equipped with an algebraic G-action.*

(1) *We denote by $Y(G)$ the set of one-parameter subgroups of G.*

(2) *When $x \in X$, we put $\Lambda_x = \{\lambda(t) \in Y(G)\colon \lim_{t\to 0} \lambda(t)\cdot x \text{ exists}\}$.*

(3) *When $\lambda \in Y(G)$, we define $P(\lambda) = \{g \in G\colon \lim_{t\to 0} \lambda(t)\cdot g \cdot \lambda(t)^{-1} \text{ exists}\}$. It is a parabolic subgroup of G.*

Definition 4.3. *We say that a subset $\Lambda \subset Y(G)$ is symmetric if given any $\lambda \in \Lambda$, there is another 1-PS $\lambda' \in \Lambda$ such that $P(\lambda) \cap P(\lambda')$ is a Levi subgroup of both $P(\lambda)$ and $P(\lambda')$.*

We need the following theorem (Theorem 1.1 of [CF12]):

Theorem 4.4. *(Casimiro–Florentino criterion) Let G be a reductive algebraic group and X be an affine G-variety. Then, a point $x \in X$ is polystable if and only if Λ_x is symmetric.*

Lemma 4.5 (rank 1 symmetry for Casimiro–Florentino)**.** *Let G be a reductive algebraic group acting on an affine G-variety X, and let $x \in X$. Assume that there exists a one-parameter subgroup $\mu \in Y(G)$ such that*

$$\Lambda_x \;=\; \{\mu_k \mid k \in \mathbb{Z}\} \,\cup\, \{0\}, \qquad \mu_k(t) := \mu(t^k),$$

where 0 denotes the trivial 1-PS. Then Λ_x is symmetric in the sense of Definition 4.3. Consequently, by the Casimiro–Florentino criterion (Theorem 4.4), the point x is polystable.

Proof. Fix $k \neq 0$. It is standard that $P(\mu_k)$ and $P(\mu_{-k})$ are opposite parabolic subgroups and that

$$P(\mu_k) \cap P(\mu_{-k}) \;=\; C_G\big(\mu(\mathbb{G}_m)\big)$$

which is a Levi subgroup of both $P(\mu_k)$ and $P(\mu_{-k})$. The inclusion $C_G\big(\mu(\mathbb{G}_m)\big) \subset P(\mu_k) \cap P(\mu_{-k})$ is immediate: if g centralizes $\mu(\mathbb{G}_m)$, then $\mu_k(t)\, g\, \mu_k(t)^{-1} = g$ for all $t \in \mathbb{G}_m$, so both limits as $t \to 0$ and $t \to \infty$ exist. For the reverse inclusion, take $g \in P(\mu_k) \cap P(\mu_{-k})$ and consider

$$\phi : \mathbb{G}_m \longrightarrow G, \qquad \phi(t) := \mu_k(t)\, g\, \mu_k(t)^{-1}.$$

By assumption, the limits $\lim_{t\to 0} \phi(t)$ and $\lim_{t\to\infty} \phi(t)$ both exist (the latter because $g \in P(\mu_{-k})$ and $\phi(1/s) = \mu_{-k}(s)\, g\, \mu_{-k}(s)^{-1}$ for $s \to 0$). Hence, ϕ extends to a morphism $\tilde{\phi} : \mathbb{P}^1 \to G$. As G is affine, $\tilde{\phi}$ is constant; in particular,

$$\mu_k(t)\, g\, \mu_k(t)^{-1} = \phi(t) = \phi(1) = g \quad \text{for all } t \in \mathbb{G}_m.$$

Thus, g centralizes $\mu_k(\mathbb{G}_m) = \mu(\mathbb{G}_m)$, i.e. $g \in C_G\big(\mu(\mathbb{G}_m)\big)$. This proves $P(\mu_k) \cap P(\mu_{-k}) = C_G\big(\mu(\mathbb{G}_m)\big)$.

Based on the hypothesis $\mu_k, \mu_{-k} \in \Lambda_x$, so the requirement in Definition 4.3 is satisfied for every $\lambda = \mu_k$. Hence, Λ_x is symmetric. The last assertion follows from Theorem 4.4 (Casimiro–Florentino). $\square$

Convention 4.6. *Throughout this section, we fix the following setup and notation.*

(1) ***Coordinates and the maximal torus.*** *We work on $W = \mathrm{Sym}^3\mathbb{C}^7$ with homogeneous coordinates $(x_0, \dots, x_6)$. We fix the diagonal maximal torus $T \subset SL(7)$ acting by $\mathrm{diag}(\mu_0, \dots, \mu_6)$ with $\prod_i \mu_i = 1$. A one-parameter subgroup (1-PS) is written as*

$$\lambda(t) = \mathrm{diag}\big(t^{a_0}, \dots, t^{a_6}\big), \qquad \sum_{i=0}^{6} a_i = 0.$$

(2) ***1-PS limits and centralizer reduction.*** *For each $k \in \{1, \dots, 21\}$ let λ_k correspond to r_k in Proposition 2.3, and let f_k be the generic member from Theorem 3.2. We set the 1-PS limit*

$$\phi_k := \lim_{t\to 0} \lambda_k(t) \cdot f_k.$$

Put $H := \lambda_k(\mathbb{G}_m)$ and write $C_G(H)$ for the centralizer in $G = SL(7)$. Let $W^H \subset W$ be the H-fixed subspace. By Luna's centralizer reduction [Lun75], closedness of $SL(7)\cdot\phi_k$ in $\mathbb{P}(W)^{ss}$ is equivalent to closedness of $C_G(H)\cdot\phi_k$ in W^H.

(3) ***Two criteria for polystability.*** *If $C_G(H)$ is a torus, we certify closedness by the* convex-hull criterion *(Theorem 4.1). If $C_G(H)$ is non-toric, we use the Casimiro–Florentino criterion (Theorem 4.4) as follows: for $\lambda \in Y(C_G(H))$ we write $\mathrm{wt}_\lambda(x_i)$ for the λ-weight on x_i, and $w(m)$ for the induced weight of a monomial m. We denote by S the linear constraint coming from* $\det = 1$ *on $C_G(H)$ (the "trace" of block weights), so $S = 0$ for every λ. We then exhibit a positive linear identity*

$$\sum_j c_j\, w(m_j) \;=\; C\cdot S \qquad (c_j > 0,\ C > 0).$$

If $\lambda \in \Lambda_{\phi_k}$, then every $w(m_j) \ge 0$ and $S = 0$; hence, all $w(m_j) = 0$; solving yields a symmetric 1-PS μ_k, so Λ_{ϕ_k} is symmetric and ϕ_k is polystable.

(4) ***Normal forms and coefficient normalizations.*** *Passing to "normal form," we are allowed to: (i) multiply by a nonzero scalar; (ii) act by $C_G(H)/H$ (e.g., block $GL(2)$, $GL(3)$ actions) to diagonalize blocks; and (iii) use diagonal elements of T (with $\prod \mu_i = 1$) to normalize nonzero coefficients to 1. Parameters $(\alpha, \rho, \sigma, \dots)$ record the residual moduli.*

(5) ***Dimension count.*** *Component dimensions are computed as*

$$\dim(W^H) - \dim_{\text{eff}}\bigl(C_G(H)\bigr) - 1,$$

where $\dim_{\text{eff}}$ is the dimension of the effective $C_G(H)$-action on W^H (central tori acting trivially are subtracted). We state explicitly when a central factor acts trivially.

(6) ***Weights and symbols.*** *We freely reuse symbols a_i for nonzero coefficients of ϕ_k prior to normalization. For 1-PS families obtained in the CF-check, we write $\mu_k(t)$. All such conventions are in force throughout § 4.*

Let us now determine, for each $k = 1, 2, \dots, 21$, a polynomial whose SL(7)-orbit is closed. The procedure is uniform across all cases. First, we take a 1-PS limit to produce a candidate ϕ_k for a polystable point. Next, to apply Luna's criterion, we take the stabilizer $H \subset G = \mathrm{SL}(7)$ of ϕ_k; we choose H as large as possible so that its centralizer $C_G(H)$ is as small as possible, which simplifies the closedness check. Write W^H for the H-fixed locus in the ambient representation $W = \mathrm{Sym}^3\mathbb{C}^7$. If $C_G(H)$ is a torus, we apply the convex-hull criterion (Theorem4.1) to show that $C_G(H)\cdot\phi_k$ is closed in W^H; if $C_G(H)$ is not a torus, we instead apply the Casimiro–Florentino criterion (Theorem 4.4) to obtain the same conclusion. In either case, $C_G(H)\cdot\phi_k$ is closed in W^H; hence, by Luna's criterion, the orbit $\mathrm{SL}(7)\cdot\phi_k$ is closed in $\mathbb{P}(\mathrm{Sym}^3\mathbb{C}^7)^{ss}$. Lastly, by determining a normal form of ϕ_k under the action of $C_G(H)/H$, we fix the dimension of the corresponding component of the moduli space for that k.

4.1 Case $k = 1$

1-PS limit. Set

$$\lambda_1(t) = \operatorname{diag}\big(t^8, t^3, t^2, t^{-1}, t^{-2}, t^{-4}, t^{-6}\big), \quad t \in \mathbb{G}_m.$$

For a generic f_1 as in Section 3, the 1-PS limit is

$$\phi_1 := \lim_{t\to 0} \lambda_1(t)\cdot f_1 = a_1\, x_2x_3^2 + a_2\, x_1x_3x_4 + a_3\, x_2^2x_5 + a_4\, x_0x_5^2 + a_5\, x_1^2x_6 + a_6\, x_0x_4x_6.$$

H and $C_G(H)$. Let $H = \lambda_1(\mathbb{G}_m)$. The diagonal weights on $\langle x_0, \dots, x_6\rangle$ are pairwise distinct; hence, $C_G(H) = T$ (the maximal diagonal torus). Each monomial of ϕ_1 has H-weight 0, so $\phi_1 \in W^H$.

Polystability (Luna + convex-hull). By Luna's criterion (see [Lun75]), closedness of the $SL(7)$-orbit of ϕ_1 is equivalent to closedness of the T-orbit in W^H. By the convex-hull criterion (Theorem 4.1), it suffices to check that 0 is an interior point of $\mathrm{Conv}(\mathrm{Supp}(\phi_1)) \subset X(T)_{\mathbb{R}} \simeq \mathbb{R}^7/\mathbb{R}(1,\dots,1)$. This holds because the exponent vectors of the six monomials satisfy

$$\begin{aligned} &2(0,0,1,2,0,0,0) + 2(0,1,0,1,1,0,0) + 2(0,0,2,0,0,1,0) \\ &\quad + 2(1,0,0,0,0,2,0) + 2(0,2,0,0,0,0,1) + 4(1,0,0,0,1,0,1) = 6(1,1,1,1,1,1,1), \end{aligned} \tag{1}$$

Hence, ϕ_1 is polystable and $SL(7)\cdot\phi_1$ is closed.

Normal form and component dimension. Let $\mathrm{diag}(\mu_0, \dots, \mu_6) \in T$ with $\prod_{i=0}^{6} \mu_i = 1$. Along with the overall projective scaling, this acts on the six coefficients of ϕ_1 via the characters determined by the exponent vectors in (1); we may normalize all six coefficients to 1 simultaneously. Thus, a normal form is

$$\phi_1^{\mathrm{nf}} = x_2x_3^2 + x_1x_3x_4 + x_2^2x_5 + x_0x_5^2 + x_1^2x_6 + x_0x_4x_6.$$

The residual T-stabilizer is finite; hence, the corresponding component of the moduli is zero-dimensional.

4.2 Case $k = 2$

1-PS limit. Set

$$\lambda_2(t) = \operatorname{diag}\big(t^6, t^4, t, t^{-1}, t^{-2}, t^{-3}, t^{-5}\big), \qquad t \in \mathbb{G}_m.$$

For a generic f_2 as in Section 3, the 1-PS limit is

$$\phi_2 := \lim_{t\to 0} \lambda_2(t)\cdot f_2 = a_1x_2^2x_4 + a_2x_1x_4^2 + a_3x_1x_3x_5 + a_4x_0x_5^2 + a_5x_1x_2x_6 + a_6x_0x_3x_6.$$

H and $C_G(H)$. Let $H = \lambda_2(\mathbb{G}_m)$. The diagonal weights on $\langle x_0, \dots, x_6\rangle$ are pairwise distinct; hence, $C_G(H) = T$ (the maximal diagonal torus). Each monomial of ϕ_2 has H-weight 0, so $\phi_2 \in W^H$.

Polystability (Luna + convex-hull). By Luna's criterion, closedness of the $SL(7)$-orbit of ϕ_2 is equivalent to closedness of the T-orbit in the H-fixed subspace. By the convex-hull criterion (Theorem 4.1), it suffices to check that 0 lies in the interior of $\mathrm{Conv}(\mathrm{Supp}(\phi_2)) \subset X(T)_{\mathbb{R}} \simeq \mathbb{R}^7/\mathbb{R}(1,\dots,1)$. This holds because

$$\begin{aligned} &2(0,0,2,0,1,0,0) + 2(0,1,0,0,2,0,0) + 2(0,1,0,1,0,1,0) \\ &\quad + 2(1,0,0,0,0,2,0) + 2(0,1,1,0,0,0,1) + 4(1,0,0,1,0,0,1) = 6(1,1,1,1,1,1,1), \end{aligned} \tag{2}$$

written in terms of the exponent vectors of the six monomials of ϕ_2. Hence, ϕ_2 is polystable and $SL(7)\cdot\phi_2$ is closed.

Normal form and component dimension. A diagonal scaling $\mathrm{diag}(\mu_0,\dots,\mu_6) \in T$ with $\prod \mu_i = 1$, together with an overall scalar, acts on the six coefficients via the characters determined by the exponent vectors in (2); thus we may normalize all six coefficients to 1 simultaneously. Therefore, a normal form is

$$\phi_2^{\mathrm{nf}} = x_2^2x_4 + x_1x_4^2 + x_1x_3x_5 + x_0x_5^2 + x_1x_2x_6 + x_0x_3x_6.$$

The residual T-stabilizer is finite; hence, the corresponding component Φ_2 of the moduli is zero-dimensional.

4.3 Case $k = 3$

1-PS limit. Set

$$\lambda_3(t) = \mathrm{diag}\big(t^4,\, t^2,\, t,\, t^{-1},\, t^{-1},\, t^{-2},\, t^{-3}\big), \qquad t \in \mathbb{G}_m.$$

For a generic f_3 as in Section 3, the 1-PS limit is

$$\begin{aligned} \phi_3 := \lim_{t\to 0} \lambda_3(t)\cdot f_3 = a_1\, x_1x_3^2 + a_2\, x_1x_3x_4 + a_3\, x_1x_4^2 \\ + a_4\, x_2^2x_5 + a_5\, x_0x_5^2 + a_6\, x_1x_2x_6 + a_7\, x_0x_3x_6 + a_8\, x_0x_4x_6. \end{aligned}$$

H **and** $C_G(H)$**.** Let $H = \lambda_3(\mathbb{G}_m)$. The multiplicities of the diagonal weights on $\langle x_0,\dots,x_6\rangle$ are 1 on x_0,x_1,x_2,x_5,x_6 and 2 on $\langle x_3,x_4\rangle$; hence,

$$C_G(H) = \left\{ \begin{array}{l} \mathrm{diag}(\alpha_0,\alpha_1,\alpha_2) \oplus A \oplus \mathrm{diag}(\alpha_5,\alpha_6) : \\ \quad \alpha_i \in \mathbb{G}_m,\ A \in \mathrm{GL}(2),\ \alpha_0\alpha_1\alpha_2\, \det(A)\, \alpha_5\alpha_6 = 1 \end{array} \right\} \cong \mathrm{SL}(2)\times\mathbb{G}_m^5.$$

Each monomial of ϕ_3 has H-weight 0, so ϕ_3 is H-fixed.

Polystability (Luna + Casimiro–Florentino). By Luna's slice/centralizer reduction, the closedness of the SL(7)-orbit of ϕ_3 is equivalent to polystability for the $C_G(H)$-action on the H-fixed subspace. After conjugating inside the SL(2)-block, any $\lambda \in Y(C_G(H))$ may be chosen with weights

$$\mathrm{wt}(x_0,\dots,x_6) = (a_0,\, a_1,\, a_2,\, c{+}n,\, c{-}n,\, a_5,\, a_6), \qquad S := a_0{+}a_1{+}a_2{+}2c{+}a_5{+}a_6 = 0$$

(as in Convention 4.6. Let w_i be the λ-weight of the i-th monomial of ϕ_3. Then

$$\begin{aligned} &w_1 = a_1 + 2(c+n), && w_2 = a_1 + 2c, && w_3 = a_1 + 2(c-n), \\ &w_4 = 2a_2 + a_5, && w_5 = a_0 + 2a_5, && w_6 = a_1 + a_2 + a_6, \\ &w_7 = a_0 + (c+n) + a_6, && w_8 = a_0 + (c-n) + a_6. \end{aligned}$$

A direct computation yields the positive linear identity

$$w_1 + w_2 + 2w_3 + 2w_4 + 2w_5 + 2w_6 + 3w_7 + w_8 \;=\; 6S. \tag{3}$$

If $\lambda \in \Lambda_{\phi_3}$, then all $w_i \geq 0$ and $S = 0$; by (3) they must all vanish. Solving gives

$$n = 0, \quad a_1 = -2c, \quad a_5 = -2a_2, \quad a_0 = 4a_2, \quad a_6 = 2c - a_2, \quad a_2 = -c.$$

Thus, with $k \in \mathbb{Z}$,

$$\mu_k(t) := \operatorname{diag}\bigl(t^{4k},\, t^{2k},\, t^{k},\, t^{-k},\, t^{-}k,\, t^{-2k},\, t^{-3k}\bigr), \qquad \Lambda_{\phi_3} = \{\mu_k \mid k \in \mathbb{Z}\} \cup \{0\}.$$

Therefore Λ_{ϕ_3} is symmetric, and by the Casimiro–Florentino criterion ϕ_3 is polystable; in particular, $\mathrm{SL}(7) \cdot \phi_3$ is closed.

Normal form and component dimension. On the H-fixed slice, the eight weight-zero monomials are

$$x_1x_3^2,\; x_1x_3x_4,\; x_1x_4^2,\; x_2^2x_5,\; x_0x_5^2,\; x_1x_2x_6,\; x_0x_3x_6,\; x_0x_4x_6,$$

Therefore, W^H denotes their span. i.e.

$$W^H = \underbrace{\langle x_1\rangle \otimes \mathrm{Sym}^2\langle x_3, x_4\rangle}_{\text{(I)}} \oplus \underbrace{\mathrm{Sym}^2\langle x_2\rangle \otimes x_5}_{\text{(II)}} \oplus \underbrace{x_0 \otimes \mathrm{Sym}^2\langle x_5\rangle}_{\text{(III)}} \oplus \underbrace{x_0 \otimes \langle x_3, x_4\rangle \otimes x_6}_{\text{(IV)}}.$$

The centralizer is

$$C_G(H) \cong \mathrm{SL}(2) \times \mathbb{G}_m^5,$$

acting by $\mathrm{SL}(2)$ on $\langle x_3, x_4\rangle$ and by a diagonal torus on $\langle x_0, x_1, x_2, x_5, x_6\rangle$ (subject to the product-one condition).

(I) binary quadratic on $\langle x_3, x_4\rangle$. Write the x_1-part as a binary quadratic $Q = a_1x_3^2 + a_2x_3x_4 + a_3x_4^2 \in \mathrm{Sym}^2\langle x_3, x_4\rangle$. As $\Delta(Q)$ is $\mathrm{SL}(2)$-invariant while rescaling x_1 scales Q (and hence Δ) homogenously, for a generic (nondegenerate) Q, there exists $A \in \mathrm{SL}(2)$ and a rescaling of x_1 such that $Q \sim x_3^2 + x_3x_4 + x_4^2$. Thus, the block (I) is fixed to $x_1x_3^2 + x_1x_3x_4 + x_1x_4^2$.

(II)(III)(IV) torus normalizations. Let $T' = \{\operatorname{diag}(\mu_0, \mu_1, \mu_2) \oplus I_2 \oplus \operatorname{diag}(\mu_5, \mu_6) :\ \mu_0\mu_1\mu_2\mu_5\mu_6 = 1\}$. Under T', the coefficients transform by the characters determined by exponent vectors: $x_2^2x_5$ by $\mu_2^2\mu_5$, $x_0x_5^2$ by $\mu_0\mu_5^2$, $x_1x_2x_6$ by $\mu_1\mu_2\mu_6$, while $x_0x_3x_6$ and $x_0x_4x_6$ both by $\mu_0\mu_6$. Using $\mu_0, \mu_1, \mu_2, \mu_5, \mu_6$ (together with an overall scalar), we set $x_2^2x_5$, $x_0x_5^2$, and $x_1x_2x_6$ to have coefficient 1.

The remaining ratio. The pair $(x_0x_3x_6,\ x_0x_4x_6)$ is scaled by the same torus character $\mu_0\mu_6$, and the stabilizer in SL(2) of $x_3^2 + x_3x_4 + x_4^2$ is finite; hence, only the ratio remains. This yields the one-parameter normal form

$$\phi_3^{\mathrm{nf}}(\alpha) = x_1x_3^2+x_1x_3x_4+x_1x_4^2+x_2^2x_5+x_0x_5^2+x_1x_2x_6+x_0x_3x_6+\alpha\, x_0x_4x_6, \quad \alpha \in \mathbb{C}^\times.$$

For general α the residual stabilizer is finite, and the corresponding component Φ_3 is one-dimensional.

4.4 Case $k = 4$

1-PS limit.

$$\lambda_4(t) = \mathrm{diag}\big(t^3,\, t^2,\, t,\, 1,\, t^{-1},\, t^{-2},\, t^{-3}\big)\,, \quad t \in \mathbb{G}_m.$$

For a generic f_4 as in Section 3, the 1-PS limit is

$$\begin{aligned}\phi_4 &:= \lim_{t\to 0} \lambda_4(t)\cdot f_4\\ &= a_1\, x_3^3 + a_2\, x_2x_3x_4 + a_3\, x_1x_4^2 + a_4\, x_2^2x_5 + a_5\, x_1x_3x_5 + a_6\, x_0x_4x_5 + a_7\, x_1x_2x_6 + a_8\, x_0x_3x_6.\end{aligned}$$

H **and** *$C_G(H)$***.** Let $H = \lambda_4(\mathbb{G}_m)$. The diagonal weights on $\langle x_0,\dots,x_6\rangle$ are pairwise distinct; hence, $C_G(H) = T$ (the maximal diagonal torus). Each monomial of ϕ_4 has H-weight 0, so $\phi_4 \in W^H$.

Polystability (Luna + convex-hull). By Luna's criterion, closedness of the $SL(7)$-orbit of ϕ_4 is equivalent to closedness of the T-orbit in W^H. By the convex-hull criterion (Theorem 4.1), it suffices to check that 0 is an interior point of $\mathrm{Conv}(\mathrm{Supp}(\phi_4)) \subset X(T)_{\mathbb{R}} \cong \mathbb{R}^7/\mathbb{R}(1,\dots,1)$, which holds because the exponent vectors satisfy

$$\begin{aligned}&(0,0,0,3,0,0,0)+(0,0,1,1,1,0,0)+(0,1,0,0,2,0,0)+(0,0,2,0,0,1,0)\\ &\quad+2(0,1,0,1,0,1,0)+6(1,0,0,0,1,1,0)+6(0,1,1,0,0,0,1)\\ &\quad+3(1,0,0,1,0,0,1) \;=\; 9(1,1,1,1,1,1,1).\end{aligned} \tag{4}$$

Hence, ϕ_4 is polystable and $SL(7)\cdot\phi_4$ is closed.

Normal form and component dimension. A diagonal scaling $\mathrm{diag}(\mu_0,\dots,\mu_6) \in T$ with $\prod \mu_i = 1$, together with an overall scalar, acts on the eight coefficients via the characters determined by the exponent vectors above; we may normalize seven of them to 1. Thus, a normal form is

$$\phi_4^{\mathrm{nf}}(\alpha) = x_3^3+x_2x_3x_4+x_1x_4^2+x_2^2x_5+x_1x_3x_5+x_0x_4x_5+x_1x_2x_6+\alpha\, x_0x_3x_6, \qquad \alpha \in \mathbb{C}^*.$$

The residual T-stabilizer is finite; hence, the corresponding component Φ_4 of the moduli is one-dimensional.

4.5 Case $k = 5$

1-PS limit. Set

$$\lambda_5(t) = \operatorname{diag}\bigl(t^4,\, t^2,\, t,\, 1,\, t^{-1},\, t^{-2},\, t^{-4}\bigr), \qquad t \in \mathbb{G}_m.$$

For a generic f_5 as in Section 3, the 1-PS limit is

$$\phi_5 := \lim_{t\to 0} \lambda_5(t)\cdot f_5 = a_1 x_3^3 + a_2 x_2 x_3 x_4 + a_3 x_1 x_4^2 + a_4 x_2^2 x_5 + a_5 x_1 x_3 x_5 + a_6 x_0 x_5^2 + a_7 x_1^2 x_6 + a_8 x_0 x_3 x_6.$$

H and $C_G(H)$. Let $H = \lambda_5(\mathbb{G}_m)$. The diagonal weights on $\langle x_0, \dots, x_6\rangle$ are pairwise distinct; hence, $C_G(H) = T$ (the maximal diagonal torus). Each monomial of ϕ_5 has H-weight 0, so $\phi_5 \in W^H$.

Polystability (Luna + convex-hull). By Luna's criterion, closedness of the $SL(7)$-orbit of ϕ_5 is equivalent to closedness of the T-orbit in the H-fixed subspace. By the convex-hull criterion (Theorem 4.1), it suffices to check that 0 lies in the interior of $\mathrm{Conv}(\mathrm{Supp}(\phi_5)) \subset X(T)_{\mathbb{R}} \cong \mathbb{R}^7/\mathbb{R}(1, \dots, 1)$. This holds because the exponent vectors of the eight monomials satisfy

$$\begin{aligned}
&(0,0,0,3,0,0,0) + (0,0,1,1,1,0,0) \\
&\quad + 10(0,1,0,0,2,0,0) + 10(0,0,2,0,0,1,0) \\
&\quad + (0,1,0,1,0,1,0) + 5(1,0,0,0,0,2,0) \\
&\quad + 5(0,2,0,0,0,0,1) + 16(1,0,0,1,0,0,1) \\
&= 21(1,1,1,1,1,1,1),
\end{aligned} \tag{5}$$

Hence, ϕ_5 is polystable and $SL(7)\cdot\phi_5$ is closed.

Normal form and component dimension. A diagonal scaling $\operatorname{diag}(\mu_0, \dots, \mu_6) \in T$ with $\prod \mu_i = 1$, together with projective rescaling, acts on the eight coefficients via the characters determined by the exponent vectors in (5); we can normalize seven of them to 1. A convenient normal form is

$$\phi_5^{\mathrm{nf}}(\alpha) = x_3^3 + x_2 x_3 x_4 + x_1 x_4^2 + x_2^2 x_5 + x_1 x_3 x_5 + x_0 x_5^2 + x_1^2 x_6 + \alpha\, x_0 x_3 x_6, \qquad \alpha \in \mathbb{C}^\times.$$

The residual T-stabilizer is finite; hence, the corresponding component Φ_5 of the moduli is one-dimensional.

4.6 Case $k = 6$

1-PS limit. Set

$$\lambda_6(t) = \operatorname{diag}\bigl(t^6,\, t^4,\, t^2,\, t,\, t^{-2},\, t^{-3},\, t^{-8}\bigr), \qquad t \in \mathbb{G}_m.$$

For a generic f_6 as in Section 3, the 1-PS limit is

$$\phi_6 := \lim_{t\to 0} \lambda_6(t)\cdot f_6 = a_1\, x_3^2 x_4 + a_2\, x_1 x_4^2 + a_3\, x_2 x_3 x_5 + a_4\, x_0 x_5^2 + a_5\, x_1^2 x_6 + a_6\, x_0 x_2 x_6.$$

H and $C_G(H)$. Let $H = \lambda_6(\mathbb{G}_m)$. The diagonal weights on $\langle x_0, \ldots, x_6 \rangle$ are pairwise distinct; hence, $C_G(H) = T$ (the maximal diagonal torus). Each monomial of ϕ_6 has H-weight 0, so ϕ_6 lies in the H-fixed subspace.

Polystability (Luna + convex-hull). By Luna's criterion, closedness of the SL(7)-orbit of ϕ_6 is equivalent to closedness of the T-orbit in the H-fixed subspace. By the convex-hull criterion (Theorem 4.1), it suffices to check that 0 is an interior point of $\mathrm{Conv}(\mathrm{Supp}(\phi_6)) \subset X(T)_{\mathbb{R}} \cong \mathbb{R}^7/\mathbb{R}(1, \ldots, 1)$. This holds because

$$\begin{aligned}&(0,0,0,2,1,0,0) + (0,1,0,0,2,0,0) + (0,0,1,1,0,1,0)\\&\quad + (1,0,0,0,0,2,0) + (0,2,0,0,0,0,1) + 2(1,0,1,0,0,0,1)\\&= 3(1,1,1,1,1,1,1),\end{aligned} \tag{6}$$

written in terms of the exponent vectors of the six monomials of ϕ_6. Hence, ϕ_6 is polystable and $\mathrm{SL}(7) \cdot \phi_6$ is closed.

Normal form and component dimension. A diagonal scaling $\mathrm{diag}(\mu_0, \ldots, \mu_6) \in T$ with $\prod \mu_i = 1$, together with an overall scalar, acts on the six coefficients via the characters determined by the exponent vectors in (6); we may normalize all six coefficients to 1 simultaneously. Thus, a normal form is

$$\phi_6^{\mathrm{nf}} = x_3^2 x_4 + x_1 x_4^2 + x_2 x_3 x_5 + x_0 x_5^2 + x_1^2 x_6 + x_0 x_2 x_6.$$

The residual T-stabilizer is finite; hence, the corresponding component Φ_6 of the moduli is zero-dimensional.

4.7 Case $k = 7$

1-PS limit. Set

$$\lambda_7(t) = \mathrm{diag}\big(t^5,\, t^3,\, t^2,\, t,\, t^{-1},\, t^{-4},\, t^{-6}\big), \quad t \in \mathbb{G}_m.$$

For a generic f_7 as in Section 3, the 1-PS limit is

$$\phi_7 := \lim_{t\to 0} \lambda_7(t) \cdot f_7 = a_1\, x_2 x_4^2 + a_2\, x_2^2 x_5 + a_3\, x_1 x_3 x_5 + a_4\, x_0 x_4 x_5 + a_5\, x_1^2 x_6 + a_6\, x_0 x_3 x_6.$$

H and $C_G(H)$. Let $H = \lambda_7(\mathbb{G}_m)$. The diagonal weights on $\langle x_0, \ldots, x_6 \rangle$ are pairwise distinct; hence, $C_G(H) = T$ (the maximal diagonal torus). Each monomial of ϕ_7 has H-weight 0, so $\phi_7 \in W^H$.

Polystability (Luna + convex-hull). Based on Luna's criterion, closedness of the $SL(7)$-orbit of ϕ_7 is equivalent to closedness of the T-orbit in the H-fixed subspace. By Theorem 4.1, it suffices to check that 0 is an interior point of

$\mathrm{Conv}(\mathrm{Supp}(\phi_7)) \subset X(T)_{\mathbb{R}} \cong \mathbb{R}^7/\mathbb{R}(1,\ldots,1)$. This holds because the exponent vectors of the six monomials of ϕ_7 satisfy the positive relation

$$\begin{aligned}&(0,0,1,0,2,0,0)+(0,0,2,0,0,1,0)+(0,1,0,1,0,1,0)\\ &\qquad+(1,0,0,0,1,1,0)+(0,2,0,0,0,0,1)+2(1,0,0,1,0,0,1)\\ &=3(1,1,1,1,1,1,1).\end{aligned} \tag{7}$$

Hence, ϕ_7 is polystable and $SL(7)\cdot\phi_7$ is closed.

Normal form and component dimension. A diagonal scaling $\mathrm{diag}(\mu_0,\ldots,\mu_6) \in T$ with $\prod \mu_i = 1$, together with an overall scalar, acts on the six coefficients via the characters determined by the exponent vectors in (7); hence, we may normalize all six coefficients to 1 simultaneously. A normal form is therefore

$$\phi_7^{\mathrm{nf}} = x_2x_4^2 + x_2^2x_5 + x_1x_3x_5 + x_0x_4x_5 + x_1^2x_6 + x_0x_3x_6.$$

The residual T-stabilizer is finite; hence, the corresponding component Φ_7 of the moduli is zero-dimensional.

4.8 Case $k = 8$

1-PS limit. Set

$$\lambda_8(t) = \mathrm{diag}\big(t^4,\, t,\, t,\, 1,\, t^{-2},\, t^{-2},\, t^{-2}\big), \qquad t\in\mathbb{G}_m.$$

For a generic f_8 as in Section 3, the 1-PS limit is

$$\begin{aligned}\phi_8 := \lim_{t\to 0}\lambda_8(t)\cdot f_8 &= a_1x_3^3 + a_2x_1^2x_4 + a_3x_1x_2x_4 + a_4x_2^2x_4 + a_6x_1^2x_5 + a_7x_1x_2x_5 + a_8x_2^2x_5\\ &\quad + a_{11}x_1^2x_6 + a_{12}x_1x_2x_6 + a_{13}x_2^2x_6 + a_5x_0x_4^2 + a_9x_0x_4x_5 + a_{14}x_0x_4x_6\\ &\quad + a_{10}x_0x_5^2 + a_{15}x_0x_5x_6 + a_{16}x_0x_6^2.\end{aligned}$$

***H* and $C_G(H)$.** Let $H = \lambda_8(\mathbb{G}_m)$. The weights on $\langle x_0,\ldots,x_6\rangle$ are $(4,1,1,0,-2,-2,-2)$ with multiplicities $(1,2,1,3)$; hence,

$$C_G(H) = \Big\{\mathrm{diag}(\alpha)\oplus A\oplus \mathrm{diag}(\beta)\oplus B:\ \alpha,\beta\in\mathbb{G}_m,\ A\in\mathrm{GL}(2),\ B\in\mathrm{GL}(3),\ \alpha\beta\det(A)\det(B)=1\Big\}.$$

Thus $C_G(H) \cong \big(\mathbb{G}_m\times\mathrm{GL}(2)\times\mathbb{G}_m\times\mathrm{GL}(3)\big)\cap\mathrm{SL}(7)$ and $\dim C_G(H) = 14$. Every monomial of ϕ_8 has H-weight 0; hence, $\phi_8\in W^H$.

Polystability (Luna + Casimiro–Florentino). Based on Luna's reduction, the closedness of the SL(7)-orbit of ϕ_8 is equivalent to polystability for the $C_G(H)$-action on W^H. After conjugating inside the GL(2)- and GL(3)-blocks, any $\lambda\in Y(C_G(H))$ may be taken with

$$\mathrm{wt}(x_0,\ldots,x_6) = (\alpha,\, s+u,\, s-u,\, \beta,\, \gamma+v_1,\, \gamma+v_2,\, \gamma+v_3),$$

where $\alpha, \beta, s, u, \gamma, v_i \in \mathbb{Z}$, $v_1 + v_2 + v_3 = 0$, and

$$S := \alpha + \beta + 2s + 3\gamma = 0$$

is the SL-constraint fixed in Convention 4.6. Let $w(\cdot)$ be the weight of a monomial. A direct computation yields the positive linear identity

$$\sum_{j=4}^{6}\Big(w(x_1^2x_j) + w(x_2^2x_j) + 2\,w(x_1x_2x_j)\Big) + 3w(x_0x_4^2) + 3w(x_0x_5^2) + 3w(x_0x_6^2)$$
$$+w(x_0x_4x_5) + w(x_0x_4x_6) + w(x_0x_5x_6) + 4\,w(x_3^3) = 12S. \tag{8}$$

If $\lambda \in \Lambda_{\phi_8}$, then all the above weights are ≥ 0 and $S = 0$; therefore, by (8) they must all vanish. Solving gives

$$\beta = 0, \quad v_1 = v_2 = v_3 = 0, \quad u = 0, \quad 2s + \gamma = 0, \quad \alpha + 2\gamma = 0.$$

Putting $s = k \in \mathbb{Z}$, we obtain

$$\mu_k(t) = \mathrm{diag}\big(t^{4k},\, t^k,\, t^k,\, 1,\, t^{-2k},\, t^{-2k},\, t^{-2k}\big)\,, \qquad \Lambda_{\phi_8} = \{\mu_k \mid k \in \mathbb{Z}\} \cup \{0\},$$

which is symmetric. Hence, by the Casimiro–Florentino criterion, ϕ_8 is polystable.

Normal form and component dimension. On the H-fixed subspace

$$W^H = \underbrace{\mathrm{Sym}^2\langle x_1, x_2\rangle \otimes \langle x_4, x_5, x_6\rangle}_{\text{(I)}} \;\oplus\; \underbrace{x_0 \otimes \mathrm{Sym}^2\langle x_4, x_5, x_6\rangle}_{\text{(II)}} \;\oplus \underbrace{\langle x_3^3\rangle}_{\text{(III)}},$$

use $\mathrm{GL}(3)$ on $\langle x_4, x_5, x_6\rangle$ to diagonalize the quadratic, then the left/right actions on $\mathrm{Sym}^2\langle x_1, x_2\rangle \otimes \langle x_4, x_5, x_6\rangle$ to diagonalize the 3×3 block (SVD-type reduction under left $\mathrm{Sym}^2\mathrm{GL}(2)$ and right $\mathrm{O}(U, q)$, here $U = \langle x_4, x_5, x_6\rangle$ and $q = x_4^2 + x_5^3 + x_6^2$) and normalize one diagonal entry to 1; the remaining two appear as parameters ρ, σ. A normal form is

$$\phi_8^{\mathrm{nf}}(\rho, \sigma) = x_3^3 + x_0x_4^2 + x_0x_5^2 + x_0x_6^2 + x_1^2x_4 + \rho\, x_1x_2x_5 + \sigma\, x_2^2x_6, \qquad (\rho, \sigma) \in (\mathbb{C}^\times)^2.$$

As $\dim W^H = 16$ and the effective action has dimension 13 (after projectivizing), the closed component has dimension $16 - 13 - 1 = 2$.

Remark 4.7. *Here $\mathrm{O}(U, q)$ denotes the orthogonal group of the quadratic space (U, q), i.e. $\mathrm{O}(U, q) = \{\, g \in \mathrm{GL}(U) \mid q(gu) = q(u)\ \forall u \in U\}$. In the chosen basis $U = \langle x_4, x_5, x_6\rangle$ with $q = x_4^2 + x_5^2 + x_6^2$, this is $\{\, B \in \mathrm{GL}_3 \mid B^{\mathsf{T}}B = I_3\,\}$.*

The residual T-stabilizer is finite; hence, the corresponding component Φ_8 of the moduli is two-dimensional.

4.9 Case $k = 9$

1-PS limit. Set

$$\lambda_9(t) = \operatorname{diag}\bigl(t^2,\, t^2,\, 1,\, 1,\, t^{-1},\, t^{-1},\, t^{-2}\bigr), \qquad t \in \mathbb{G}_m.$$

For a generic f_9 as in Section 3, the 1-PS limit is

$$\begin{aligned}\phi_9 := \lim_{t\to 0} \lambda_9(t)\cdot f_9 = {} & a_1x_2^3 + a_2x_2^2x_3 + a_3x_2x_3^2 + a_4x_3^3 \\ & + a_5x_0x_4^2 + a_6x_1x_4^2 + a_7x_0x_4x_5 + a_8x_1x_4x_5 \\ & + a_9x_0x_5^2 + a_{10}x_1x_5^2 \\ & + a_{11}x_0x_2x_6 + a_{12}x_1x_2x_6 + a_{13}x_0x_3x_6 + a_{14}x_1x_3x_6.\end{aligned}$$

(All monomials have H-weight 0.)

H and $C_G(H)$. Let $H = \lambda_9(\mathbb{G}_m)$. The H-weights on $\langle x_0, \dots, x_6\rangle$ are $(2, 2, 0, 0, -1, -1, -2)$ with block decomposition $\langle x_0, x_1\rangle$, $\langle x_2, x_3\rangle$, $\langle x_4, x_5\rangle$, $\langle x_6\rangle$. Hence,

$$\begin{aligned}C_G(H) &= \Bigl\{\, A\oplus B\oplus C\oplus\gamma :\ A, B, C \in \mathrm{GL}(2),\ \gamma \in \mathbb{G}_m,\ \det(A)\det(B)\det(C)\gamma = 1 \,\Bigr\} \\ &\cong\ \bigl(\mathrm{GL}(2)^3 \times \mathbb{G}_m\bigr) \cap \mathrm{SL}(7).\end{aligned}$$

Polystability (Luna + Casimiro–Florentino). By Luna's reduction, the closedness of the SL(7)-orbit of ϕ_9 is equivalent to polystability for the $C_G(H)$-action on the H-fixed locus. After conjugating inside each GL(2)-block, any $\lambda \in Y\bigl(C_G(H)\bigr)$ may be taken with

$$\mathrm{wt}(x_0, \dots, x_6) = (a{+}u,\ a{-}u,\ b{+}v,\ b{-}v,\ c{+}w,\ c{-}w,\ d), \qquad S := 2a{+}2b{+}2c{+}d = 0.$$

A direct computation yields the positive linear identity

$$\begin{aligned}&\bigl[w(x_2^3) + w(x_2^2x_3) + w(x_2x_3^2) + w(x_3^3)\bigr] \\ &\quad + 2\bigl[w(x_0x_4^2) + w(x_0x_4x_5) + w(x_0x_5^2) + w(x_1x_4^2) + w(x_1x_4x_5) + w(x_1x_5^2)\bigr] \\ &\quad + 3\bigl[w(x_0x_2x_6) + w(x_1x_2x_6) + w(x_0x_3x_6) + w(x_1x_3x_6)\bigr] = 12\,S. \qquad (9)\end{aligned}$$

If $\lambda \in \Lambda_{\phi_9}$, then all the above weights are ≥ 0 and $S = 0$; hence, by (9) they all vanish. Solving gives

$$b = v = u = w = 0, \qquad a + 2c = 0, \qquad a + d = 0.$$

Writing $a = 2k$ with $k \in \mathbb{Z}$ we obtain

$$\mu_k(t) := \operatorname{diag}\bigl(t^{2k},\, t^{2k},\, 1,\, 1,\, t^{-k},\, t^{-k},\, t^{-2k}\bigr), \qquad \Lambda_{\phi_9} = \{\mu_k \mid k \in \mathbb{Z}\} \cup \{0\}.$$

Thus Λ_{ϕ_9} is symmetric, and by the Casimiro–Florentino criterion ϕ_9 is polystable; in particular $\mathrm{SL}(7)\cdot\phi_9$ is closed.

Normal form and component dimension. From now on, we normalize coefficients under the $C_G(H)$-action. Decompose

$$W^H = \underbrace{\mathrm{Sym}^3\langle x_2, x_3\rangle}_{\text{(I)}} \oplus \underbrace{\langle x_0, x_1\rangle \otimes \mathrm{Sym}^2\langle x_4, x_5\rangle}_{\text{(II)}} \oplus \underbrace{\langle x_0, x_1\rangle \otimes \langle x_2, x_3\rangle \otimes \langle x_6\rangle}_{\text{(III)}}.$$

(I) Binary cubic $\mathrm{Sym}^3\langle x_2, x_3\rangle$ A general element is $\mathrm{GL}(\langle x_2, x_3\rangle)$-equivalent (up to overall scaling) to

$$x_2^2 x_3 + \tau\, x_2 x_3^2 \qquad (\tau \in \mathbb{C}^\times),$$

leaving one parameter τ.

(II) $\langle x_0, x_1\rangle \otimes \mathrm{Sym}^2\langle x_4, x_5\rangle$ **(2×3 block)** Via the right $\mathrm{GL}(\langle x_4, x_5\rangle)$ (through Sym^2) diagonalize a reference quadratic to $q = x_4^2 + x_5^2$; the residual right group is $O(U, q)$ on $U = \langle x_4, x_5\rangle$. Using the left $\mathrm{GL}(\langle x_0, x_1\rangle)$ together with this right orthogonal action (an SVD-type reduction), eliminate the x_4x_5 cross term and equalize the x_4^2 entries. After central torus/projective scalings,

$$x_0(x_4^2 + \rho\, x_5^2) + x_1(x_4^2 + x_5^2), \qquad \rho \in \mathbb{C}^\times.$$

(III) $\langle x_0, x_1\rangle \otimes \langle x_2, x_3\rangle \otimes \langle x_6\rangle$ **(2×2 block)** With $\mathrm{GL}(\langle x_0, x_1\rangle)$, $\mathrm{GL}(\langle x_2, x_3\rangle)$ (respecting the choice in (I)), and scaling x_6, we diagonalize to

$$x_0x_2x_6 + x_1x_3x_6,$$

and normalize the coefficients to 1.

Combining the three steps yields the normal form

$$\varphi_9^{\mathrm{nf}}(\tau, \rho) = x_2^2x_3 + \tau\, x_2x_3^2 + x_0x_4^2 + \rho\, x_0x_5^2 + x_1x_4^2 + x_1x_5^2 + x_0x_2x_6 + x_1x_3x_6, (\tau, \rho) \in (\mathbb{C}^\times)^2.$$

As $\dim W^H = 16$ and $\dim C_G(H) = 14$, the effective action has dimension 13; after projectivizing, we obtain $16 - 13 - 1 = 2$. The residual T-stabilizer is finite; hence, the corresponding component Φ_9 of the moduli is two-dimensional.

4.10 Case $k = 10$

1-PS limit. Set

$$\lambda_{10}(t) = \mathrm{diag}\big(t^2,\, t,\, 1,\, 1,\, t^{-1},\, t^{-1},\, t^{-1}\big), \qquad t \in \mathbb{G}_m.$$

For a generic f_{10} as in Section 3, the 1-PS limit is

$$\begin{aligned}\phi_{10} := \lim_{t\to 0} \lambda_{10}(t) \cdot f_{10} = {} & a_1x_2^3 + a_2x_2^2x_3 + a_3x_2x_3^2 + a_4x_3^3 \\ & + a_5x_1x_2x_4 + a_6x_1x_3x_4 + a_8x_1x_2x_5 + a_9x_1x_3x_5 \\ & + a_{12}x_1x_2x_6 + a_{13}x_1x_3x_6 \\ & + a_7x_0x_4^2 + a_{10}x_0x_4x_5 + a_{11}x_0x_5^2 + a_{14}x_0x_4x_6 + a_{15}x_0x_5x_6 + a_{16}x_0x_6^2.\end{aligned}$$

H and $C_G(H)$. Let $H = \lambda_{10}(\mathbb{G}_m)$. The H-weights on $\langle x_0, \ldots, x_6 \rangle$ are $(2, 1, 0, 0, -1, -1, -1)$ with block multiplicities $(1, 1, 2, 3)$; hence,

$$\begin{aligned} C_G(H) &= \Big\{ \operatorname{diag}(\alpha) \oplus \operatorname{diag}(\beta) \oplus A \oplus B : \ \alpha, \beta \in \mathbb{G}_m,\ A \in \mathrm{GL}(2),\ B \in \mathrm{GL}(3),\ \alpha\beta \det(A)\det(B) = 1 \Big\} \\ &\cong \big(\mathbb{G}_m \times \mathbb{G}_m \times \mathrm{GL}(2) \times \mathrm{GL}(3)\big) \cap \mathrm{SL}(7). \end{aligned} \tag{10}$$

Every monomial of ϕ_{10} has H-weight 0, so $\phi_{10} \in W^H$.

Polystability (Luna + Casimiro–Florentino). By Luna's reduction, the closedness of the SL(7)-orbit of ϕ_{10} is equivalent to polystability for the $C_G(H)$-action on W^H. After conjugating within the GL(2)- and GL(3)-blocks, any $\lambda \in Y\big(C_G(H)\big)$ may be taken as

$$\mathrm{wt}(x_0, \ldots, x_6) = (\alpha,\ \beta,\ s + u,\ s - u,\ \gamma + v_1,\ \gamma + v_2,\ \gamma + v_3),$$

where $\alpha, \beta, s, u, \gamma, v_i \in \mathbb{Z}$ with $v_1 + v_2 + v_3 = 0$, and by Convention 4.6 the SL-constraint is

$$S := \alpha + \beta + 2s + 3\gamma = 0.$$

A direct computation gives the positive linear identity

$$\begin{aligned} &\big[w(x_2^3) + w(x_2^2 x_3) + w(x_2 x_3^2) + w(x_3^3)\big] \\ &\quad + 2\big[w(x_0 x_4^2) + w(x_0 x_4 x_5) + w(x_0 x_5^2) + w(x_0 x_4 x_6) + w(x_0 x_5 x_6) + w(x_0 x_6^2)\big] \\ &\quad + 2\big[w(x_1 x_2 x_4) + w(x_1 x_3 x_4) + w(x_1 x_2 x_5) + w(x_1 x_3 x_5) + w(x_1 x_2 x_6) + w(x_1 x_3 x_6)\big] \ = \ 12\,S. \end{aligned} \tag{11}$$

If $\lambda \in \Lambda_{\phi_{10}}$, then all the weights above are ≥ 0 and $S = 0$; hence, by (11), they all vanish. Solving yields

$$\beta = -\gamma, \qquad s = 0, \qquad u = 0, \qquad v_1 = v_2 = v_3 = 0, \qquad \alpha = -2\gamma.$$

Writing $\gamma = -k$ with $k \in \mathbb{Z}$ we obtain

$$\mu_k(t) := \operatorname{diag}\big(t^{2k},\ t^k,\ 1,\ 1,\ t^{-k},\ t^{-k},\ t^{-k}\big), \qquad \Lambda_{\phi_{10}} = \{\mu_k \mid k \in \mathbb{Z}\} \cup \{0\}.$$

Thus $\Lambda_{\phi_{10}}$ is symmetric, and by the Casimiro–Florentino criterion, ϕ_{10} is polystable; in particular $\mathrm{SL}(7) \cdot \phi_{10}$ is closed.

Normal form and component dimension. We decompose the H-fixed locus as

$$W^H = \underbrace{\mathrm{Sym}^3\langle x_2, x_3\rangle}_{\text{(I)}} \ \oplus\ \underbrace{x_1 \otimes \langle x_2, x_3\rangle \otimes \langle x_4, x_5, x_6\rangle}_{\text{(II)}} \ \oplus\ \underbrace{x_0 \otimes \mathrm{Sym}^2\langle x_4, x_5, x_6\rangle}_{\text{(III)}}.$$

We normalize coefficients block by block under the $C_G(H)$-action.

(I) Binary cubic $\mathrm{Sym}^3\langle x_2, x_3\rangle$ Using $\mathrm{GL}(\langle x_2, x_3\rangle)$ and overall scaling, a general binary cubic is equivalent to

$$x_2^2x_3 + \tau\, x_2x_3^2 \qquad (\tau \in \mathbb{C}^\times),$$

which fixes one modulus τ.

(III) Three-variable quadratic $x_0 \otimes \mathrm{Sym}^2\langle x_4, x_5, x_6\rangle$ Acting by $\mathrm{GL}(\langle x_4, x_5, x_6\rangle)$ on the Sym^2-representation, a nondegenerate quadratic form can be diagonalized to the identity. Using the central torus and projective scaling, we normalize the coefficients to 1:

$$x_0x_4^2 + x_0x_5^2 + x_0x_6^2.$$

After this step, the residual right action is the isometry group of the diagonal form (orthogonal group).

(II) The 2×3 **block** $x_1 \otimes \langle x_2, x_3\rangle \otimes \langle x_4, x_5, x_6\rangle$ The coefficients in this block can be arranged as a 2×3 matrix M. After (III), the right group is the isometry group of the diagonal quadratic on $\langle x_4, x_5, x_6\rangle$, and the left group is $\mathrm{GL}(\langle x_2, x_3\rangle)$. For a general element (rank $M = 2$), a simultaneous (left $\mathrm{GL}(2)$, right isometry) SVD-type reduction yields

$$M \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \end{pmatrix} \iff x_1x_2x_4 + \rho\, x_1x_3x_5,$$

where $\rho \in \mathbb{C}^\times$. Here, the cross terms are eliminated by the right isometry, column choices are coordinated by the left action, and the remaining nonzero entries are scaled to the displayed normal form.

Combining (I)-(III), the convenient normal form for Case $k = 10$ is

$$\varphi_{10}^{\mathrm{nf}}(\tau, \rho) = x_2^2x_3 + \tau\, x_2x_3^2 + x_0x_4^2 + x_0x_5^2 + x_0x_6^2 + x_1x_2x_4 + \rho\, x_1x_3x_5, \; (\tau, \rho) \in (\mathbb{C}^\times)^2.$$

As $\dim W^H = 16$ and $\dim C_G(H) = 14$, the effective action has dimension 13; after projectivizing, we obtain $16 - 13 - 1 = 2$. *The residual T-stabilizer is finite; hence, the corresponding component Φ_{10} of the moduli is two-dimensional.*

4.11 Case $k = 11$

1-PS limit. Set

$$\lambda_{11}(t) = \mathrm{diag}\big(t^2, 1, 1, 1, 1, t^{-1}, t^{-1}\big), \qquad t \in \mathbb{G}_m.$$

For a generic f_{11} as in Section 3, the 1-PS limit is

$$\begin{aligned} \phi_{11} := \lim_{t \to 0} \lambda_{11}(t) \cdot f_{11} = {} & a_1x_1^3 + a_2x_1^2x_2 + a_3x_1x_2^2 + a_4x_2^3 + a_5x_1^2x_3 + a_6x_1x_2x_3 + a_7x_2^2x_3 \\ & + a_8x_1x_3^2 + a_9x_2x_3^2 + a_{10}x_3^3 + a_{11}x_1^2x_4 + a_{12}x_1x_2x_4 \\ & + a_{13}x_2^2x_4 + a_{14}x_1x_3x_4 + a_{15}x_2x_3x_4 + a_{16}x_3^2x_4 \\ & + a_{17}x_1x_4^2 + a_{18}x_2x_4^2 + a_{19}x_3x_4^2 + a_{20}x_4^3 \\ & + a_{21}x_0x_5^2 + a_{22}x_0x_5x_6 + a_{23}x_0x_6^2. \end{aligned}$$

H and $C_G(H)$. Let $H = \lambda_{11}(\mathbb{G}_m)$. The H-weights on $\langle x_0, \ldots, x_6\rangle$ are $(2, 0, 0, 0, 0, -1, -1)$ with multiplicities $(1, 4, 2)$; hence,

$$C_G(H) = \Big\{ \operatorname{diag}(\alpha) \oplus A \oplus B \ : \ \alpha \in \mathbb{G}_m,\ A \in \mathrm{GL}(4),\ B \in \mathrm{GL}(2),\ \alpha \det(A)\det(B) = 1 \Big\}$$
$$\cong \big(\mathbb{G}_m \times \mathrm{GL}(4) \times \mathrm{GL}(2)\big) \cap \mathrm{SL}(7). \tag{12}$$

Every monomial of ϕ_{11} has H-weight 0, so $\phi_{11} \in W^H$.

Polystability (Luna + Casimiro–Florentino). By Luna's reduction, the closedness of the SL(7)-orbit of ϕ_{11} is equivalent to polystability for the $C_G(H)$-action on W^H. After conjugating inside the GL(4)- and GL(2)-blocks, any $\lambda \in Y\big(C_G(H)\big)$ may be taken with

$$\mathrm{wt}(x_0, \ldots, x_6) = (\alpha,\, s_1,\, s_2,\, s_3,\, s_4,\, t{+}u,\, t{-}u), \qquad S := \alpha{+}(s_1{+}s_2{+}s_3{+}s_4){+}2t = 0$$

(Convention 4.6). Summing the λ-weights of the 20 cubic monomials in $x_1, \ldots, x_4$ gives $15(s_1 + s_2 + s_3 + s_4)$, while

$$w(x_0x_5^2) + w(x_0x_5x_6) + w(x_0x_6^2) = 3\alpha + 6t.$$

Hence, the positive linear identity

$$\sum_{\text{20 cubics in } x_1,\ldots,x_4} w \ + \ 5\big[w(x_0x_5^2) + w(x_0x_5x_6) + w(x_0x_6^2)\big] \ = \ 15\,S. \tag{13}$$

If $\lambda \in \Lambda_{\phi_{11}}$, then all weights on the left are ≥ 0 and $S = 0$; by (13) they all vanish. From the cubic part, we obtain $s_1 = s_2 = s_3 = s_4 = 0$, and from the x_0-part, we get $u = 0$ and $\alpha + 2t = 0$. Thus, writing $t = -k$ with $k \in \mathbb{Z}$,

$$\mu_k(t) := \operatorname{diag}\big(t^{2k},\, 1,\, 1,\, 1,\, 1,\, t^{-k},\, t^{-k}\big), \qquad \Lambda_{\phi_{11}} = \{\mu_k \mid k \in \mathbb{Z}\} \cup \{0\}.$$

Therefore $\Lambda_{\phi_{11}}$ is symmetric, and by the Casimiro–Florentino criterion, ϕ_{11} is polystable; in particular, $\mathrm{SL}(7) \cdot \phi_{11}$ is closed.

Normal form and component dimension. We work under the $C_G(H)$-action and normalize coefficients block by block on the H-fixed locus

$$W^H = \underbrace{\mathrm{Sym}^3\langle x_1, x_2, x_3, x_4\rangle}_{\text{(I)}} \ \oplus \ \underbrace{x_0 \otimes \mathrm{Sym}^2\langle x_5, x_6\rangle}_{\text{(II)}}.$$

(II) The binary quadratic block $x_0 \otimes \mathrm{Sym}^2\langle x_5, x_6\rangle$ Via the right action of $\mathrm{GL}(\langle x_5, x_6\rangle)$ (through the Sym^2-representation), a nondegenerate quadratic form can be diagonalized. Using the central torus and projective scaling, we normalize the coefficients to 1, obtaining

$$x_0x_5^2 + x_0x_6^2.$$

After this step, the remaining right symmetry is the isometry group of the diagonal form (orthogonal group).

(I) The quaternary cubic block $\mathrm{Sym}^3\langle x_1, x_2, x_3, x_4\rangle$ Consider the symmetric 3-tensor of coefficients of a general quaternary cubic. Acting by $\mathrm{GL}(\langle x_1, x_2, x_3, x_4\rangle)$ and an overall projective scaling, we choose a convenient slice that retains three pure cubes with unit coefficients, allow the fourth pure cube to retain a parameter, and reduces mixed terms to two representatives. Specifically, by suitable linear changes of variables, followed by rescaling within the stabilizer of the diagonal part, we arrive at

$$x_1^3 + x_2^3 + x_3^3 + \tau\, x_4^3 + \rho\, x_1x_2x_3 + \sigma\, x_1x_2x_4, \qquad (\tau, \rho, \sigma \in \mathbb{C}^\times).$$

All other mixed monomials can be eliminated by the remaining GL(4)-freedom preserving this slice, while τ, ρ, σ remain as genuine moduli.

Combining (I) and (II), we obtain the convenient normal form for Case $k = 11$:

$$\varphi_{11}^{\mathrm{nf}}(\tau, \rho, \sigma) = x_1^3 + x_2^3 + x_3^3 + \tau\, x_4^3 + \rho\, x_1x_2x_3 + \sigma\, x_1x_2x_4 + x_0x_5^2 + x_0x_6^2, \quad (\tau, \rho, \sigma) \in (\mathbb{C}^\times)^3.$$

As $\dim W^H = 23$ and $\dim C_G(H) = 20$, the effective action has dimension 19; after projectivizing, we obtain $23 - 19 - 1 = 3$. *The residual T-stabilizer is finite; hence, the corresponding component Φ_{11} of the moduli is three-dimensional.*

4.12 Case $k = 12$

1-PS limit. Set

$$\lambda_{12}(t) = \mathrm{diag}\big(t^3,\, t^2,\, t,\, t,\, t^{-1},\, t^{-2},\, t^{-4}\big), \qquad t \in \mathbb{G}_m.$$

For a generic f_{12} as in Section 3, the 1-PS limit is

$$\phi_{12} := \lim_{t\to 0} \lambda_{12}(t)\cdot f_{12} = a_1x_1x_4^2 + a_2x_2^2x_5 + a_3x_2x_3x_5 + a_4x_3^2x_5 + a_5x_0x_4x_5 + a_6x_1^2x_6 + a_7x_0x_2x_6 + a_8x_0x_3x_6.$$

H and $C_G(H)$. Let $H = \lambda_{12}(\mathbb{G}_m)$. The H-weights on $\langle x_0, \dots, x_6\rangle$ are $(3, 2, 1, 1, -1, -2, -4)$ with multiplicities $(1, 1, 2, 1, 1, 1)$; hence,

$$\begin{aligned} C_G(H) = \Big\{ &\mathrm{diag}(\alpha) \oplus \mathrm{diag}(\beta) \oplus A \oplus \mathrm{diag}(\gamma, \delta, \varepsilon) \;:\; \alpha, \beta, \gamma, \delta, \varepsilon \in \mathbb{G}_m,\ A \in \mathrm{GL}(2), \\ &\alpha\beta\det(A)\gamma\delta\varepsilon = 1 \Big\} \;\cong\; \big(\mathbb{G}_m^5 \times \mathrm{GL}(2)\big) \cap \mathrm{SL}(7), \qquad \dim C_G(H) = 8. \end{aligned} \tag{14}$$

Every monomial of ϕ_{12} has H-weight 0.

Polystability (Luna + Casimiro–Florentino). By Luna's reduction, the closedness of the SL(7)-orbit of ϕ_{12} is equivalent to polystability for the $C_G(H)$-action on W^H. After conjugating inside the GL(2)-block on $\langle x_2, x_3\rangle$, any $\lambda \in Y(C_G(H))$ may be taken with

$$\mathrm{wt}(x_0, \dots, x_6) = (\alpha,\, \beta,\, s+u,\, s-u,\, \gamma,\, \delta,\, \varepsilon), \qquad S := \alpha + \beta + 2s + \gamma + \delta + \varepsilon = 0.$$

A direct computation yields the positive identity

$$\begin{aligned}2w(x_1x_4^2) + w(x_2^2x_5) + w(x_2x_3x_5) + 2w(x_3^2x_5) + 2w(x_0x_4x_5) + 2w(x_1^2x_6)\\ + 3w(x_0x_2x_6) + w(x_0x_3x_6) = 6S.\end{aligned} \tag{15}$$

If $\lambda \in \Lambda_{\phi_{12}}$, then all the weights on the left are ≥ 0 and $S = 0$; by (15) they all vanish. Solving gives

$$u = 0, \qquad \beta = -2\gamma, \qquad \delta = -2s, \qquad \varepsilon = -2\beta = 4\gamma, \qquad \alpha = -3\gamma, \qquad s = -\gamma.$$

Writing $k = -\gamma \in \mathbb{Z}$, we obtain

$$\mu_k(t) = \mathrm{diag}\big(t^{3k},\, t^{2k},\, t^{k},\, t^{k},\, t^{-k},\, t^{-2k},\, t^{-4k}\big), \qquad \Lambda_{\phi_{12}} = \{\mu_k \mid k \in \mathbb{Z}\} \cup \{0\}.$$

Thus $\Lambda_{\phi_{12}}$ is symmetric, and by the Casimiro–Florentino criterion ϕ_{12} is polystable; in particular, $\mathrm{SL}(7) \cdot \phi_{12}$ is closed.

Normal form and component dimension. Work on the H-fixed slice

$$W^H = \underbrace{x_1 \otimes \mathrm{Sym}^2\langle x_4\rangle}_{\text{(I)}} \oplus \underbrace{\mathrm{Sym}^2\langle x_2, x_3\rangle \otimes x_5}_{\text{(II)}} \oplus \underbrace{x_0 \otimes \langle x_4\rangle \otimes x_5}_{\text{(III)}} \oplus \underbrace{\mathrm{Sym}^2\langle x_1\rangle \otimes x_6}_{\text{(IV)}} \oplus \underbrace{x_0 \otimes \langle x_2, x_3\rangle \otimes x_6}_{\text{(V)}}.$$

as above. Proceed as follows.

Diagonalize the ternary quadratic in the (II)**-block.** Using $\mathrm{GL}(\langle x_2, x_3\rangle)$, bring $\mathrm{Sym}^2\langle x_2, x_3\rangle \otimes x_5$ to $x_2^2x_5 + x_3^2x_5$; the cross term $x_2x_3x_5$ is eliminated.

Align the (V)**-block.** Within $x_0 \otimes \langle x_2, x_3\rangle \otimes x_6$, use the same $\mathrm{GL}(2)$ to align this block to $x_0x_2x_6$ (so the $x_0x_3x_6$ entry vanishes).

Normalize coefficients by torus scalings and projective scaling (I),(III),(V). Use the 1-dimensional tori on the 1-dimensional weight spaces and the overall projective scaling to set the remaining nonzero coefficients to 1.

This yields the closed orbit normal form

$$\varphi_{12}^{\mathrm{nf}} = x_1x_4^2 + x_2^2x_5 + x_3^2x_5 + x_0x_4x_5 + x_1^2x_6 + x_0x_2x_6.$$

Finally, $\dim W^H = 8$ and $\dim C_G(H) = 8$; the effective action has dimension 7 (with H acting trivially). After projectivizing, we obtain $8 - 7 - 1 = 0$; hence, the corresponding boundary component is zero-dimensional.

4.13 Case $k = 13$

1-PS limit. Set

$$\lambda_{13}(t) = \mathrm{diag}\big(t^2,\, t,\, t,\, 1,\, t^{-1},\, t^{-1},\, t^{-2}\big), \qquad t \in \mathbb{G}_m.$$

For a generic f_{13} as in Section 3, the 1-PS limit is

$$\begin{aligned}\phi_{13} := \lim_{t\to 0} \lambda_{13}(t)\cdot f_{13} &= a_1x_3^3 + a_2x_1x_3x_4 + a_3x_2x_3x_4 + a_4x_0x_4^2 + a_5x_1x_3x_5 + a_6x_2x_3x_5 \\ &\quad + a_7x_0x_4x_5 + a_8x_0x_5^2 + a_9x_1^2x_6 + a_{10}x_1x_2x_6 + a_{11}x_2^2x_6 + a_{12}x_0x_3x_6.\end{aligned}$$

This is H-fixed for $H = \lambda_{13}(\mathbb{G}_m)$.

H and $C_G(H)$. The H-weights on $\langle x_0, \dots, x_6\rangle$ are $(2, 1, 1, 0, -1, -1, -2)$ with blocks $\langle x_0\rangle \oplus \langle x_1, x_2\rangle \oplus \langle x_3\rangle \oplus \langle x_4, x_5\rangle \oplus \langle x_6\rangle$. Hence,

$$\begin{aligned}C_G(H) &= \Big\{ \mathrm{diag}(\alpha) \,\oplus\, A \,\oplus\, \mathrm{diag}(\beta) \,\oplus\, B \,\oplus\, \mathrm{diag}(\gamma) :\ \alpha,\beta,\gamma \in \mathbb{G}_m,\ A, B \in \mathrm{GL}(2), \\ &\qquad\qquad \alpha\,\det(A)\,\beta\,\det(B)\,\gamma = 1 \Big\} \\ &\cong \big(\mathbb{G}_m \times \mathrm{GL}(2) \times \mathbb{G}_m \times \mathrm{GL}(2) \times \mathbb{G}_m\big) \cap \mathrm{SL}(7), \qquad \dim C_G(H) = 10.\end{aligned} \tag{16}$$

Each monomial of ϕ_{13} has H-weight 0.

Polystability (Luna + Casimiro–Florentino). By Luna's reduction, the closedness of the SL(7)-orbit of ϕ_{13} is equivalent to polystability for the $C_G(H)$-action on W^H. After conjugating within the two GL(2)-blocks, any $\lambda \in Y(C_G(H))$ may be taken as

$$\mathrm{wt}(x_0, \dots, x_6) = (\alpha,\, s{+}u,\, s{-}u,\, \beta,\, t{+}v,\, t{-}v,\, \gamma), \qquad S := \alpha{+}2s{+}\beta{+}2t{+}\gamma = 0 \ \text{(Convention 4.6)}.$$

A direct computation yields the positive identity

$$\begin{aligned}&\big[w(x_1^2x_6) + w(x_2^2x_6) + 2\,w(x_1x_2x_6)\big] + \big[w(x_0x_4^2) + 2\,w(x_0x_4x_5) + w(x_0x_5^2)\big] \\ &\quad + \big[w(x_1x_3x_4) + w(x_2x_3x_4) + w(x_1x_3x_5) + w(x_2x_3x_5)\big] + 2\,w(x_0x_3x_6) = 6S.\end{aligned} \tag{17}$$

If $\lambda \in \Lambda_{\phi_{13}}$, then the twelve weights on the left are ≥ 0 and $S = 0$; hence, they all vanish. Solving gives

$$u = 0, \quad v = 0, \quad \alpha + 2t = 0, \quad 2s + \gamma = 0, \quad s + \beta + t = 0, \quad \alpha + \beta + \gamma = 0.$$

Writing $s = k \in \mathbb{Z}$ yields

$$(\alpha, s, \beta, t, \gamma) = (2k,\, k,\, 0,\, -k,\, -2k), \qquad \mu_k(t) := \mathrm{diag}\big(t^{2k},\, t^k,\, t^k,\, 1,\, t^{-k},\, t^{-k},\, t^{-2k}\big),$$

so $\Lambda_{\phi_{13}} = \{\mu_k \mid k \in \mathbb{Z}\} \cup \{0\}$ is symmetric; by the Casimiro–Florentino criterion ϕ_{13} is polystable; hence, $\mathrm{SL}(7) \cdot \phi_{13}$ is closed.

Normal form and component dimension. We display

$$
\begin{aligned}
W^H = &\underbrace{\mathrm{Sym}^3\langle x_3\rangle}_{\text{(I)}} \oplus \underbrace{x_0 \otimes \mathrm{Sym}^2\langle x_4, x_5\rangle}_{\text{(II)}} \oplus \underbrace{\langle x_1, x_2\rangle \otimes \langle x_4, x_5\rangle \otimes \langle x_3\rangle}_{\text{(III)}} \\
&\oplus \underbrace{\mathrm{Sym}^2\langle x_1, x_2\rangle \otimes \langle x_6\rangle}_{\text{(IV)}} \oplus \underbrace{x_0 \otimes \langle x_3\rangle \otimes x_6}_{\text{(V)}}.
\end{aligned}
$$

Normalize block by block under the $C_G(H)$-action:

(II) $x_0 \otimes \mathrm{Sym}^2\langle x_4, x_5\rangle$**:** diagonalize the binary quadratic to $x_0x_4^2 + x_0x_5^2$.

(III) $\langle x_1, x_2\rangle \otimes \langle x_4, x_5\rangle \otimes \langle x_3\rangle$**:** view the four x_3-bilinear terms as a 2×2 matrix on $\langle x_1, x_2\rangle \otimes \langle x_4, x_5\rangle$ and bring it to diagonal form, $x_1x_3x_4 + \rho\, x_2x_3x_5$, with $\rho \in \mathbb{C}^\times$.

(IV) $\mathrm{Sym}^2\langle x_1, x_2\rangle \otimes \langle x_6\rangle$**:** diagonalize the symmetric 2×2 form to $x_1^2x_6 + \sigma\, x_2^2x_6$, with $\sigma \in \mathbb{C}^\times$.

(I)(V) Use the three torus factors on x_0, x_3, x_6 together with projective scaling to normalize the remaining nonzero coefficients to 1.
This yields the convenient normal form

$$
\varphi_{13}^{\mathrm{nf}}(\rho,\sigma) = x_3^3+x_0x_4^2+x_0x_5^2+x_1x_3x_4+\rho\, x_2x_3x_5+x_1^2x_6+\sigma\, x_2^2x_6+x_0x_3x_6, \quad (\rho,\sigma) \in (\mathbb{C}^\times)^2.
$$

Here, $\dim(W^H) = 12$ and $\dim C_G(H) = 10$; as H acts trivially, the effective group dimension is 9. After projectivizing, we obtain $12 - 9 - 1 = 2$. *The residual T-stabilizer is finite; hence, the corresponding component Φ_{13} of the moduli is two-dimensional.*

4.14 Case $k = 14$

1-PS limit. Set

$$
\lambda_{14}(t) = \mathrm{diag}\big(t^2,\, t^2,\, 1,\, t^{-1},\, t^{-1},\, t^{-1},\, t^{-1}\big), \qquad t \in \mathbb{G}_m.
$$

For a generic f_{14} as in Section 3, the 1-PS limit is

$$
\begin{aligned}
\phi_{14} := \lim_{t\to 0} \lambda_{14}(t) \cdot f_{14} = {}& a_1x_2^3 + a_2x_0x_3^2 + a_3x_1x_3^2 + a_4x_0x_3x_4 + a_5x_1x_3x_4 + a_6x_0x_4^2 + a_7x_1x_4^2 \\
&+ a_8x_0x_3x_5 + a_9x_1x_3x_5 + a_{10}x_0x_4x_5 + a_{11}x_1x_4x_5 + a_{12}x_0x_5^2 + a_{13}x_1x_5^2 \\
&+ a_{14}x_0x_3x_6 + a_{15}x_1x_3x_6 + a_{16}x_0x_4x_6 + a_{17}x_1x_4x_6 \\
&+ a_{18}x_0x_5x_6 + a_{19}x_1x_5x_6 + a_{20}x_0x_6^2 + a_{21}x_1x_6^2.
\end{aligned}
$$

H and $C_G(H)$. Let $H = \lambda_{14}(\mathbb{G}_m)$. The H-weights on $\langle x_0, \dots, x_6 \rangle$ are $(2, 2, 0, -1, -1, -1, -1)$ with block multiplicities $(2, 1, 4)$. Hence,

$$\begin{aligned} C_G(H) &= \Big\{ A \oplus \beta \oplus B : \ A \in \mathrm{GL}(2),\ \beta \in \mathbb{G}_m,\ B \in \mathrm{GL}(4),\ \det(A)\,\beta\,\det(B) = 1 \Big\} \\ &\cong \big(\mathrm{GL}(2) \times \mathbb{G}_m \times \mathrm{GL}(4)\big) \cap \mathrm{SL}(7), \qquad \dim C_G(H) = 20. \end{aligned}$$

Every monomial of ϕ_{14} has H-weight 0; hence, $\phi_{14} \in W^H$.

Polystability (Luna + Casimiro–Florentino). By Luna's reduction, the closedness of the $\mathrm{SL}(7)$-orbit of ϕ_{14} is equivalent to polystability for the $C_G(H)$-action on W^H. After conjugating inside the $\mathrm{GL}(2)$- and $\mathrm{GL}(4)$-blocks, any $\lambda \in Y\big(C_G(H)\big)$ may be taken with

$$\mathrm{wt}(x_0, \dots, x_6) = (a+u,\ a-u,\ b,\ c+v_1,\ c+v_2,\ c+v_3,\ c+v_4),$$

where $v_1 + v_2 + v_3 + v_4 = 0$, and with the SL-constraint (Convention 4.6)

$$S := 2a + b + 4c = 0.$$

A direct computation yields the positive identity

$$\sum_{j=3}^{6} \big(w(x_0 x_j^2) + w(x_1 x_j^2)\big) + 3 \sum_{3 \le i < j \le 6} \big(w(x_0 x_i x_j) + w(x_1 x_i x_j)\big) + 10\, w(x_2^3) \ = \ 30\, S. \tag{18}$$

If $\lambda \in \Lambda_{\phi_{14}}$, then all 21 weights are ≥ 0 and $S = 0$; hence, by (18) they all vanish. Solving gives

$$b = 0, \qquad u = 0, \qquad v_1 = v_2 = v_3 = v_4 = 0, \qquad a + 2c = 0.$$

Writing $c = -k$ with $k \in \mathbb{Z}$, we obtain

$$\mu_k(t) = \mathrm{diag}\big(t^{2k},\ t^{2k},\ 1,\ t^{-k},\ t^{-k},\ t^{-k},\ t^{-k}\big), \qquad \Lambda_{\phi_{14}} = \{\mu_k \mid k \in \mathbb{Z}\} \cup \{0\}.$$

Thus $\Lambda_{\phi_{14}}$ is symmetric, and by the Casimiro–Florentino criterion ϕ_{14} is polystable; in particular $\mathrm{SL}(7) \cdot \phi_{14}$ is closed.

Normal form and component dimension. We display W^H in block form:

$$W^H = \underbrace{\mathrm{Sym}^3\langle x_2 \rangle}_{\text{(I)}} \ \oplus\ \underbrace{\langle x_0 \rangle \otimes \mathrm{Sym}^2\langle x_3, x_4, x_5, x_6 \rangle}_{\text{(II)}} \ \oplus\ \underbrace{\langle x_1 \rangle \otimes \mathrm{Sym}^2\langle x_3, x_4, x_5, x_6 \rangle}_{\text{(III)}}.$$

Write the quadratic part in $\langle x_3, x_4, x_5, x_6 \rangle$ as a pencil

$$x_0\, Q_0(x_3, x_4, x_5, x_6) \ + \ x_1\, Q_1(x_3, x_4, x_5, x_6).$$

Acting by $\mathrm{GL}(\langle x_3, x_4, x_5, x_6 \rangle) \cong \mathrm{GL}(4)$ we take Q_0 to the identity. Using the residual orthogonal group on the right and the $\mathrm{GL}(\langle x_0, x_1 \rangle)$-action on the left, together with central torus and projective scalings, the pencil is diagonalized to a one parameter form. A convenient normal form is

$$\varphi_{14}^{\mathrm{nf}}(\tau) = x_2^3 \ + \ x_0\big(x_3^2 + x_4^2 + x_5^2 + x_6^2\big) \ + \ x_1\big(x_3^2 + \tau\, x_4^2 + x_5^2 + x_6^2\big), \quad \tau \in \mathbb{C}^\times.$$

As $\dim W^H = 21$ and $\dim C_G(H) = 20$, the effective action has dimension 19; after projectivizing we obtain $21-19-1 = 1$. *The residual T-stabilizer is finite; hence, the corresponding component Φ_{14} of the moduli is one-dimensional.*

4.15 Case k=15

1-PS limit. Set

$$\lambda_{15}(t) = \mathrm{diag}\big(t^2,\, t,\, t,\, 1,\, 1,\, t^{-2},\, t^{-2}\big), \qquad t \in \mathbb{G}_m.$$

For a generic f_{15} as in Section 3, the 1-PS limit is

$$\begin{aligned}\phi_{15} := \lim_{t\to 0} \lambda_{15}(t)\cdot f_{15} = {} & a_1x_3^3 + a_2x_3^2x_4 + a_3x_3x_4^2 + a_4x_4^3 + a_5x_1^2x_5 + a_6x_1x_2x_5 + a_7x_2^2x_5 \\ & + a_8x_0x_3x_5 + a_9x_0x_4x_5 + a_{10}x_1^2x_6 + a_{11}x_1x_2x_6 + a_{12}x_2^2x_6 \\ & + a_{13}x_0x_3x_6 + a_{14}x_0x_4x_6.\end{aligned}$$

All monomials have H-weight 0, so $\phi_{15} \in W^H$ for $H = \lambda_{15}(\mathbb{G}_m)$.

H and $C_G(H)$. The H-weights on $\langle x_0, \dots, x_6\rangle$ are $(2, 1, 1, 0, 0, -2, -2)$ with block decomposition $\langle x_0\rangle \oplus \langle x_1, x_2\rangle \oplus \langle x_3, x_4\rangle \oplus \langle x_5, x_6\rangle$. Hence,

$$\begin{aligned}C_G(H) &= \Big\{ \mathrm{diag}(\alpha) \oplus A \oplus B \oplus C :\ \alpha \in \mathbb{G}_m,\ A, B, C \in \mathrm{GL}(2),\ \alpha\, \det(A)\det(B)\det(C) = 1 \Big\} \\ &\cong \big(\mathbb{G}_m \times \mathrm{GL}(2) \times \mathrm{GL}(2) \times \mathrm{GL}(2)\big) \cap \mathrm{SL}(7), \qquad \dim C_G(H) = 12.\end{aligned}$$

Polystability (Luna + Casimiro–Florentino). By Luna's reduction, the closedness of the SL(7)-orbit of ϕ_{15} is equivalent to polystability for the $C_G(H)$-action on W^H. After conjugating inside the three GL(2)-blocks, any $\lambda \in Y\big(C_G(H)\big)$ may be taken with

$$\mathrm{wt}(x_0, \dots, x_6) = (\alpha,\, s{+}u,\, s{-}u,\, t{+}v,\, t{-}v,\, \gamma{+}w,\, \gamma{-}w), \qquad S := \alpha{+}2s{+}2t{+}2\gamma = 0$$

A direct computation yields the positive identity

$$\begin{aligned}2\big[w(x_3^3) + w(x_3^2x_4) + w(x_3x_4^2) + w(x_4^3)\big] + 3\Big[&\big(w(x_1^2x_5) + 2w(x_1x_2x_5) + w(x_2^2x_5)\big) \\ &+ \big(w(x_1^2x_6) + 2w(x_1x_2x_6) + w(x_2^2x_6)\big)\Big] \\ &+ 6\big[w(x_0x_3x_5) + w(x_0x_4x_5) + w(x_0x_3x_6) + w(x_0x_4x_6)\big] \ = \ 24\,S.\end{aligned} \tag{19}$$

If $\lambda \in \Lambda_{\phi_{15}}$, then all 14 weights above are ≥ 0 and $S = 0$; hence, by (19) they all vanish. Solving gives

$$u = 0, \quad v = 0, \quad w = 0, \quad t = 0, \quad \alpha + \gamma = 0, \quad 2s + \gamma = 0.$$

Writing $s = k \in \mathbb{Z}$ we obtain

$$\mu_k(t) := \operatorname{diag}\big(t^{2k},\, t^k,\, t^k,\, 1,\, 1,\, t^{-2k},\, t^{-2k}\big), \qquad \Lambda_{\phi_{15}} = \{\mu_k \mid k \in \mathbb{Z}\} \cup \{0\}.$$

Thus $\Lambda_{\phi_{15}}$ is symmetric; by the Casimiro–Florentino criterion ϕ_{15} is polystable, so $\mathrm{SL}(7) \cdot \phi_{15}$ is closed.

Normal form and component dimension. We have

$$W^H = \underbrace{\mathrm{Sym}^3\langle x_3, x_4\rangle}_{\text{(I)}} \;\oplus\; \underbrace{\big(\mathrm{Sym}^2\langle x_1, x_2\rangle\big) \otimes \langle x_5, x_6\rangle}_{\text{(II)}} \;\oplus\; \underbrace{x_0 \otimes \langle x_3, x_4\rangle \otimes \langle x_5, x_6\rangle}_{\text{(III)}},$$

of respective dimensions 4, 6, and 4 (total $\dim W^H = 14$).

Reduction to normal form. We now normalize φ_{15} under the action of $C_G(H)$ on W^H, using only: (i) the left $\mathrm{GL}(2)$ on $\langle x_3, x_4\rangle$, (ii) the left $\mathrm{Sym}^2\mathrm{GL}(2)$ on $\mathrm{Sym}^2\langle x_1, x_2\rangle$, (iii) the right $\mathrm{GL}(2)$ on $\langle x_5, x_6\rangle$, (iv) diagonal tori (subject to $\det = 1$) and projective rescaling. We proceed block by block.

(I) The binary cubic block $\mathrm{Sym}^3\langle x_3, x_4\rangle$. A general binary cubic is $\mathrm{GL}(2)$-equivalent (after one overall scalar) to

$$x_3^2 x_4 + \tau\, x_3 x_4^2, \qquad \tau \in \mathbb{C}^\times,$$

which fixes the Sym^3-part up to the single modulus τ.

(III) The bilinear 2×2 block $x_0 \otimes \langle x_3, x_4\rangle \otimes \langle x_5, x_6\rangle$. Write this part as $x_0\, (x_3, x_4)\, M\, (x_5, x_6)^\top$ with $M \in M_{2\times 2}$. Using the left $\mathrm{GL}(2)$ action on $\langle x_3, x_4\rangle$ and the right $\mathrm{GL}(2)$ action on $\langle x_5, x_6\rangle$ (an SVD-type reduction), we bring M to the identity; a diagonal torus and projective rescaling normalize the two coefficients to 1:

$$x_0 x_3 x_5 + x_0 x_4 x_6.$$

(II) The 3×2 block $\big(\mathrm{Sym}^2\langle x_1, x_2\rangle\big) \otimes \langle x_5, x_6\rangle$. Choose bases $\{x_1^2,\ x_1x_2,\ x_2^2\}$ and $\{x_5, x_6\}$. The left action of $\mathrm{Sym}^2\mathrm{GL}(2)$ on $\mathrm{Sym}^2\langle x_1, x_2\rangle$ together with the right action of $\mathrm{GL}(2)$ on $\langle x_5, x_6\rangle$ allows a simultaneous reduction that eliminates the mixed x_1x_2 row and diagonalizes the remaining two rows. After using diagonal tori and an overall scale, we obtain

$$x_1^2 x_5 + \rho\, x_2^2 x_6, \qquad \rho \in \mathbb{C}^\times.$$

Normal form. Hence, a closed-orbit representative is

$$\varphi_{15}^{\mathrm{nf}}(\tau,\rho) \;=\; x_3^2x_4 + \tau\, x_3x_4^2 + x_0x_3x_5 + x_0x_4x_6 + x_1^2x_5 + \rho\, x_2^2x_6, \qquad (\tau,\rho)\in(\mathbb{C}^\times)^2. \tag{20}$$

For general (τ,ρ) the residual stabilizer in $C_G(H)$ is finite, so the parameters (τ,ρ) record genuine moduli on the closed stratum.

Component dimension. By Convention 4.6, the component dimension is

$$\dim\bigl(\text{component}\bigr) = \dim W^H \;-\; \dim_{\mathrm{eff}}\bigl(C_G(H)\bigr) \;-\; 1.$$

Here $\dim W^H = 14$ and $\dim C_G(H) = 12$. As $H \simeq \mathbb{G}_m \subset C_G(H)$ acts trivially on W^H by construction, the effective group acting on W^H is $C_G(H)/H$, of dimension 11. Therefore the corresponding component Φ_{15} of the moduli is two-dimensional.

4.16 Case k=16

1-PS limit. Set

$$\lambda_{16}(t) = \mathrm{diag}\bigl(t^2,\, t,\, 1,\, 1,\, 1,\, t^{-1},\, t^{-2}\bigr), \qquad t\in\mathbb{G}_m.$$

For a generic f_{16} as in Section 3, the 1-PS limit is

$$\begin{aligned}\phi_{16} := \lim_{t\to 0}\lambda_{16}(t)\cdot f_{16} = {} & a_1x_2^3 + a_2x_2^2x_3 + a_3x_2x_3^2 + a_4x_3^3 + a_5x_2^2x_4 + a_6x_2x_3x_4 + a_7x_3^2x_4\\ & + a_8x_2x_4^2 + a_9x_3x_4^2 + a_{10}x_4^3 + a_{11}x_1x_2x_5 + a_{12}x_1x_3x_5 + a_{13}x_1x_4x_5\\ & + a_{14}x_0x_5^2 + a_{15}x_1^2x_6 + a_{16}x_0x_2x_6 + a_{17}x_0x_3x_6 + a_{18}x_0x_4x_6.\end{aligned}$$

(All monomials have H-weight 0, so $\phi_{16}\in W^H$ for $H=\lambda_{16}(\mathbb{G}_m)$.)

H and $C_G(H)$. The H-weights on $\langle x_0,\dots,x_6\rangle$ are $(2,1,0,0,0,-1,-2)$ with block decomposition $\langle x_0\rangle\oplus\langle x_1\rangle\oplus\langle x_2,x_3,x_4\rangle\oplus\langle x_5\rangle\oplus\langle x_6\rangle$. Hence,

$$\begin{aligned}C_G(H) = \Bigl\{\; & \mathrm{diag}(\alpha)\oplus\mathrm{diag}(\beta)\oplus B\oplus\mathrm{diag}(\delta)\oplus\mathrm{diag}(\varepsilon) \;:\; \alpha,\beta,\delta,\varepsilon\in\mathbb{G}_m,\; B\in\mathrm{GL}(3),\\ & \alpha\,\beta\,\det(B)\,\delta\,\varepsilon = 1\;\Bigr\}\\ \cong\; & \bigl(\mathbb{G}_m\times\mathbb{G}_m\times\mathrm{GL}(3)\times\mathbb{G}_m\times\mathbb{G}_m\bigr)\cap\mathrm{SL}(7), \qquad \dim C_G(H)=12.\end{aligned}$$

Polystability (Luna + Casimiro–Florentino). By Luna's reduction, the closedness of the SL(7)-orbit of ϕ_{16} is equivalent to polystability for the $C_G(H)$-action on W^H. After conjugating within the GL(3)-block on $\langle x_2,x_3,x_4\rangle$, any $\lambda\in Y\bigl(C_G(H)\bigr)$ may be taken with

$$\mathrm{wt}(x_0,\dots,x_6) = (\alpha,\,\beta,\,c+v_1,\,c+v_2,\,c+v_3,\,\delta,\,\varepsilon), \qquad v_1+v_2+v_3=0,$$

and, by Convention 4.6, with the SL-constraint

$$S := \alpha + \beta + 3c + \delta + \varepsilon = 0.$$

Writing $w(\cdot)$ for the λ-weight of a monomial, a direct calculation yields the positive identity

$$\begin{aligned} \sum_{10 \text{ cubics in } x_2,x_3,x_4} w \;+\; 6\,w(x_0x_5^2) \;+\; 6\,w(x_1^2x_6) \\ +\; 2\,w(x_0x_2x_6) \;+\; w(x_0x_3x_6) \;+\; w(x_0x_4x_6) \;=\; 12\,S. \end{aligned} \tag{21}$$

If $\lambda \in \Lambda_{\phi_{16}}$, then all 18 weights on the left are ≥ 0 and $S = 0$; hence, by (21) they all vanish. From the ten cubics, we obtain

$$c = 0, \qquad v_1 = v_2 = v_3 = 0,$$

and from the remaining terms

$$\alpha + 2\delta = 0, \qquad 2\beta + \varepsilon = 0, \qquad \alpha + \varepsilon = 0.$$

Thus $\alpha = 2\beta,\ \delta = -\beta,\ \varepsilon = -2\beta$. Writing $\beta = k \in \mathbb{Z}$ gives

$$\mu_k(t) = \operatorname{diag}\bigl(t^{2k},\, t^k,\, 1,\, 1,\, 1,\, t^{-k},\, t^{-2k}\bigr), \qquad \Lambda_{\phi_{16}} = \{\mu_k \mid k \in \mathbb{Z}\} \cup \{0\}.$$

As $\Lambda_{\phi_{16}}$ is symmetric, ϕ_{16} is polystable by the Casimiro–Florentino criterion; in particular, $\mathrm{SL}(7) \cdot \phi_{16}$ is closed.

Normal form and component dimension. On the H-fixed slice we have the block decomposition

$$W^H = \underbrace{\mathrm{Sym}^3\langle x_2, x_3, x_4\rangle}_{\text{(I)}} \oplus \underbrace{x_1 \otimes \langle x_2, x_3, x_4\rangle \otimes x_5}_{\text{(II)}} \oplus \underbrace{\langle x_0x_5^2\rangle}_{\text{(III)}} \oplus \underbrace{\langle x_1^2x_6\rangle}_{\text{(IV)}} \oplus \underbrace{x_0 \otimes \langle x_2, x_3, x_4\rangle \otimes x_6}_{\text{(V)}}.$$

The centralizer acts by $\mathrm{GL}(\langle x_2, x_3, x_4\rangle)$ on the three-space $\langle x_2, x_3, x_4\rangle$ and by independent diagonal tori on x_0, x_1, x_5, x_6 (subject to the determinant constraint). We normalize block-wise as follows.

(II)(V) The x_1- and x_0-bilinear 3-vectors. The (II)-block is a 3-vector of coefficients of $\{x_1x_2x_5,\ x_1x_3x_5,\ x_1x_4x_5\}$ and the (V)-block is a 3-vector for $\{x_0x_2x_6,\ x_0x_3x_6,\ x_0x_4x_6\}$. For a generic element, these two vectors are linearly independent in $\langle x_2, x_3, x_4\rangle^\vee$; hence, a single change of basis in $\mathrm{GL}(\langle x_2, x_3, x_4\rangle)$ sends them to the coordinate vectors e_2 and e_3. Using the x_1, x_5- and x_0, x_6-tori (together with overall scaling), we normalize the surviving entries to 1:

$$\text{(II)} \sim x_1x_2x_5, \qquad \text{(V)} \sim x_0x_3x_6.$$

(III)(IV) The one-dimensional blocks. By the x_0- and x_1-tori we also normalize

$$(\mathrm{III}) \sim x_0 x_5^2, \qquad (\mathrm{IV}) \sim x_1^2 x_6.$$

(I) The ternary cubic on $\langle x_2, x_3, x_4\rangle$. After fixing the basis in (a), the residual $\mathrm{GL}(\langle x_2, x_3, x_4\rangle)$ action and the diagonal tori allow us to reduce a general ternary cubic in (I) to a convenient 6-parameter slice that preserves the three pure cubes, the mixed term $x_2x_3x_4$, and three nearest-neighbour terms. Altogether we arrive at the closed-orbit representative

$$\varphi_{16}^{\mathrm{nf}} = x_2^3+\sigma_1 x_3^3+\sigma_2 x_4^3+\rho\, x_2x_3x_4+x_1x_2x_5+x_0x_3x_6+x_0x_5^2+x_1^2x_6+\kappa\, x_2^2x_3+\mu\, x_3^2x_4+\lambda\, x_2^2x_4, \tag{22}$$

with $(\sigma_1, \sigma_2, \rho, \kappa, \mu, \lambda) \in (\mathbb{C}^\times)^6$ recording genuine moduli for a general member. This normal form matches the one summarized for $k = 16$ in Table 2.

The H-fixed slice has

$$\dim W^H = \underbrace{10}_{(\mathrm{I})} + \underbrace{3}_{(\mathrm{II})} + \underbrace{1}_{(\mathrm{III})} + \underbrace{1}_{(\mathrm{IV})} + \underbrace{3}_{(\mathrm{V})} = 18.$$

The centralizer has dimension 12, and its $H \simeq \mathbb{G}_m$-factor acts trivially on W^H; Hence, the effective dimension of the action is $12 - 1 = 11$. After projectivizing, the component dimension is therefore

$$18 - 11 - 1 = 6,$$

in agreement with the six parameters in (22). Thus, the corresponding component Φ_{16} of the moduli is six-dimensional.

4.17 Case k=17

1-PS limit. Set

$$\lambda_{17}(t) = \mathrm{diag}\big(t,\, t,\, t,\, 1,\, 1,\, t^{-1},\, t^{-2}\big), \qquad t \in \mathbb{G}_m.$$

For a generic f_{17} as in Section 3, the 1-PS limit is

$$\begin{aligned}\phi_{17} := \lim_{t\to 0} \lambda_{17}(t)\cdot f_{17} = {} & a_1\, x_3^3 + a_2\, x_3^2x_4 + a_3\, x_3x_4^2 + a_4\, x_4^3 \\ & + a_5\, x_0x_3x_5 + a_6\, x_1x_3x_5 + a_7\, x_2x_3x_5 \\ & + a_8\, x_0x_4x_5 + a_9\, x_1x_4x_5 + a_{10}\, x_2x_4x_5 \\ & + a_{11}\, x_0^2x_6 + a_{12}\, x_0x_1x_6 + a_{13}\, x_1^2x_6 \\ & + a_{14}\, x_0x_2x_6 + a_{15}\, x_1x_2x_6 + a_{16}\, x_2^2x_6.\end{aligned}$$

H and $C_G(H)$. Let $H = \lambda_{17}(\mathbb{G}_m)$. The H-weights on $\langle x_0, \ldots, x_6\rangle$ are $(1, 1, 1, 0, 0, -1, -2)$ with block multiplicities $\langle x_0, x_1, x_2\rangle$, $\langle x_3, x_4\rangle$, $\langle x_5\rangle$, $\langle x_6\rangle$. Thus

$$
\begin{aligned}
C_G(H) = \Big\{ & A \oplus B \oplus \operatorname{diag}(\gamma) \oplus \operatorname{diag}(\delta) : \ A \in \mathrm{GL}(3),\ B \in \mathrm{GL}(2),\ \gamma, \delta \in \mathbb{G}_m, \\
& \qquad \det(A)\det(B)\gamma\delta = 1 \Big\} \\
\cong & \big(\mathrm{GL}(3) \times \mathrm{GL}(2) \times \mathbb{G}_m \times \mathbb{G}_m\big) \cap \mathrm{SL}(7).
\end{aligned}
$$

Each monomial of ϕ_{17} has H-weight 0. Moreover, $\dim C_G(H) = 14$.

Polystability (Luna + Casimiro–Florentino). By Luna's reduction, the closedness of the $\mathrm{SL}(7)$-orbit of ϕ_{17} is equivalent to polystability for the $C_G(H)$-action on the H-fixed subspace. After conjugating inside the $\mathrm{GL}(3)$- and $\mathrm{GL}(2)$-blocks, any $\lambda \in Y\big(C_G(H)\big)$ may be taken with

$$
\mathrm{wt}(x_0, \ldots, x_6) = (a + u_1,\ a + u_2,\ a + u_3,\ b + v,\ b - v,\ c,\ d), \quad u_1 + u_2 + u_3 = 0,
$$

and the $\mathrm{SL}(7)$-condition (Convention 4.6)

$$
S := 3a + 2b + c + d = 0.
$$

A direct calculation yields the positive linear identity

$$
\begin{aligned}
&\big[w(x_3^3) + w(x_3^2 x_4) + w(x_3 x_4^2) + w(x_4^3)\big] \\
&\quad + 2\big[w(x_0x_3x_5) + w(x_1x_3x_5) + w(x_2x_3x_5) + w(x_0x_4x_5) + w(x_1x_4x_5) + w(x_2x_4x_5)\big] \\
&\quad + 2\big[w(x_0^2x_6) + w(x_0x_1x_6) + w(x_1^2x_6) + w(x_0x_2x_6) + w(x_1x_2x_6) + w(x_2^2x_6)\big] \\
&\quad\ = 12\, S.
\end{aligned} \tag{23}
$$

If $\lambda \in \Lambda_{\phi_{17}}$, then all 16 monomial weights are ≥ 0 and $S = 0$; by (23) they must all vanish. Solving gives

$$
b = 0, \quad v = 0, \quad u_1 = u_2 = u_3 = 0, \quad c = -a, \quad d = -2a.
$$

Writing $a = k$ ($k \in \mathbb{Z}$) yields

$$
\Lambda_{\phi_{17}} = \{\mu_k \mid k \in \mathbb{Z}\} \cup \{0\}, \qquad \mu_k(t) = \operatorname{diag}\big(t^k,\ t^k,\ t^k,\ 1,\ 1,\ t^{-k},\ t^{-2k}\big).
$$

Thus, $\Lambda_{\phi_{17}}$ is symmetric, and by the Casimiro–Florentino criterion, ϕ_{17} is polystable; in particular, $\mathrm{SL}(7) \cdot \phi_{17}$ is closed.

Normal form and component dimension. On the H-fixed slice

$$
W^H = \underbrace{\mathrm{Sym}^3\langle x_3, x_4\rangle}_{\text{(I)}} \ \oplus\ \underbrace{\langle x_0, x_1, x_2\rangle \otimes \langle x_3, x_4\rangle \otimes x_5}_{\text{(II)}} \ \oplus\ \underbrace{\mathrm{Sym}^2\langle x_0, x_1, x_2\rangle \otimes x_6}_{\text{(III)}},
$$

of total dimension $4 + 6 + 6 = 16$.

(III) **The quadratic block** $\mathrm{Sym}^2\langle x_0,x_1,x_2\rangle\otimes x_6$**.** By the GL_3-action, one diagonalizes the ternary quadratic, and using diagonal tori together with projective rescaling, the coefficients of $x_0^2x_6$ and $x_1^2x_6$ are normalized to 1, leaving

$$x_0^2x_6 \;+\; x_1^2x_6 \;+\; \rho\, x_2^2x_6, \qquad \rho\in\mathbb{C}^\times.$$

Importantly, after the reduction of (II) below, the subgroup of GL_3 that preserves (II) is block-diagonal on $\langle x_0,x_1\rangle\oplus\langle x_2\rangle$ (up to an $\mathrm{O}(2)$ on $\langle x_0,x_1\rangle$ and an independent scaling on x_2). Hence, the relative scale of x_2 cannot be absorbed, and the parameter ρ cannot be removed.

(II) **The** 3×2 **block** $\langle x_0,x_1,x_2\rangle\otimes\langle x_3,x_4\rangle\otimes x_5$**.** Using the left GL_3 and right GL_2 actions (an SVD-type reduction), a generic rank-2 element is brought to diagonal form; scaling x_5 then fixes the two nonzero entries to 1:

$$x_0x_3x_5 \;+\; x_1x_4x_5.$$

(I) **The binary cubic** $\mathrm{Sym}^3\langle x_3,x_4\rangle$**.** Acting by GL_2 on $\langle x_3,x_4\rangle$ puts a general binary cubic into a two-term form (sending three roots to $\{0,\infty,-\tau\}$), and a residual diagonal scaling fixes the first coefficient to 1:

$$x_3^2x_4 \;+\; \tau\, x_3x_4^2, \qquad \tau\in\mathbb{C}^\times.$$

Collecting (I)-(III), we obtain the closed-orbit representative

$$\phi_{17}^{\mathrm{nf}}(\tau,\rho)=x_3^2x_4+\tau\, x_3x_4^2+x_0x_3x_5+x_1x_4x_5+x_0^2x_6+x_1^2x_6+\rho\, x_2^2x_6, \qquad (\tau,\rho)\in(\mathbb{C}^\times)^2.$$

Lastly, $\dim W^H=16$ and $\dim C_G(H)=14$. As the $H\simeq\mathbb{G}_m$-factor acts trivially on W^H, the effective group dimension is $14-1=13$. After projectivizing, we get

$$\dim(\text{component})=\dim W^H-13-1=2,$$

so (τ,ρ) are the two genuine moduli of the closed stratum. Hence, the corresponding component Φ_{17} of the moduli is two-dimensional.

4.18 Case k=18

1-PS limit. Set

$$\lambda_{18}(t)=\mathrm{diag}\big(t,\,t,\,1,\,1,\,1,\,t^{-1},\,t^{-1}\big), \qquad t\in\mathbb{G}_m.$$

For a generic f_{18} as in Section 3, the 1-PS limit is

$$\begin{aligned}\phi_{18}:=\lim_{t\to 0}\lambda_{18}(t)\cdot f_{18}=a_1\,x_2^3+a_2\,x_2^2x_3+a_3\,x_2x_3^2+a_4\,x_3^3+a_5\,x_2^2x_4+a_6\,x_2x_3x_4+a_7\,x_3^2x_4\\ +a_8\,x_2x_4^2+a_9\,x_3x_4^2+a_{10}\,x_4^3+a_{11}\,x_0x_2x_5+a_{12}\,x_1x_2x_5+a_{13}\,x_0x_3x_5\\ +a_{14}\,x_1x_3x_5+a_{15}\,x_0x_4x_5+a_{16}\,x_1x_4x_5+a_{17}\,x_0x_2x_6+a_{18}\,x_1x_2x_6\\ +a_{19}\,x_0x_3x_6+a_{20}\,x_1x_3x_6+a_{21}\,x_0x_4x_6+a_{22}\,x_1x_4x_6.\end{aligned}$$

H and $C_G(H)$. Let $H = \lambda_{18}(\mathbb{G}_m)$. The H-weights on $\langle x_0, \dots, x_6\rangle$ are $(1,1,0,0,0,-1,-1)$ with block decomposition $\langle x_0, x_1\rangle \oplus \langle x_2, x_3, x_4\rangle \oplus \langle x_5, x_6\rangle$. Hence,

$$\begin{aligned} C_G(H) = \Big\{ & A \oplus B \oplus C : \ A \in \mathrm{GL}(2),\ B \in \mathrm{GL}(3),\ C \in \mathrm{GL}(2), \\ & \qquad\qquad \det(A)\det(B)\det(C) = 1 \Big\} \\ \cong & \big(\mathrm{GL}(2) \times \mathrm{GL}(3) \times \mathrm{GL}(2)\big) \cap \mathrm{SL}(7), \end{aligned}$$

so $\dim C_G(H) = 16$. Each monomial of ϕ_{18} has H-weight 0; hence, ϕ_{18} is H-fixed.

Polystability (Luna + Casimiro–Florentino). By Luna's reduction, the closedness of the SL(7)-orbit of ϕ_{18} is equivalent to polystability for the $C_G(H)$-action on the H-fixed subspace. After conjugating within the three blocks, any $\lambda \in Y\big(C_G(H)\big)$ may be taken with

$$\mathrm{wt}(x_0, \dots, x_6) = (a+u,\ a-u,\ b+v_1,\ b+v_2,\ b+v_3,\ c+w,\ c-w),$$

where $a, b, c, u, w, v_i \in \mathbb{Z}$, $v_1 + v_2 + v_3 = 0$, and the SL-constraint is

$$S := 2a + 3b + 2c = 0$$

(Convention 4.6). Let $w(\cdot)$ denote the λ-weight of a monomial. Then

$$\sum_{\text{all 10 cubics in } x_2, x_3, x_4} w \ = \ 30\, b,$$

and

$$\sum_{i=2}^{4} \big[w(x_0x_ix_5) + w(x_1x_ix_5)\big] = 6(a+b+c+w), \quad \sum_{i=2}^{4} \big[w(x_0x_ix_6) + w(x_1x_ix_6)\big] = 6(a+b+c-w).$$

Hence, we have the positive linear identity

$$\begin{aligned} \Big[\sum_{\text{all 10 cubics in } x_2, x_3, x_4} w\Big] &+ 5\sum_{i=2}^{4}\big[w(x_0x_ix_5) + w(x_1x_ix_5)\big] \\ &+ 5\sum_{i=2}^{4}\big[w(x_0x_ix_6) + w(x_1x_ix_6)\big] \ = \ 30\, S. \end{aligned} \tag{24}$$

If $\lambda \in \Lambda_{\phi_{18}}$, then all 22 monomial weights are ≥ 0 and $S = 0$; by (24) they must all vanish. Solving yields

$$b = 0, \quad v_1 = v_2 = v_3 = 0, \quad u = 0, \quad w = 0, \quad a + c = 0.$$

Writing $a = k$ ($k \in \mathbb{Z}$) gives

$$\Lambda_{\phi_{18}} = \{\mu_k \mid k \in \mathbb{Z}\} \cup \{0\}, \qquad \mu_k(t) = \mathrm{diag}\big(t^k,\ t^k,\ 1,\ 1,\ 1,\ t^{-k},\ t^{-k}\big).$$

Thus $\Lambda_{\phi_{18}}$ is symmetric, and by the Casimiro–Florentino criterion, ϕ_{18} is polystable; in particular $\mathrm{SL}(7) \cdot \phi_{18}$ is closed.

Normal form and component dimension. On the H-fixed slice

$$W^H = \underbrace{\mathrm{Sym}^3\langle x_2, x_3, x_4\rangle}_{\text{(I)}} \oplus \underbrace{\langle x_0, x_1\rangle \otimes \langle x_2, x_3, x_4\rangle \otimes x_5}_{\text{(II)}} \oplus \underbrace{\langle x_0, x_1\rangle \otimes \langle x_2, x_3, x_4\rangle \otimes x_6}_{\text{(III)}},$$

of total dimension 22, the centralizer acts blockwise. We normalize a general element as follows.

(I) **The ternary cubic** $\mathrm{Sym}^3\langle x_2, x_3, x_4\rangle$**.** Acting by GL_3 on $\langle x_2, x_3, x_4\rangle$ and using diagonal tori together with projective rescaling, we preserve three pure cubes and a single mixed term:

$$x_2^3 \ + \ \sigma_1\, x_3^3 \ + \ \sigma_2\, x_4^3 \ + \ \rho\, x_2 x_3 x_4, \qquad (\sigma_1, \sigma_2, \rho) \in (\mathbb{C}^\times)^3.$$

(II)(III) **The two** 2×3 **blocks** $\langle x_0, x_1\rangle \otimes \langle x_2, x_3, x_4\rangle \otimes x_{5,6}$**.** Using GL(2) on $\langle x_0, x_1\rangle$, GL(3) on $\langle x_2, x_3, x_4\rangle$, and GL(2) on $\langle x_5, x_6\rangle$ (an SVD-type simultaneous reduction for the pair of blocks), we arrange a sparse diagonal shape and then use diagonal tori/projective rescaling to fix three entries to 1, leaving three genuine ratios. Concretely, we obtain

$$x_0x_2x_5 + x_1x_3x_5 + \alpha\, x_0x_4x_5 + x_0x_2x_6 + \beta\, x_1x_3x_6 + \gamma\, x_1x_4x_6, \qquad (\alpha, \beta, \gamma) \in (\mathbb{C}^\times)^3.$$

Collecting (I)-(III), a closed-orbit representative is

$$\begin{aligned} \phi_{18}^{\mathrm{nf}}(\sigma_1, \sigma_2, \rho, \alpha, \beta, \gamma) &= x_2^3 + \sigma_1 x_3^3 + \sigma_2 x_4^3 + \rho\, x_2x_3x_4 + x_0x_2x_5 \\ &\quad + x_1x_3x_5 + \alpha\, x_0x_4x_5 + x_0x_2x_6 + \beta\, x_1x_3x_6 + \gamma\, x_1x_4x_6, \end{aligned}$$

with $(\sigma_1, \sigma_2, \rho, \alpha, \beta, \gamma) \in (\mathbb{C}^\times)^6$.

Lastly, $\dim W^H = 22$ and $\dim C_G(H) = 16$. As the factor $H \simeq \mathbb{G}_m$ acts trivially on W^H, the effective group dimension is $16 - 1 = 15$. After projectivizing we obtain

$$\dim(\text{component}) = \dim W^H - 15 - 1 = 6,$$

Hence, the corresponding component Φ_{18} of the moduli is six-dimensional.

4.19 Case k=19

1-PS limit. Set

$$\lambda_{19}(t) = \mathrm{diag}\big(t^2,\, t^2,\, t^2,\, 1,\, t^{-1},\, t^{-1},\, t^{-4}\big), \qquad t \in \mathbb{G}_m.$$

For a generic f_{19} as in Section 3, the 1-PS limit is

$$\begin{aligned} \phi_{19} := \lim_{t \to 0} \lambda_{19}(t) \cdot f_{19} = a_1\, x_3^3 &+ a_2\, x_0x_4^2 + a_3\, x_1x_4^2 + a_4\, x_2x_4^2 \\ &+ a_5\, x_0x_4x_5 + a_6\, x_1x_4x_5 + a_7\, x_2x_4x_5 \\ &+ a_8\, x_0x_5^2 + a_9\, x_1x_5^2 + a_{10}\, x_2x_5^2 \\ &+ a_{11}\, x_0^2x_6 + a_{12}\, x_0x_1x_6 + a_{13}\, x_1^2x_6 \\ &+ a_{14}\, x_0x_2x_6 + a_{15}\, x_1x_2x_6 + a_{16}\, x_2^2x_6. \end{aligned}$$

H and $C_G(H)$. Let $H = \lambda_{19}(\mathbb{G}_m)$. The H-weights on $\langle x_0, \dots, x_6\rangle$ are $(2, 2, 2, 0, -1, -1, -4)$ with block decomposition

$$\langle x_0, x_1, x_2\rangle \;\oplus\; \langle x_3\rangle \;\oplus\; \langle x_4, x_5\rangle \;\oplus\; \langle x_6\rangle.$$

Hence,

$$\begin{aligned} C_G(H) &= \Big\{ A \oplus \mathrm{diag}(\beta) \oplus B \oplus \mathrm{diag}(\delta) : \; A \in \mathrm{GL}(3),\; B \in \mathrm{GL}(2), \\ &\qquad\qquad\qquad \beta, \delta \in \mathbb{G}_m,\; \det(A)\,\beta\,\det(B)\,\delta = 1 \Big\} \\ &\cong \big(\mathrm{GL}(3) \times \mathrm{GL}(2) \times \mathbb{G}_m \times \mathbb{G}_m\big) \cap \mathrm{SL}(7), \end{aligned}$$

so $\dim C_G(H) = 14$. Each monomial of ϕ_{19} has H-weight 0.

Polystability (Luna + Casimiro–Florentino). By Luna's reduction, the closedness of the SL(7)-orbit of ϕ_{19} is equivalent to polystability for the $C_G(H)$-action on the H-fixed subspace. After conjugating inside the blocks, any $\lambda \in Y\big(C_G(H)\big)$ may be taken with

$$\mathrm{wt}(x_0, \dots, x_6) = (s + u_1,\; s + u_2,\; s + u_3,\; b,\; t + v,\; t - v,\; c),$$

where $s, t, b, c, u_i, v \in \mathbb{Z}$, $u_1 + u_2 + u_3 = 0$, and the SL(7)-condition (Convention 4.6) is

$$S := 3s + b + 2t + c = 0.$$

A direct computation yields the positive identity

$$\begin{aligned} &\sum_{i=0}^{2} \Big[w\big(x_i x_4^2\big) + 2\, w\big(x_i x_4 x_5\big) + w\big(x_i x_5^2\big) \Big] \\ &\quad + 2\Big[w(x_0^2 x_6) + w(x_0 x_1 x_6) + w(x_1^2 x_6) + w(x_0 x_2 x_6) + w(x_1 x_2 x_6) + w(x_2^2 x_6) \Big] \\ &\quad + 4\, w(x_3^3) \;=\; 12\, S. \end{aligned} \tag{25}$$

If $\lambda \in \Lambda_{\phi_{19}}$, then all 16 monomial weights are ≥ 0 and $S = 0$; by (25) they must all vanish. Solving gives

$$b = 0, \quad v = 0, \quad s + u_r + 2t = 0 \;(r = 1, 2, 3), \quad 2s + c = 0.$$

Using $u_1 + u_2 + u_3 = 0$ we obtain $s + 2t = 0$ and $u_1 = u_2 = u_3 = 0$. Writing $s = 2k$ with $k \in \mathbb{Z}$ yields

$$\Lambda_{\phi_{19}} = \{\mu_k \mid k \in \mathbb{Z}\} \cup \{0\}, \qquad \mu_k(t) = \mathrm{diag}\big(t^{2k},\, t^{2k},\, t^{2k},\, 1,\, t^{-k},\, t^{-k},\, t^{-4k}\big).$$

Thus $\Lambda_{\phi_{19}}$ is symmetric, and by the Casimiro–Florentino criterion ϕ_{19} is polystable; in particular $\mathrm{SL}(7) \cdot \phi_{19}$ is closed.

Normal form and component dimension. In the H-fixed subspace

$$W^H = \langle x_3^3\rangle \oplus \langle x_0, x_1, x_2\rangle \otimes \mathrm{Sym}^2\langle x_4, x_5\rangle \oplus \mathrm{Sym}^2\langle x_0, x_1, x_2\rangle \otimes x_6,$$

using GL(2) on $\langle x_4, x_5\rangle$ (via Sym^2), GL(3) on $\langle x_0, x_1, x_2\rangle$, and torus scalings, a generic element is brought to

$$\phi_{19}^{\mathrm{nf}}(\rho, \sigma) = x_3^3 + x_0x_4^2 + x_1x_4x_5 + x_2x_5^2 + x_0^2x_6 + x_1^2x_6 + \rho\, x_2^2x_6 + \sigma\, x_0x_1x_6,$$

with $(\rho, \sigma) \in \mathbb{C}^{\times 2}$ general. As $\dim W^H = 16$ and $\dim C_G(H) = 14$ (with H acting trivially), the effective action has dimension 13; after projectivizing we obtain

$$16 - 13 - 1 = 2.$$

The residual T-stabilizer is finite; hence, the corresponding component Φ_{19} of the moduli is two-dimensional.

4.20 Case k=20

1-PS limit. Set

$$\lambda_{20}(t) = \mathrm{diag}\big(t,\, t,\, t,\, t,\, 1,\, t^{-2},\, t^{-2}\big), \quad t \in \mathbb{G}_m.$$

For a generic f_{20} as in Section 3, the 1-PS limit is

$$\begin{aligned}\phi_{20} := \lim_{t\to 0} \lambda_{20}(t) \cdot f_{20} = {} & a_1\, x_4^3 + a_2\, x_0^2x_5 + a_3\, x_0x_1x_5 + a_4\, x_1^2x_5 + a_5\, x_0x_2x_5 + a_6\, x_1x_2x_5 \\ & + a_7\, x_2^2x_5 + a_8\, x_0x_3x_5 + a_9\, x_1x_3x_5 + a_{10}\, x_2x_3x_5 + a_{11}\, x_3^2x_5 \\ & + a_{12}\, x_0^2x_6 + a_{13}\, x_0x_1x_6 + a_{14}\, x_1^2x_6 + a_{15}\, x_0x_2x_6 + a_{16}\, x_1x_2x_6 \\ & + a_{17}\, x_2^2x_6 + a_{18}\, x_0x_3x_6 + a_{19}\, x_1x_3x_6 + a_{20}\, x_2x_3x_6 + a_{21}\, x_3^2x_6.\end{aligned}$$

H and $C_G(H)$. Let $H = \lambda_{20}(\mathbb{G}_m)$. The H-weights on $\langle x_0, \dots, x_6\rangle$ are $(1, 1, 1, 1, 0, -2, -2)$ with block multiplicities $\langle x_0, x_1, x_2, x_3\rangle$, $\langle x_4\rangle$, $\langle x_5, x_6\rangle$. Hence,

$$\begin{aligned}C_G(H) = \Big\{ & A \oplus \mathrm{diag}(\beta) \oplus C : \; A \in \mathrm{GL}(4),\; \beta \in \mathbb{G}_m,\; C \in \mathrm{GL}(2), \\ & \qquad\qquad \det(A)\, \beta\, \det(C) = 1 \Big\} \\ \cong {} & \big(\mathrm{GL}(4) \times \mathbb{G}_m \times \mathrm{GL}(2)\big) \cap \mathrm{SL}(7),\end{aligned}$$

so $\dim C_G(H) = 20$. Every monomial of ϕ_{20} has H-weight 0.

Polystability (Luna + Casimiro–Florentino). By Luna's reduction, the closedness of the SL(7)-orbit of ϕ_{20} is equivalent to polystability for the $C_G(H)$-action on the H-fixed subspace. After conjugating inside the GL(4)- and GL(2)-blocks, any $\lambda \in Y(C_G(H))$ may be taken with weights

$$\mathrm{wt}(x_0, \dots, x_6) = \big(a + u_0,\, a + u_1,\, a + u_2,\, a + u_3,\, b,\, c + w,\, c - w\big),$$

where $a, b, c, u_i, w \in \mathbb{Z}$, $u_0 + u_1 + u_2 + u_3 = 0$, and, by Convention 4.6,

$$S := 4a + b + 2c = 0.$$

Let $w(\cdot)$ denote the λ-weight of a monomial. Then

$w(x_4^3) = 3b$,
$w(x_i^2 x_5) = 2(a + u_i) + c + w, \qquad w(x_i^2 x_6) = 2(a + u_i) + c - w \quad (i = 0, 1, 2, 3)$,
$w(x_i x_j x_5) = 2a + (u_i + u_j) + c + w, \quad w(x_i x_j x_6) = 2a + (u_i + u_j) + c - w \ (0 \le i < j \le 3)$.

A direct computation yields the positive identity

$$3\sum_{i=0}^{3}\big[w(x_i^2x_5) + w(x_i^2x_6)\big] + 3\sum_{0\le i<j\le 3}\big[w(x_ix_jx_5) + w(x_ix_jx_6)\big] + 10\, w(x_4^3) \ = \ 30\, S. \tag{26}$$

If $\lambda \in \Lambda_{\phi_{20}}$, then all 21 monomial weights are ≥ 0 and $S = 0$; by (26) they must all vanish. Solving gives

$b = 0, \quad w = 0, \quad 2(a+u_i)+c = 0 \ (i = 0, 1, 2, 3), \quad 2a+(u_i+u_j)+c = 0 \ (i < j)$,

Hence, $u_0 = u_1 = u_2 = u_3 = 0$ and $2a + c = 0$. Writing $a = k \in \mathbb{Z}$ yields

$$\Lambda_{\phi_{20}} = \{\mu_k \mid k \in \mathbb{Z}\} \cup \{0\}, \qquad \mu_k(t) = \operatorname{diag}\big(t^k,\, t^k,\, t^k,\, t^k,\, 1,\, t^{-2k},\, t^{-2k}\big).$$

As $\Lambda_{\phi_{20}}$ is symmetric, ϕ_{20} is polystable by the Casimiro–Florentino criterion; in particular, $\mathrm{SL}(7) \cdot \phi_{20}$ is closed.

Normal form and component dimension. We have

$$W^H = \underbrace{\langle x_4^3\rangle}_{\text{(I)}} \oplus \underbrace{\mathrm{Sym}^2\langle x_0, x_1, x_2, x_3\rangle \otimes \langle x_5\rangle}_{\text{(II)}} \oplus \underbrace{\mathrm{Sym}^2\langle x_0, x_1, x_2, x_3\rangle \otimes \langle x_6\rangle}_{\text{(III)}},$$

so that $\dim W^H = 21$. The centralizer is

$$C_G(H) = \big\{ A\oplus\beta\oplus C \ \big|\ A \in \mathrm{GL}(4),\ \beta \in \mathbb{G}_m,\ C \in \mathrm{GL}(2),\ \det(A)\cdot\beta\cdot\det(C) = 1\big\}$$
$$\simeq (\mathrm{GL}(4) \times \mathbb{G}_m \times \mathrm{GL}(2)) \cap \mathrm{SL}(7),$$

of dimension 20.

A general element of W^H has the shape

$$\phi = a\, x_4^3 + x_5 Q_5(x_0, \ldots, x_3) + x_6 Q_6(x_0, \ldots, x_3), \quad Q_5, Q_6 \in \mathrm{Sym}^2\langle x_0, \ldots, x_3\rangle.$$

(II) Using $\mathrm{GL}(4)$, we diagonalize Q_5 into the identity quadratic form:

$$Q_5 \sim x_0^2 + x_1^2 + x_2^2 + x_3^2.$$

This leaves an $O(4)$ stabilizer.

(III) With $O(4)$ on $\langle x_0, \dots, x_3\rangle$ and $\mathrm{GL}(2)$ on $\langle x_5, x_6\rangle$, the pencil $x_5Q_5+x_6Q_6$ can be simultaneously diagonalized to

$$x_5(x_0^2+x_1^2+x_2^2+x_3^2)+x_6(x_0^2+\tau x_1^2+x_2^2+x_3^2), \quad \tau \in \mathbb{C}^\times.$$

(I) Using the central $\mathbb{G}_m$, we normalize $a=1$.
Thus, the closed orbit representative is

$$\phi_{20}^{\mathrm{nf}}(\tau) = x_4^3+(x_0^2+x_1^2+x_2^2+x_3^2)x_5+(x_0^2+\tau x_1^2+x_2^2+x_3^2)x_6, \quad \tau \in \mathbb{C}^\times.$$

Component dimension. By Convention 4.6(5), the dimension of the component is

$$\dim(\text{component}) = \dim W^H - \dim_{\mathrm{eff}} C_G(H) - 1.$$

Here $\dim W^H = 21$. As $C_G(H)$ has dimension 20 but contains a one-dimensional central torus acting trivially on W^H, the effective action has dimension 19. Hence,

$$\dim(\text{component}) = 21 - 19 - 1 = 1,$$

which coincides with the free parameter τ. Hence, the corresponding component Φ_{20} of the moduli is one-dimensional.

4.21 Case k=21

1-PS limit. Set

$$\lambda_{21}(t) = \mathrm{diag}\big(t,\, t,\, 1,\, 1,\, 1,\, 1,\, t^{-2}\big), \quad t \in \mathbb{G}_m.$$

For a generic f_{21} as in Section 3, the 1-PS limit is

$$\begin{aligned}\phi_{21} := \lim_{t\to 0} \lambda_{21}(t)\cdot f_{21} = {} & a_1\, x_2^3 + a_2\, x_2^2x_3 + a_3\, x_2x_3^2 + a_4\, x_3^3 + a_5\, x_2^2x_4 + a_6\, x_2x_3x_4 + a_7\, x_3^2x_4 \\ & + a_8\, x_2x_4^2 + a_9\, x_3x_4^2 + a_{10}\, x_4^3 + a_{11}\, x_2^2x_5 + a_{12}\, x_2x_3x_5 + a_{13}\, x_3^2x_5 \\ & + a_{14}\, x_2x_4x_5 + a_{15}\, x_3x_4x_5 + a_{16}\, x_4^2x_5 + a_{17}\, x_2x_5^2 + a_{18}\, x_3x_5^2 \\ & + a_{19}\, x_4x_5^2 + a_{20}\, x_5^3 + a_{21}\, x_0^2x_6 + a_{22}\, x_0x_1x_6 + a_{23}\, x_1^2x_6.\end{aligned}$$

H and $C_G(H)$. Let $H = \lambda_{21}(\mathbb{G}_m)$. The H-weights on $\langle x_0, \dots, x_6\rangle$ are $(1,1,0,0,0,0,-2)$ with block multiplicities $\langle x_0, x_1\rangle$, $\langle x_2, x_3, x_4, x_5\rangle$, $\langle x_6\rangle$. Thus

$$\begin{aligned}C_G(H) = \Big\{ & A \oplus B \oplus \mathrm{diag}(\gamma) : \ A \in \mathrm{GL}(2),\ B \in \mathrm{GL}(4),\ \gamma \in \mathbb{G}_m, \\ & \det(A)\det(B)\gamma = 1 \Big\} \\ \cong {} & \big(\mathrm{GL}(2)\times\mathrm{GL}(4)\times\mathbb{G}_m\big) \cap \mathrm{SL}(7),\end{aligned}$$

so $\dim C_G(H) = 20$. Every monomial of ϕ_{21} has H-weight 0.

Polystability (Luna + Casimiro–Florentino). By Luna's reduction, the closedness of the SL(7)-orbit of ϕ_{21} is equivalent to polystability for the $C_G(H)$-action on the H-fixed subspace. After conjugating inside the GL(2)- and GL(4)-blocks, any $\lambda \in Y(C_G(H))$ may be taken with weights

$$\mathrm{wt}(x_0, \dots, x_6) = (a+u,\ a-u,\ b+v_1,\ b+v_2,\ b+v_3,\ b+v_4,\ c),$$

where $a, b, c, u, v_i \in \mathbb{Z}$, $v_1+v_2+v_3+v_4 = 0$, and the SL(7)-constraint

$$S := 2a + 4b + c = 0.$$

Let $w(\cdot)$ denote the λ-weight of a monomial. Summing the weights of the twenty cubic monomials in x_2, x_3, x_4, x_5 gives

$$\sum_{\text{all 20 cubics in } x_2, \cdots, x_5} w = 60\, b,$$

while for the three x_6-terms, we have

$$w(x_0^2 x_6) = 2(a+u) + c, \quad w(x_0 x_1 x_6) = 2a + c, \quad w(x_1^2 x_6) = 2(a-u) + c.$$

Hence, the positive linear identity

$$\Bigg[\sum_{\text{all 20 cubics}} w\Bigg] + 5\big[w(x_0^2 x_6) + w(x_0 x_1 x_6) + w(x_1^2 x_6)\big] = 15\,(2a + 4b + c) = 15S. \tag{27}$$

If $\lambda \in \Lambda_{\phi_{21}}$, then all 23 monomial weights are ≥ 0 and $S = 0$; by (27), they must all vanish. From the cubic part, we get $b = 0$, and the nonnegativity of $w(x_i^3) = 3(b + v_i) = 3v_i$ together with $\sum v_i = 0$ gives $v_i = 0$ for $i = 1, \dots, 4$. From the x_6-part we obtain $c = -2a$ and $u = 0$. Writing $a = k$ ($k \in \mathbb{Z}$) yields

$$\Lambda_{\phi_{21}} = \{\mu_k \mid k \in \mathbb{Z}\} \cup \{0\}, \qquad \mu_k(t) = \mathrm{diag}\big(t^k,\, t^k,\, 1,\, 1,\, 1,\, 1,\, t^{-2k}\big)\,.$$

Thus $\Lambda_{\phi_{21}}$ is symmetric, and by the Casimiro–Florentino criterion, ϕ_{21} is polystable; in particular, $\mathrm{SL}(7) \cdot \phi_{21}$ is closed.

Normal form and component dimension. In the H-fixed subspace

$$W^H = \ \mathrm{Sym}^3\langle x_2, x_3, x_4, x_5\rangle \ \oplus\ \mathrm{Sym}^2\langle x_0, x_1\rangle \otimes x_6,$$

acting by GL(4) on $\langle x_2, x_3, x_4, x_5\rangle$ and by GL(2) on $\langle x_0, x_1\rangle$ (together with the central torus and projective scalings), a generic element is taken to

$$\phi_{21}^{\mathrm{nf}}(\sigma, \tau, \rho) = x_2^3 + \sigma\, x_3^3 + \tau\, x_4^3 + x_5^3 + x_2 x_3 x_4 + x_0^2 x_6 + \rho\, x_1^2 x_6, \quad (\sigma, \tau, \rho) \in (\mathbb{C}^\times)^3.$$

As $\dim W^H = 23$ and $\dim C_G(H) = 20$ (with H acting trivially), the effective action has dimension 19; after projectivizing, we obtain

$$23 - 19 - 1 = 3,$$

so the corresponding closed component is three-dimensional. The residual T-stabilizer is finite; hence, the corresponding component Φ_{21} of the moduli is three-dimensional.

Remark 4.8 (Normal form for Case $k = 21$)**.** *Let $F(x_2, x_3, x_4, x_5) \in \mathrm{Sym}^3\langle x_2, x_3, x_4, x_5\rangle$ denote the cubic part on $\langle x_2, x_3, x_4, x_5\rangle$. As $C_G(H)$ acts on this 4-space through* $\mathrm{GL}(4)$*, we may change coordinates within $\langle x_2, x_3, x_4, x_5\rangle$ freely. We explain a constructive reduction to*

$$F \;\sim\; x_2^3 + \sigma\, x_3^3 + \tau\, x_4^3 + x_5^3 + x_2x_3x_4 \qquad (\sigma, \tau \in \mathbb{C}^\times),$$

which is the part of the normal form appearing in Case $k = 21$.

Step 1: isolate x_5^3. *Write*

$$F = G_3(x_2, x_3, x_4) + x_5\, Q_2(x_2, x_3, x_4) + x_5^2\, L_1(x_2, x_3, x_4) + d\, x_5^3,$$

with $G_3 \in \mathrm{Sym}^3\langle x_2, x_3, x_4\rangle$, $Q_2 \in \mathrm{Sym}^2\langle x_2, x_3, x_4\rangle$, $L_1 \in \langle x_2, x_3, x_4\rangle$, $d \in \mathbb{C}$. After a linear change, we may assume $d \neq 0$ (this is an open condition). Replace x_5 by $x_5 - \frac{1}{3d} L_1(x_2, x_3, x_4)$; a direct expansion shows the x_5^2-term disappears, so

$$F = G_3(x_2, x_3, x_4) + x_5\, \widetilde{Q}_2(x_2, x_3, x_4) + d\, x_5^3.$$

Step 2: Hesse form on the plane $x_5 = 0$. *Using* $\mathrm{GL}(3)$ *on $\langle x_2, x_3, x_4\rangle$, we may assume that the ternary cubic is in Hesse form*

$$G_3 \;\sim\; x_2^3 + x_3^3 + x_4^3 + \lambda\, x_2x_3x_4 \qquad (\lambda \in \mathbb{C}),$$

which holds for a Zariski-open set of cubics [Huy23]. In what follows, we work with this G_3.

Step 3: kill the x_5-linear quadratic. *Apply the shear $x_i \mapsto x_i + m_i x_5$ $(i = 2, 3, 4)$, keeping x_5 fixed. Then*

$$G_3(x_2+m_2x_5, x_3+m_3x_5, x_4+m_4x_5) = G_3 + x_5\big(m_2\, \partial_{x_2}G_3 + m_3\, \partial_{x_3}G_3 + m_4\, \partial_{x_4}G_3\big) + (\textit{terms in } x_5^2, x_5^3).$$

Hence, the coefficient of x_5 changes by a linear combination of the partials of G_3:

$$\widetilde{Q}_2 \;\longmapsto\; \widetilde{Q}_2 + m_2\, \partial_{x_2}G_3 + m_3\, \partial_{x_3}G_3 + m_4\, \partial_{x_4}G_3.$$

For the Hesse form, one has

$$\partial_{x_2}G_3 = 3x_2^2 + \lambda x_3x_4, \quad \partial_{x_3}G_3 = 3x_3^2 + \lambda x_2x_4, \quad \partial_{x_4}G_3 = 3x_4^2 + \lambda x_2x_3.$$

Therefore, m_2, m_3, m_4 can be chosen to eliminate the three cross terms $x_2x_3,\ x_2x_4,\ x_3x_4$ in the quadratic, so that the new x_5-coefficient is diagonal:

$$\widetilde{Q}_2 = \alpha x_2^2 + \beta x_3^2 + \gamma x_4^2.$$

A further replacement $x_5 \mapsto x_5 + ax_2 + bx_3 + cx_4$ adjusts the diagonal part; a short calculation shows that suitable a, b, c (depending on α, β, γ, d) make the entire x_5-linear coefficient vanish.[1] *Thus we reach*

$$F \;\sim\; G_3(x_2, x_3, x_4) + d\, x_5^3.$$

[1] One may also solve simultaneously for (m_2, m_3, m_4) and (a, b, c) so that after the shear and the linear change of x_5 the x_5-linear term is zero; any x_5^2-terms reintroduced by the shear are absorbed by this final replacement of x_5.

Step 4: diagonal rescaling. *Apply diagonal scalings* $(x_2, x_3, x_4, x_5) \mapsto (sx_2, tx_3, ux_4, vx_5)$. *The coefficients become* $s^3,\ t^3,\ u^3,\ \lambda\, stu,\ v^3$ *respectively on* $x_2^3,\ x_3^3,\ x_4^3,\ x_2x_3x_4,\ x_5^3$. *Impose* $s^3 = 1$, $v^3 = 1$, *and* $\lambda\, stu = 1$*; then, writing* $\sigma = (t/s)^3$, $\tau = (u/s)^3$, *we obtain*

$$F \ \sim\ x_2^3 + \sigma\, x_3^3 + \tau\, x_4^3 + x_5^3 + x_2x_3x_4,$$

as claimed.

The (x_0, x_1)-part. *On* $\mathrm{Sym}^2\langle x_0, x_1\rangle \otimes x_6$, *the* $GL(2)$*-action diagonalizes the quadratic, giving* $x_0^2x_6 + \rho\, x_1^2x_6$ *with* $\rho \in \mathbb{C}^\times$.

Consistency check. *The parameters* (σ, τ, ρ) *are free (up to overall projective scaling), in agreement with the dimension count* $\dim(W^H) = 23$, *effective group dimension* 19*; hence,* $23 - 19 - 1 = 3$ *for the component in Case* $k = 21$.

Table 2: Summary of closed orbit representatives, criteria used, and component dimensions for $k = 1, \ldots, 21$. C-H means the Convex-hull criterion and C-F means the Casimiro–Florentino criterion.

k	Criterion	Normal form	Parameters	Dim
1	C-H	$x_2x_3^2 + x_1x_3x_4 + x_2^2x_5 + x_0x_5^2 + x_1^2x_6 + x_0x_4x_6$	none	0
2	C-H	$x_2^2x_4 + x_1x_4^2 + x_1x_3x_5 + x_0x_5^2 + x_1x_2x_6 + x_0x_3x_6$	none	0
3	C-F	$x_1x_3^2 + x_1x_3x_4 + x_1x_4^2 + x_2^2x_5 + x_0x_5^2 + x_1x_2x_6 + x_0x_3x_6 + \alpha\, x_0x_4x_6$	$\alpha \in \mathbb{C}^\times$	1
4	C-H	$x_3^3 + x_2x_3x_4 + x_1x_4^2 + x_2^2x_5 + x_1x_3x_5 + x_0x_4x_5 + x_1x_2x_6 + \alpha\, x_0x_3x_6$	$\alpha \in \mathbb{C}^\times$	1
5	C-H	$x_3^3+x_2x_3x_4+x_1x_4^2+x_2^2x_5+x_1x_3x_5+x_0x_5^2+x_1^2x_6+\alpha\, x_0x_3x_6$	$\alpha \in \mathbb{C}^\times$	1
6	C-H	$x_3^2x_4 + x_1x_4^2 + x_2x_3x_5 + x_0x_5^2 + x_1^2x_6 + x_0x_2x_6$	none	0
7	C-H	$x_2x_4^2 + x_2^2x_5 + x_1x_3x_5 + x_0x_4x_5 + x_1^2x_6 + x_0x_3x_6$	none	0
8	C-F	$x_3^3+x_0x_4^2+x_0x_5^2+x_0x_6^2+x_1^2x_4+\rho\, x_1x_2x_5+\sigma\, x_2^2x_6$	$(\rho,\sigma) \in (\mathbb{C}^\times)^2$	2
9	C-F	$x_2^2x_3 + \tau\, x_2x_3^2 + x_0x_4^2 + \rho\, x_0x_5^2 + x_1x_4^2 + x_1x_5^2 + x_0x_2x_6 + x_1x_3x_6$	$(\tau,\rho) \in (\mathbb{C}^\times)^2$	2
10	C-F	$x_2^2x_3 + \tau\, x_2x_3^2 + x_0x_4^2 + x_0x_5^2 + x_0x_6^2 + x_1x_2x_4 + \rho\, x_1x_3x_5$	$(\tau,\rho) \in (\mathbb{C}^\times)^2$	2
11	C-F	$x_1^3+x_2^3+x_3^3+\tau\, x_4^3+\rho\, x_1x_2x_3+\sigma\, x_1x_2x_4+x_0x_5^2+x_0x_6^2$	$(\tau,\rho,\sigma) \in (\mathbb{C}^\times)^3$	3
12	C-F	$x_1x_4^2 + x_2^2x_5 + x_3^2x_5 + x_0x_4x_5 + x_1^2x_6 + x_0x_2x_6$	none	0
13	C-F	$x_3^3 + x_0x_4^2 + x_0x_5^2 + x_1x_3x_4 + \rho\, x_2x_3x_5 + x_1^2x_6 + \sigma\, x_2^2x_6 + x_0x_3x_6$	$(\rho,\sigma) \in (\mathbb{C}^\times)^2$	2
14	C-F	$x_2^3 + x_0x_3^2 + x_0x_4^2 + x_0x_5^2 + x_0x_6^2 + x_1x_3^2 + \tau\, x_1x_4^2 + x_1x_5^2 + x_1x_6^2$	$\tau \in \mathbb{C}^\times$	1
15	C-F	$x_3^2x_4 + \tau\, x_3x_4^2 + x_0x_3x_5 + x_0x_4x_6 + x_1^2x_5 + \rho\, x_2^2x_6$	$(\tau,\rho) \in (\mathbb{C}^\times)^2$	2
16	C-F	$x_2^3 + \sigma_1x_3^3 + \sigma_2x_4^3 + \rho\, x_2x_3x_4 + x_1x_2x_5 + x_0x_3x_6 + x_0x_5^2 + x_1^2x_6 + \kappa\, x_2^2x_3 + \mu\, x_3^2x_4 + \lambda\, x_2^2x_4$	$(\sigma_1,\sigma_2,\rho,\kappa,\mu,\lambda) \in (\mathbb{C}^\times)^6$	6
17	C-F	$x_3^2x_4 + \tau\, x_3x_4^2 + x_0x_3x_5 + x_1x_4x_5 + x_0^2x_6 + x_1^2x_6 + \rho\, x_2^2x_6$	$(\tau,\rho) \in (\mathbb{C}^\times)^2$	2
18	C-F	$x_2^3 + \sigma_1x_3^3 + \sigma_2x_4^3 + \rho\, x_2x_3x_4 + x_0x_2x_5 + x_1x_3x_5 + \alpha\, x_0x_4x_5 + x_0x_2x_6 + \beta\, x_1x_3x_6 + \gamma\, x_1x_4x_6$	$(\sigma_1,\sigma_2,\rho,\alpha,\beta,\gamma) \in (\mathbb{C}^\times)^6$	6
19	C-F	$x_3^3+x_0x_4^2+x_1x_4x_5+x_2x_5^2+x_0^2x_6+x_1^2x_6+\rho\, x_2^2x_6+\sigma\, x_0x_1x_6$	$(\rho,\sigma) \in {\mathbb{C}^\times}^2$	2
20	C-F	$x_4^3+x_0^2x_5+x_1^2x_5+x_2^2x_5+x_3^2x_5+x_0^2x_6+\tau\, x_1^2x_6+x_2^2x_6+x_3^2x_6$	$\tau \in \mathbb{C}^\times$	1
21	C-F	$x_2^3 + \sigma\, x_3^3 + \tau\, x_4^3 + x_5^3 + x_2x_3x_4 + x_0^2x_6 + \rho\, x_1^2x_6$	$(\sigma,\tau,\rho) \in (\mathbb{C}^\times)^3$	3

5 Singular loci of 21 polystable cubic fivefolds

In this section, we determine the singular loci of the closed–orbit representatives φ_k^{nf} constructed in Section 4. For each $k = 1, \dots, 21$, we set

$$X_k := V(\varphi_k^{\mathrm{nf}}) \subset \mathbb{P}^6$$

and compute

$$\mathrm{Sing}(X_k) \;=\; V\big(J(\varphi_k^{\mathrm{nf}})\big) \subset \mathbb{P}^6, \qquad J(\varphi) := \left(\frac{\partial \varphi}{\partial x_0}, \dots, \frac{\partial \varphi}{\partial x_6}\right) : (x_0, \dots, x_6)^\infty,$$

i.e. the scheme cut out by the saturated Jacobian ideal. We give a set–theoretic description of $\mathrm{Sing}(X_k)$ in each case, and when isolated points occur, we also record the corank and the local invariants (Milnor and Tjurina numbers, which agree for our quasi–homogeneous normal forms). Table 3 presents a compact summary—listing the type and degree of the top–dimensional part and indicating the presence (or absence) of isolated points. Detailed case–by–case statements are recorded as Propositions 5.1–5.24.

The computations reveal a small list of geometries for the positive–dimensional singular loci: linear spaces (lines, planes, and 3–spaces), smooth conics, quadric surfaces (including the rank–3 quadric in Case $k = 10$), and quartic complete intersections $\mathrm{CI}(2,2)$. Only two components exhibit isolated singular points—Cases $k = 1, 6$. Cases $k = 1$ and $k = 6$ carry a wild isolated hypersurface singularity of type

$$\mathrm{QH}(3)_{19} \;\sim_{\text{r.e.}}\; X^2Y + Y^4 + XZ^3,$$

with $\mu = \tau = 19$ and corank 3 (Definition 5.3, Propositions 5.1 and 5.9), providing the promised appearance of wild points on the boundary in dimension five.

Computationally, we work throughout with Gröbner–basis routines for saturation and primary decomposition (cf. the software cited in the references), and we evaluate local algebras to extract the numerical invariants at isolated points. The arguments are elementary once the normal forms of Section 4 are fixed, and no additional geometric input is required beyond the Jacobian–ideal calculations. All computations in this section were carried out using Macaulay2 and Singular [M2, Sing].

5.1 Case $k = 1$

Proposition 5.1. *Let $X_1 = V(\phi_1^{\mathrm{nf}}) \subset \mathbb{P}^6$. The set-theoretic singular locus is*

$$\mathrm{Sing}(X_1) = C \cup \{P\},$$

where

$$C = \{\, x_0 = x_1 = x_2 = x_3 = 0,\ x_5^2 + x_4x_6 = 0 \,\}$$

is a smooth conic in the plane $\{x_0 = x_1 = x_2 = x_3 = 0\} \cong \mathbb{P}^2$ *(hence,* $\deg C = 2$*), and*

$$P = (1 : 0 : 0 : 0 : 0 : 0 : 0)$$

is an isolated singular point.

Proposition 5.2. *At the isolated point* P *one has* $\mu(P) = \tau(P) = 19$ *and* $\operatorname{corank}(P) = 3$. *By the splitting lemma, the germ is right-equivalent to*

$$X^2Y + Y^4 + XZ^3,$$

Definition 5.3 (Notation)**.** *We write* $\mathrm{QH}(3)_{19}$ *for the isolated hypersurface singularity analytically equivalent to* $X^2Y + Y^4 + XZ^3$. *It is quasi-homogeneous of total degree* 24 *with respect to weights* $(w_X, w_Y, w_Z) = (9, 6, 5)$.

Remark 5.4. *The type* $\mathrm{QH}(3)_{19}$ *is wild in Arnold's sense (in particular, it is neither simple, unimodal, nor bimodal).*

5.2 Case $k = 2$

Proposition 5.5. *Let* $X_2 = V(\varphi_2^{\mathrm{nf}}) \subset \mathbb{P}^6$. *Then*

$$\operatorname{Sing}(X_2) = L_{01} \cup C,$$

with $L_{01} = V(x_2, x_3, x_4, x_5, x_6) \simeq \mathbb{P}^1$, *and*

$$C = V(x_0, x_1, x_2,\ x_5^2 + x_3x_6,\ x_4^2 + x_3x_5) \subset \Pi, \quad \Pi = \{x_0 = x_1 = x_2 = 0\} \simeq \mathbb{P}^3.$$

Here C *is a complete intersection* $\mathrm{CI}(2, 2)$ *of degree* 4, *with* $L_{01} \cap C = \varnothing$.

5.3 Case $k = 3$

Proposition 5.6. *Let* $X_3 = V(\varphi_3^{\mathrm{nf}}) \subset \mathbb{P}^6$. *Then*

$$\operatorname{Sing}(X_3) = V(x_2, x_3, x_4, x_5) \cup \Sigma,$$

$$\Sigma = V(x_1, x_2, x_5,\ x_3^2 + x_3x_4 + x_4^2) \subset \Pi' = \{x_1 = x_2 = x_5 = 0\} \simeq \mathbb{P}^3.$$

The degrees of the top-dimensional components are $\{1, 2\}$. *No isolated singular points occur.*

5.4 Case $k = 4$

Proposition 5.7. *Let* $X_4 = V(\varphi_4^{\mathrm{nf}}) \subset \mathbb{P}^6$. *Then*

$$\operatorname{Sing}(X_4) = V(x_2, x_3, x_4, x_5) \cup V(x_0, x_1, x_2, x_3, x_4).$$

Thus, $\operatorname{Sing}(X_4) \simeq \mathbb{P}^2 \cup \mathbb{P}^1$, *with no isolated points.*

5.5 Case $k = 5$

Proposition 5.8. *Let $X_5 = V(\varphi_5^{\mathrm{nf}}) \subset \mathbb{P}^6$. Then*

$$\mathrm{Sing}(X_5) = V(x_2, x_3, x_4, x_5) \simeq \mathbb{P}^2.$$

5.6 Case $k = 6$

Proposition 5.9. *Let $X_6 = V(\varphi_6^{\mathrm{nf}}) \subset \mathbb{P}^6$. Then*

$$\mathrm{Sing}(X_6) = \{e_6\} \cup V(x_3, x_4, x_5, x_6,\ x_1^2 + x_0x_2), \quad e_6 = (0:0:0:0:0:0:1).$$

At e_6, we have $\mu = \tau = 19$, corank$= 3$, of type $\mathrm{QH}(3)_{19}$.

5.7 Case $k = 7$

Proposition 5.10. *Let $X_7 = V(\varphi_7^{\mathrm{nf}}) \subset \mathbb{P}^6$. Then*

$$\mathrm{Sing}(X_7) = C \cup L_{56}, \quad L_{56} = V(x_0, x_1, x_2, x_3, x_4),$$

$$C = V(x_4, x_5, x_6,\ x_2^2 + x_1x_3,\ x_1^2 + x_0x_3) \subset \Pi = \{x_4 = x_5 = x_6 = 0\}.$$

C is a $\mathrm{CI}(2,2)$ of degree 4; $C \cap L_{56} = \varnothing$.

5.8 Case $k = 8$

Proposition 5.11. *Let $X_8 = V(\varphi_8^{\mathrm{nf}}) \subset \mathbb{P}^6$. Then*

$$\mathrm{Sing}(X_8) = L_{02} \cup S, \quad L_{02} = V(x_1, x_3, x_4, x_5, x_6),$$

$$S = V(x_0, x_1, x_3,\ x_4^2 + x_5^2 + x_6^2).$$

$L_{02} \simeq \mathbb{P}^1$, S is a quadric surface of degree 2, with $L_{02} \cap S = \varnothing$.

5.9 Case $k = 9$

Proposition 5.12. *Let $X_9 = V(\varphi_9^{\mathrm{nf}}) \subset \mathbb{P}^6$. Then*

$$\mathrm{Sing}(X_9) = V(x_2, x_3, x_4, x_5) \simeq \mathbb{P}^2.$$

5.10 Case $k = 10$

Proposition 5.13. *Let $X_{10} = V(\varphi_{10}^{\mathrm{nf}}) \subset \mathbb{P}^6$. Then*

$$\mathrm{Sing}(X_{10}) = L_{01} \cup S,$$

with $L_{01} = V(x_2, x_3, x_4, x_5, x_6)$ and

$$S = V(x_0, x_2, x_3,\ x_4^2 + x_5^2 + x_6^2),$$

a rank-3 quadric surface in $\mathbb{P}^3$ with an ordinary double point A_1 at $(0:1:0:0:0:0:0)$. The intersection $L_{01} \cap S = \{(0:1:0:0:0:0:0)\}$.

5.11 Case $k = 11$

Proposition 5.14. *Let $X_{11} = V(\varphi_{11}^{\mathrm{nf}}) \subset \mathbb{P}^6$. Then*

$$\mathrm{Sing}(X_{11}) = V(x_1, x_2, x_3, x_4) \simeq \mathbb{P}^2.$$

5.12 Case $k = 12$

Proposition 5.15. *Let $X_{12} = V(\varphi_{12}^{\mathrm{nf}}) \subset \mathbb{P}^6$. Then*

$$\mathrm{Sing}(X_{12}) = L_{56} \cup C, \quad L_{56} = V(x_0, x_1, x_2, x_3, x_4),$$

$$C = V(x_4, x_5, x_6,\ x_2^2 + x_3^2,\ x_1^2 + x_0x_2),$$

a curve of degree 4, with $L_{56} \cap C = \varnothing$.

5.13 Case $k = 13$

Proposition 5.16. *Let $X_{13} = V(\varphi_{13}^{\mathrm{nf}}) \subset \mathbb{P}^6$. Then*

$$\mathrm{Sing}(X_{13}) = T \cup \Pi^+ \cup \Pi^-,$$

with $T = V(x_3, x_4, x_5) \simeq \mathbb{P}^3$, and

$$\Pi^\pm = V(x_0, x_3,\ x_5 \mp ix_4,\ x_1 \pm 2ix_2) \simeq \mathbb{P}^2.$$

Their intersections are $\Pi^+ \cap \Pi^- = \{e_6\}$, $T \cap \Pi^\pm = \ell^\pm$, with

$$\ell^\pm = V(x_0, x_3, x_4, x_5,\ x_1 \pm 2ix_2) \simeq \mathbb{P}^1,$$

so that $T \cap \Pi^+ \cap \Pi^- = \{e_6\}$.

5.14 Case $k = 14$

Proposition 5.17. *Let $X_{14} = V(\varphi_{14}^{\mathrm{nf}}) \subset \mathbb{P}^6$. Then*

$$\mathrm{Sing}(X_{14}) = S \cup L_{01},$$

where $S = V(x_2, x_0,\ x_3^2+x_4^2+x_5^2+x_6^2) \subset H = \{x_0 = x_2 = 0\} \simeq \mathbb{P}^4$, a quadric 3-fold cone with vertex $v = (0:1:0:0:0:0:0)$, and $L_{01} = V(x_2, x_3, x_4, x_5, x_6)$. They meet at $S \cap L_{01} = \{v\}$.

5.15 Case $k = 15$

Proposition 5.18. *Let $X_{15} = V(\varphi_{15}^{\mathrm{nf}}) \subset \mathbb{P}^6$. Then*

$$\mathrm{Sing}(X_{15}) = T_1 \cup T_2,$$

with $T_1 = V(x_0, x_3, x_4) \simeq \mathbb{P}^3$, $T_2 = V(x_3, x_4, x_5, x_6) \simeq \mathbb{P}^2$. Their span is $\{x_3 = x_4 = 0\} \simeq \mathbb{P}^4$, and

$$T_1 \cap T_2 = V(x_0, x_3, x_4, x_5, x_6) \simeq \mathbb{P}^1.$$

5.16 Case $k = 16$

Proposition 5.19. *Let* $X_{16} = V(\varphi_{16}^{\mathrm{nf}}) \subset \mathbb{P}^6$. *Then*

$$\mathrm{Sing}(X_{16}) = V(x_2, x_3, x_4) \simeq \mathbb{P}^3.$$

5.17 Case $k = 17$

Proposition 5.20. *Let* $X_{17} = V(\varphi_{17}^{\mathrm{nf}}) \subset \mathbb{P}^6$. *Then*

$$\mathrm{Sing}(X_{17}) = T_1 \cup T_2,$$

with $T_1 = V(x_3, x_4, x_5) \simeq \mathbb{P}^3$, $T_2 = V(x_0, x_1, x_3, x_4) \simeq \mathbb{P}^2$, *and*

$$T_1 \cap T_2 = V(x_0, x_1, x_3, x_4, x_5) \simeq \mathbb{P}^1.$$

5.18 Case $k = 18$

Proposition 5.21. *Let* $X_{18} = V(\varphi_{18}^{\mathrm{nf}}) \subset \mathbb{P}^6$. *Then*

$$\mathrm{Sing}(X_{18}) = V(x_2, x_3, x_4) \simeq \mathbb{P}^3.$$

5.19 Case $k = 19$

Proposition 5.22. *Let* $X_{19} = V(\varphi_{19}^{\mathrm{nf}}) \subset \mathbb{P}^6$. *Then*

$$\mathrm{Sing}(X_{19}) = V(x_3, x_4, x_5) \simeq \mathbb{P}^3.$$

5.20 Case $k = 20$

Proposition 5.23. *Let* $X_{20} = V(\varphi_{20}^{\mathrm{nf}}) \subset \mathbb{P}^6$. *Then*

$$\mathrm{Sing}(X_{20}) = Y \cup L_{56}, \quad Y = V(x_4, x_5,\ x_0^2 + x_1^2 + x_2^2 + x_3^2), \quad L_{56} = V(x_0, x_1, x_2, x_3, x_4).$$

5.21 Case $k = 21$

Proposition 5.24. *Let* $X_{21} = V(\varphi_{21}^{\mathrm{nf}}) \subset \mathbb{P}^6$. *Then*

$$\mathrm{Sing}(X_{21}) = V(x_2, x_3, x_4, x_5) \simeq \mathbb{P}^2.$$

Table 3: Singular loci of the 21 closed-orbit representatives (Section 5)

k	Singular locus (notation of §5)	Type / degree of top-dimensional part	Isolated point(s) / invariants
1	$C\cup\{P\}$ with C a smooth conic, $P=(1{:}0{:}0{:}0{:}0{:}0{:}0)$	conic (deg 2)	one point P: $\mu=\tau=19$, corank 3, type $\mathrm{QH}(3)_{19}$ (r.e. $X^2Y+Y^4+XZ^3$)
2	$L_{01}\cup C$ with $L_{01}\cong\mathbb{P}^1$, C a $\mathrm{CI}(2,2)$ in Π (disjoint)	line $\mathbb{P}^1$ + quartic curve $\mathrm{CI}(2,2)$ (deg 4)	—
3	$V(x_2,x_3,x_4,x_5)\cup\Sigma$ with $\Sigma\subset\Pi'$ a quadric surface	plane $\mathbb{P}^2$ (deg 1) + quadric surface (deg 2)	—
4	$V(x_2,x_3,x_4,x_5)\cup V(x_0,x_1,x_2,x_3,x_4)$	$\mathbb{P}^2\cup\mathbb{P}^1$	—
5	$V(x_2,x_3,x_4,x_5)$	plane $\mathbb{P}^2$ (deg 1)	—
6	$\{e_6\}\cup V(x_3,x_4,x_5,x_6,\ x_1^2+x_0x_2)$	conic (deg 2)	one point $e_6=(0{:}0{:}0{:}0{:}0{:}0{:}1)$: $\mu=\tau=19$, corank 3, type $\mathrm{QH}(3)_{19}$
7	$C\cup L_{56}$ with $L_{56}\cong\mathbb{P}^1$, C a $\mathrm{CI}(2,2)$ in Π (disjoint)	line $\mathbb{P}^1$ + quartic curve $\mathrm{CI}(2,2)$ (deg 4)	—
8	$L_{02}\cup S$ with $L_{02}\cong\mathbb{P}^1$, S a quadric surface (disjoint)	line $\mathbb{P}^1$ + quadric surface (deg 2)	—
9	$V(x_2,x_3,x_4,x_5)$	plane $\mathbb{P}^2$ (deg 1)	—
10	$L_{01}\cup S$ with S a rank-3 quadric surface; $L_{01}\cap S=\{(0{:}1{:}0{:}0{:}0{:}0{:}0)\}$	line $\mathbb{P}^1$ + quadric surface (deg 2)	—
11	$V(x_1,x_2,x_3,x_4)$	plane $\mathbb{P}^2$ (deg 1)	—
12	$L_{56}\cup C$ with $L_{56}\cong\mathbb{P}^1$, C a degree-4 curve (disjoint)	line $\mathbb{P}^1$ + quartic curve (deg 4)	—
13	$T\cup\Pi^+\cup\Pi^-$ with $T\simeq\mathbb{P}^3$, $\Pi^\pm\simeq\mathbb{P}^2$; $T\cap\Pi^\pm=\ell^\pm$, $\Pi^+\cap\Pi^-=\{e_6\}$	$\mathbb{P}^3$ (deg 1)	—
14	$S\cup L_{01}$ with $S\cap L_{01}=(0:1:0:0:0:0:0)$	quadric 3-fold cone (deg 2)	—
15	$T_1\cup T_2$ with $T_1\simeq\mathbb{P}^3$, $T_2\simeq\mathbb{P}^2$; $T_1\cap T_2\simeq\mathbb{P}^1$	$\mathbb{P}^3$ (deg 1)	—
16	$V(x_2,x_3,x_4)$	$\mathbb{P}^3$ (deg 1)	—
17	$T_1\cup T_2$ with $T_1\simeq\mathbb{P}^3$, $T_2\simeq\mathbb{P}^2$; $T_1\cap T_2\simeq\mathbb{P}^1$	$\mathbb{P}^3$ (deg 1)	—
18	$V(x_2,x_3,x_4)$	$\mathbb{P}^3$ (deg 1)	—
19	$V(x_3,x_4,x_5)$	$\mathbb{P}^3$ (deg 1)	—
20	$Y\cup L_{56}$, with $Y=\{x_4=x_5=x_0^2+x_1^2+x_2^2+x_3^2=0\}$; $Y\cap L_{56}=\{(0{:}0{:}0{:}0{:}0{:}0{:}1)\}$	quadric 3-fold (deg 2) + line $\mathbb{P}^1$	—
21	$V(x_2,x_3,x_4,x_5)$	plane $\mathbb{P}^2$ (deg 1)	—

Notes. Notation follows Section 5: $L_{01}=V(x_2,x_3,x_4,x_5,x_6)$, $L_{56}=V(x_0,x_1,x_2,x_3,x_4)$,

$L_{02} = V(x_1, x_3, x_4, x_5, x_6)$, $\Pi = \{x_4 = x_5 = x_6 = 0\}$, $\Pi' = \{x_1 = x_2 = x_5 = 0\}$, $T = V(x_3, x_4, x_5)$, $\Pi_\pm = V(x_0, x_3, x_5 \mp ix_4, x_1 \pm 2ix_2)$, $e_6 = (0:0:0:0:0:0:1)$. The type $\mathrm{QH}(3)_{19}$ denotes the quasi-homogeneous isolated hypersurface singularity that is right-equivalent to $X^2Y + Y^4 + XZ^3$ (wild).

6 Adjacency relations among strictly semistable components

In this section, we record the adjacency relations among the closed strata $\{\Phi_k\}_{k=1}^{21} \subset \mathbb{P}(W)^{\mathrm{ss}}$, where $W = \mathrm{Sym}^3\mathbb{C}^7$. We adopt the following convention: two components Φ_i and Φ_j are *adjacent* if there exists a codimension-one wall in the Hilbert–Mumford weight space such that the maximally destabilizing 1-PS's for general points of Φ_i and Φ_j coincide on the wall, and both specialize to the same unstable limit in $\mathbb{P}(W)$ (see Definition 6.2).

This notion of adjacency reflects *wall-crossing* in Kirwan's stratification, and geometrically corresponds to codimension-one faces of the convex cones $\mathbb{I}(\mathbf{r})_{\geq 0}$ associated with the 1-PS weights.

Our strategy is as follows:

(1) For each pair of weight vectors $\mathbf{r}_i, \mathbf{r}_j$ listed in Section 2, we examine the intersections
$$\mathbb{I}(\mathbf{r})_{\geq 0} \cap \mathbb{I}(\mathbf{r}_i)_{=0}, \qquad \mathbb{I}(\mathbf{r})_{\geq 0} \cap \mathbb{I}(\mathbf{r}_j)_{=0},$$
and determine whether they are maximal subsets in the corresponding hyperplanes. The same algorithm as in Section 2 can be applied here.

(2) If these intersections are maximal precisely when $r = r_j$ (resp. $r = r_i$), then Φ_i and Φ_j admit a common degeneration.

(3) Using the explicit normal forms from Section 4, we check that generic representatives $\tilde{\phi}_i \in \Phi_i$, $\tilde{\phi}_j \in \Phi_j$ satisfy
$$\lim_{t\to 0} \lambda_j(t) \cdot \tilde{\phi}_i = \lim_{t\to 0} \lambda_i(t) \cdot \tilde{\phi}_j,$$
thereby confirming adjacency.

(4) For all other pairs, we rule out adjacency by support considerations together with the non-inclusion results of Section 7.

The main result of this section is the following classification:

Theorem 6.1. *Among the closed strata Φ_k in the strictly semistable locus, the only nonempty pairwise intersections are the following eight pairs:*
$$\{\Phi_1, \Phi_7\}, \quad \{\Phi_2, \Phi_6\}, \quad \{\Phi_3, \Phi_{12}\}, \quad \{\Phi_8, \Phi_{19}\},$$
$$\{\Phi_9, \Phi_{15}\}, \quad \{\Phi_{10}, \Phi_{17}\}, \quad \{\Phi_{11}, \Phi_{21}\}, \quad \{\Phi_{14}, \Phi_{20}\}.$$
All other pairwise intersections are empty.

Thus, the adjacency graph of the boundary consists precisely of these eight edges, realized as wall-crossings in the Hilbert–Mumford weight space (see Propositions 6.3-6.10 for the case-by-case verifications).

For the following definition, see [Kir84], [DH98], and [Tha96].

Definition 6.2. *[Adjacency via wall-crossing] Let Φ_i, Φ_j be closed strata in the strictly semistable locus. We say that Φ_i and Φ_j are* adjacent *if there exists a codimension-one wall in the Hilbert–Mumford weight space such that for general $f \in \Phi_i$ and $g \in \Phi_j$ one has $\mu(f, \lambda_i) = \mu(f, \lambda_j)$ on the wall and both specialize to the same unstable limit in $\mathbb{P}(W)$.*

The following sequence of propositions and the theorem can be verified by computations using the same algorithm as in Section 2. More precisely, we work inside the hyperplane $\mathbb{I}(\mathbf{r}_k)_{=0} \subset \mathbb{I}$ and determine the inclusion-maximal subsets of the form $\mathbb{I}(\mathbf{r})_{\geq 0} \cap \mathbb{I}(\mathbf{r}_k)_{=0}$, where $\mathbf{r} \in \mathbb{Z}^7_{(0)}$ ranges over 1-PS's of T.

Proposition 6.3. *The closed strata Φ_1 and Φ_7 are both zero-dimensional and adjacent. For $\mathbf{r} \in \mathbb{Z}^7_{(0)}$, the intersection $\mathbb{I}(\mathbf{r})_{\geq 0} \cap \mathbb{I}(\mathbf{r}_1)_{=0}$ is a maximal subset in $\mathbb{I}(\mathbf{r}_1)_{=0}$ if and only if $\mathbf{r} = \mathbf{r}_7$, and $\mathbb{I}(\mathbf{r})_{\geq 0} \cap \mathbb{I}(\mathbf{r}_7)_{=0}$ is a maximal subset in $\mathbb{I}(\mathbf{r}_7)_{=0}$ if and only if $\mathbf{r} = \mathbf{r}_1$. Moreover, for suitable specializations (with all coefficients of ϕ_k normalized to 1) $\tilde{\phi}_1, \tilde{\phi}_7$ one has*

$$\lim_{t \to 0} \lambda_7(t) \cdot \tilde{\phi}_1 \;=\; \lim_{t \to 0} \lambda_1(t) \cdot \tilde{\phi}_7 \;=\; x_0 x_5^2 + x_1^2 x_6.$$

Proposition 6.4. *The closed strata Φ_2 and Φ_6 are both zero-dimensional and adjacent. For $\mathbf{r} \in \mathbb{Z}^7_{(0)}$, the intersection $\mathbb{I}(\mathbf{r})_{\geq 0} \cap \mathbb{I}(\mathbf{r}_2)_{=0}$ is a maximal subset in $\mathbb{I}(\mathbf{r}_2)_{=0}$ if and only if $\mathbf{r} = \mathbf{r}_6$, and $\mathbb{I}(\mathbf{r})_{\geq 0} \cap \mathbb{I}(\mathbf{r}_6)_{=0}$ is a maximal subset in $\mathbb{I}(\mathbf{r}_6)_{=0}$ if and only if $\mathbf{r} = \mathbf{r}_2$. Moreover, for suitable specializations (with all coefficients of ϕ_k normalized to 1) $\tilde{\phi}_2, \tilde{\phi}_6$ one has*

$$\lim_{t \to 0} \lambda_6(t) \cdot \tilde{\phi}_2 \;=\; \lim_{t \to 0} \lambda_2(t) \cdot \tilde{\phi}_6 \;=\; x_1 x_3 x_5 + x_0 x_3 x_6.$$

Proposition 6.5. *The closed stratum Φ_3 is one-dimensional and Φ_{12} is zero-dimensional. For $\mathbf{r} \in \mathbb{Z}^7_{(0)}$, the intersection $\mathbb{I}(\mathbf{r})_{\geq 0} \cap \mathbb{I}(\mathbf{r}_3)_{=0}$ is a maximal subset in $\mathbb{I}(\mathbf{r}_3)_{=0}$ if and only if $\mathbf{r} = \mathbf{r}_{12}$, and $\mathbb{I}(\mathbf{r})_{\geq 0} \cap \mathbb{I}(\mathbf{r}_{12})_{=0}$ is a maximal subset in $\mathbb{I}(\mathbf{r}_{12})_{=0}$ if and only if $\mathbf{r} = \mathbf{r}_3$. Moreover, for suitable specializations (with all coefficients of ϕ_k normalized to 1) $\tilde{\phi}_3, \tilde{\phi}_{12}$ one has*

$$\lim_{t \to 0} \lambda_{12}(t) \cdot \tilde{\phi}_3 \;=\; \lim_{t \to 0} \lambda_3(t) \cdot \tilde{\phi}_{12} \;=\; x_1 x_4^2 + x_2^2 x_5 + x_0 x_3 x_6.$$

Proposition 6.6. *The closed strata Φ_8 and Φ_{19} are both two-dimensional and adjacent. For $\mathbf{r} \in \mathbb{Z}^7_{(0)}$, the intersection $\mathbb{I}(\mathbf{r})_{\geq 0} \cap \mathbb{I}(\mathbf{r}_8)_{=0}$ is a maximal subset in $\mathbb{I}(\mathbf{r}_8)_{=0}$ if and only if $\mathbf{r} = \mathbf{r}_{19}$, and $\mathbb{I}(\mathbf{r})_{\geq 0} \cap \mathbb{I}(\mathbf{r}_{19})_{=0}$ is a maximal subset in $\mathbb{I}(\mathbf{r}_{19})_{=0}$ if and only if $\mathbf{r} = \mathbf{r}_8$. Moreover, for suitable specializations (with all coefficients of ϕ_k normalized to 1) $\tilde{\phi}_8, \tilde{\phi}_{19}$ one has*

$$\lim_{t \to 0} \lambda_{19}(t) \cdot \tilde{\phi}_8 \;=\; \lim_{t \to 0} \lambda_8(t) \cdot \tilde{\phi}_{19} \;=\; x_3^3 + x_0 x_4^2 + x_0 x_4 x_5 + x_0 x_5^2 + x_1^2 x_6 + x_1 x_2 x_6 + x_2^2 x_6$$

Proposition 6.7. *The closed strata Φ_9 and Φ_{15} are both two-dimensional and adjacent. For $\mathbf{r} \in \mathbb{Z}^7_{(0)}$, the intersection $\mathbb{I}(\mathbf{r})_{\geq 0} \cap \mathbb{I}(\mathbf{r}_9)_{=0}$ is a maximal subset in $\mathbb{I}(\mathbf{r}_9)_{=0}$ if and only if $\mathbf{r} = \mathbf{r}_{15}$, and $\mathbb{I}(\mathbf{r})_{\geq 0} \cap \mathbb{I}(\mathbf{r}_{15})_{=0}$ is a maximal subset in $\mathbb{I}(\mathbf{r}_{15})_{=0}$ if and only if $\mathbf{r} = \mathbf{r}_9$. Moreover, for suitable specializations (with all coefficients of ϕ_k normalized to 1) $\tilde{\phi}_9, \tilde{\phi}_{15}$ one has*

$$\lim_{t \to 0} \lambda_{15}(t) \cdot \tilde{\phi}_9 \;=\; \lim_{t \to 0} \lambda_9(t) \cdot \tilde{\phi}_{15} \;=\; x_3^3 + x_0x_4x_5 + x_1x_2x_6 + x_0x_3x_6$$

Proposition 6.8. *The closed strata Φ_{10} and Φ_{17} are both two-dimensional and adjacent. For $\mathbf{r} \in \mathbb{Z}^7_{(0)}$, the intersection $\mathbb{I}(\mathbf{r})_{\geq 0} \cap \mathbb{I}(\mathbf{r}_{10})_{=0}$ is a maximal subset in $\mathbb{I}(\mathbf{r}_{10})_{=0}$ if and only if $\mathbf{r} = \mathbf{r}_{17}$, and $\mathbb{I}(\mathbf{r})_{\geq 0} \cap \mathbb{I}(\mathbf{r}_{17})_{=0}$ is a maximal subset in $\mathbb{I}(\mathbf{r}_{17})_{=0}$ if and only if $\mathbf{r} = \mathbf{r}_{10}$. Moreover, for suitable specializations (with all coefficients of ϕ_k normalized to 1) $\tilde{\phi}_{10}, \tilde{\phi}_{17}$ one has*

$$\lim_{t \to 0} \lambda_{17}(t) \cdot \tilde{\phi}_{10} \;=\; \lim_{t \to 0} \lambda_{10}(t) \cdot \tilde{\phi}_{17} \;=\; x_3^3 + x_1x_3x_5 + x_0x_4x_5 + x_1x_2x_6$$

Proposition 6.9. *The closed strata Φ_{11} and Φ_{21} are both three-dimensional and adjacent. For $\mathbf{r} \in \mathbb{Z}^7_{(0)}$, the intersection $\mathbb{I}(\mathbf{r})_{\geq 0} \cap \mathbb{I}(\mathbf{r}_{11})_{=0}$ is a maximal subset in $\mathbb{I}(\mathbf{r}_{11})_{=0}$ if and only if $\mathbf{r} = \mathbf{r}_{21}$, and $\mathbb{I}(\mathbf{r})_{\geq 0} \cap \mathbb{I}(\mathbf{r}_{21})_{=0}$ is a maximal subset in $\mathbb{I}(\mathbf{r}_{21})_{=0}$ if and only if $\mathbf{r} = \mathbf{r}_{11}$. Moreover, for suitable specializations (with all coefficients of ϕ_k normalized to 1) $\tilde{\phi}_{11}, \tilde{\phi}_{21}$ one has*

$$\lim_{t \to 0} \lambda_{21}(t) \cdot \tilde{\phi}_{11} \;=\; \lim_{t \to 0} \lambda_{11}(t) \cdot \tilde{\phi}_{21}$$

$$= x_2^3 + x_2^2x_3 + x_2x_3^2 + x_3^3 + x_2^2x_4 + x_2x_3x_4 + x_3^2x4 + x_2x_4^2 + x_3x_4^2 + x_4^3$$

Proposition 6.10. *The closed strata Φ_{14} and Φ_{20} are both one-dimensional and adjacent. For $\mathbf{r} \in \mathbb{Z}^7_{(0)}$, the intersection $\mathbb{I}(\mathbf{r})_{\geq 0} \cap \mathbb{I}(\mathbf{r}_{14})_{=0}$ is a maximal subset in $\mathbb{I}(\mathbf{r}_{14})_{=0}$ if and only if $\mathbf{r} = \mathbf{r}_{20}$, and $\mathbb{I}(\mathbf{r})_{\geq 0} \cap \mathbb{I}(\mathbf{r}_{20})_{=0}$ is a maximal subset in $\mathbb{I}(\mathbf{r}_{20})_{=0}$ if and only if $\mathbf{r} = \mathbf{r}_{14}$. Moreover, for suitable specializations (with all coefficients of ϕ_k normalized to 1) $\tilde{\phi}_{14}, \tilde{\phi}_{20}$ one has*

$$\lim_{t \to 0} \lambda_{20}(t) \cdot \tilde{\phi}_{14} \;=\; \lim_{t \to 0} \lambda_{14}(t) \cdot \tilde{\phi}_{20} \;=\; x_0x_3x_5 + x_1x_3x_5 + x_0x_3x_6 + x_1x_3x_6$$

Collecting the above, we obtain Theorem 6.1.

7 Non-inclusions and algorithmic certification

In this section, we prove Proposition 3.5. As treating all $21 \times 20 = 420$ ordered pairs (k, l) by hand is essentially impossible, we rely on computer algebra to produce a machine-checkable certificate. To this end, we first recast the problem in a computational form. All computations in this section were carried out using Magma [BCP97].

Let $P = (p_{ij})$ be a 7×7 matrix whose entries p_{ij} are algebraically independent indeterminates, and let f_k^P denote the image of f_k under the linear change of variables determined by P. Without loss of generality, we may assume that all coefficients of f_k are 1. Then

$$\mathrm{Supp}(f_k^P) \subset \mathbb{I}(\mathbf{r}_l)_{\geq 0}$$

holds if and only if every coefficient of a monomial that does not lie in $I(\mathbf{r}_l)_{\geq 0}$ (a *forbidden monomial*) vanishes in f_k^P. These coefficients are cubic polynomials in the variables p_{ij}. Let $I(k,l)$ denote the ideal generated by these coefficients in the polynomial ring in the p_{ij}.

Thus, it suffices to show that the affine variety

$$V\big(I(k,l) \ + \ \langle \det(P) - 1 \rangle\big) = \varnothing.$$

In practice, we verify this emptiness by a Gröbner basis computation: the basis reduces to 1, thereby providing a finite, machine-checkable certificate of non-inclusion. This is the overall strategy we follow below.

We shall use the following Rabinowitsch trick (see [CLO07] Section 4, Proposition 8 p.178).

Theorem 7.1 (Rabinowitsch trick)**.** *Let $R = \mathbb{C}[p_{ij}]$ be the polynomial ring in the indeterminates p_{ij}, and let $I \subset R$ be an ideal and $f \in R$. Introduce a new indeterminate t. Then the following are equivalent:*

$$V(I) \cap D(f) = \varnothing \qquad \Longleftrightarrow \qquad 1 \in I\,R[t] \ + \ \langle 1 - tf \rangle \ \subset R[t],$$

where $D(f) \subset \operatorname{Spec} R$ denotes the principal open subset $\{\mathfrak{p} \in \operatorname{Spec} R \mid f \notin \mathfrak{p}\}$.

Applying this to our situation, let $P = (p_{ij})$ and set

$$J_{k,l} \ := \ I(k,l) \ + \ \langle\, 1 - t\,\det(P)\,\rangle \ \subset R[t].$$

By Theorem 7.1, the condition

$$1 \in J_{k,l}$$

is equivalent to

$$V\big(I(k,l)\big) \ \cap \ D\big(\det(P)\big) \ = \ \varnothing.$$

Hence, it suffices to prove $1 \in J_{k,l}$. (Note that this imposes only the open condition $\det(P) \neq 0$, which is sufficient for our purposes.)

Theorem 7.2. *Running this algorithm using Gröbner bases, we verify that $1 \in J_{k,l}$ for every pair (k,l). Hence, Proposition 3.5 holds.*

Remark 7.3. *Although one can run the above algorithm verbatim, the computation time is substantial, so we consider accelerating the procedure. From the system of cubic equations in the variables p_{ij}, it follows that some of the entries p_{ij} must vanish. We record the following example as typical.*

Example. We show that f_6 is *not* contained in f_2 modulo SL(7). The monomials x_3^3, $x_3^2x_4$, $x_2x_3^2$, $x_2x_3x_4$, $x_2x_4^2$, $x_2^2x_5$ do not appear in the support of f_2, i.e., they are forbidden monomials. Embedding f_6 into f_2 would require transforming the cubic part $c(x_0, x_1, x_2, x_3)$ of f_6 into the cubic part $c(x_0, x_1, x_2)$ of f_2 via an element of SL(7). Consequently, the change of variables must have the form

$$\begin{aligned}
x_0 &= l_0(x_0, x_1, x_2), & x_1 &= l_1(x_0, x_1, x_2),\\
x_2 &= l_2(x_0, x_1, x_2), & x_3 &= l_3(x_0, x_1, x_2),\\
x_4 &= l_4(x_0, x_1, x_2, x_3, x_4, x_5, x_6), & x_5 &= l_5(x_0, x_1, x_2, x_3, x_4, x_5, x_6),\\
x_6 &= l_6(x_0, x_1, x_2, x_3, x_4, x_5, x_6), & &
\end{aligned}$$

where each l_i is a linear form. However, under such a transformation, the monomial $x_2x_3^2$ typically re-emerges from $l_2(x_0, x_1)x_4^2$; this monomial is not in the support of f_2. To avoid this, we must restrict the first three variables to

$$x_0 = l_0(x_0, x_1), \qquad x_1 = l_1(x_0, x_1), \qquad x_2 = l_2(x_0, x_1, x_2).$$

Yet new monomials such as $x_1x_5^2$, $x_0x_6^3$, $x_1x_6^2$ arise from the term $l_2(x_0, x_1)x_4^2$ and are likewise absent from the support of f_2. Hence, we must further impose $x_4 = l_4(x_3, x_4)$. With these conditions, the associated matrix $P \in \mathrm{SL}(7)$ has the block form

$$\begin{pmatrix}
p_{0,0} & p_{0,1} & 0 & 0 & 0 & 0 & 0\\
p_{1,0} & p_{1,1} & 0 & 0 & 0 & 0 & 0\\
p_{2,0} & p_{2,1} & p_{2,2} & 0 & 0 & 0 & 0\\
p_{3,0} & p_{3,1} & p_{3,2} & 0 & 0 & 0 & 0\\
p_{4,0} & p_{4,1} & p_{4,2} & p_{4,3} & p_{4,4} & 0 & 0\\
p_{5,0} & p_{5,1} & p_{5,2} & p_{5,3} & p_{5,4} & p_{5,5} & p_{5,6}\\
p_{6,0} & p_{6,1} & p_{6,2} & p_{6,3} & p_{6,4} & p_{6,5} & p_{6,6}
\end{pmatrix}.$$

Its determinant is zero, so no such matrix lies in SL(7). Therefore, f_6 cannot be contained in f_2 modulo SL(7).

Remark 7.4. *In this example, the conditions $p_{i,j} = 0$ alone already force* $\det(P) = 0$. *In general, however, such vanishing conditions by themselves do not suffice to imply* $\det(P) = 0$. *Rather, the relations $p_{i,j} = 0$ reduce the number of variables and simultaneously simplify the system of equations, so that a Gröbner basis computation becomes feasible within a reasonable amount of time, and from this computation, the conclusion* $\det(P) = 0$ *can then be drawn.*

We can decide whether $p_{ij} = 0$ is forced by using the Rabinowitsch trick as follows.

Proposition 7.5. *The following are equivalent:*

$$V\big(I_{k,l}\big) \cap D(p_{ij}) = \varnothing \qquad \Longleftrightarrow \qquad 1 \in I_{k,l} + \langle\, 1 - t\, p_{ij} \,\rangle \subset R[t].$$

Consequently, one can determine whether the vanishing $p_{ij} = 0$ is forced by a Gröbner–basis computation.

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